On Certain Infinite Products

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abstract

In this note we presents infinite products for some classical constants.

1. Introducción

Recordamos algunas constantes clásicas:

1. Constante pi:

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.14159265358979323846... \tag{1}$$

2. Constante e :

$$e = \sum_{n=0}^{\infty}\frac{1}{n!} = 2.71828182845904553488... \tag{2}$$

3. Constante γ :

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}-\ln n\right) = 0.57721566490153286060... \tag{3}$$

4. Constante $\ln 2$:

$$\ln 2 = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = 0.69314718055994528622... \tag{4}$$

5. Constante $\sqrt{2}$:

$$\sqrt{2} = 2\sum_{n=0}^{\infty}(-1)^n\binom{2n}{n}2^{-2n} = 1.41421356237309504880... \tag{5}$$

6. Constante ϕ :

$$\phi = \frac{1+\sqrt{5}}{2} = 1.61803398874989490252... \tag{6}$$

7. Constante $\zeta(3)$:

$$\zeta(3)=\sum_{n=1}^{\infty}\frac{1}{n^3}=1.202056903159594014 59... \tag{7}$$

8. Constante G :

$$G=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}=0.9159655941772190150 5... \tag{8}$$

En esta nota mostramos productos infinitos para las constantes (1)-(8) .

2. Productos Infinitos

$$\pi=\frac{3}{1}\cdot\frac{7}{6}\cdot\frac{13}{14}\cdot\frac{25}{26}\cdot\frac{51}{50}\cdot\frac{101}{102}\cdot\frac{201}{202}\cdots \tag{9}$$

$$e=\frac{2}{1}\cdot\frac{5}{4}\cdot\frac{11}{10}\cdot\frac{21}{22}\cdot\frac{43}{42}\cdot\frac{87}{86}\cdot\frac{173}{174}\cdots \tag{10}$$

$$\gamma=\frac{1}{1}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{6}\cdot\frac{9}{10}\cdot\frac{19}{18}\cdot\frac{37}{38}\cdots \tag{11}$$

$$\ln 2=\frac{1}{1}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{6}\cdot\frac{11}{10}\cdot\frac{23}{22}\cdot\frac{45}{46}\cdots \tag{12}$$

$$\sqrt{2}=\frac{1}{1}\cdot\frac{3}{2}\cdot\frac{5}{6}\cdot\frac{11}{10}\cdot\frac{23}{22}\cdot\frac{45}{46}\cdot\frac{91}{90}\cdots \tag{13}$$

$$\phi=\frac{1}{1}\cdot\frac{3}{2}\cdot\frac{7}{6}\cdot\frac{13}{14}\cdot\frac{25}{26}\cdot\frac{51}{50}\cdot\frac{103}{102}\cdots \tag{14}$$

$$\zeta(3)=\frac{1}{1}\cdot\frac{3}{2}\cdot\frac{5}{6}\cdot\frac{9}{10}\cdot\frac{19}{18}\cdot\frac{39}{38}\cdot\frac{77}{78}\cdots \tag{15}$$

$$G=\frac{1}{1}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{7}{6}\cdot\frac{15}{14}\cdot\frac{29}{30}\cdot\frac{59}{58}\cdots \tag{16}$$

Otros ejemplos son:

$$\pi+e=\frac{5}{1}\cdot\frac{11}{10}\cdot\frac{23}{22}\cdot\frac{47}{46}\cdot\frac{93}{94}\cdot\frac{187}{186}\cdot\frac{375}{374}\cdots \tag{17}$$

$$\sqrt[3]{2}=\frac{1}{1}\cdot\frac{3}{2}\cdot\frac{5}{6}\cdot\frac{11}{10}\cdot\frac{21}{22}\cdot\frac{41}{42}\cdot\frac{81}{82}\cdots \quad (18)$$

$$\sqrt{2+\sqrt{2}}=\frac{1}{1}\cdot\frac{3}{2}\cdot\frac{7}{6}\cdot\frac{15}{14}\cdot\frac{29}{30}\cdot\frac{59}{58}\cdot\frac{119}{118}\cdots \quad (19)$$

$$e^{\pi}=\frac{23}{1}\cdot\frac{47}{46}\cdot\frac{93}{94}\cdot\frac{185}{186}\cdot\frac{371}{370}\cdot\frac{741}{742}\cdots \quad (20)$$

3. La función fpm

Sean $a, N \in \mathbb{N}=\{1,2,3,4,...\}$, $\lambda>0$ tales que: $a<\lambda<a+1$, se define la función fpm como sigue:

$$fpm(a,\lambda,N):=\left|\begin{array}{c} imput\ a,\lambda,N \\ x(1)=a, b=a+1; \\ for\ m=2:N \\ pm=(a+b)/2; \\ [n,d]=numden(sym(pm)); \\ x(m)=n; \\ if\ \ pm<\lambda \\ a=pm; \\ else\ b=pm; \\ end \\ end \\ output\ x(m), m=1..N \end{array}\right. \quad (21)$$

Descripción: (21) está escrita en código Matlab.

(i) Variables de entrada: a,λ,N

(ii) Variable de salida: $x(m), m=1..N$

(iii) Función numden:entrega el numerador y denominador de la fracción pm .

(iv) Los números $x(m)\in\mathbb{N}$ son impares.

(v) Es válida la siguiente igualdad;

$$\lambda = \frac{x(1)}{1} \cdot \frac{x(2)}{2x(1)} \cdot \frac{x(3)}{2x(2)} \cdot \frac{x(4)}{2x(3)} \cdot \frac{x(5)}{2x(4)} \cdots \qquad (22)$$

4. Algunos Ejemplos

$$fpm(3,\pi,15) = \\ = \{3,7,13,25,51,101,201,403,805,1609,3217,6433,12867,25735,51471,...\} \qquad (23)$$

$$fpm(2,e,15) = \\ = \{2,5,11,21,43,87,173,347,695,1391,2783,5567,11135,22269,44537,...\} \qquad (24)$$

$$fpm(0,\gamma,15) = \\ = \{1,1,3,5,9,19,37,73,147,295,591,1183,2365,4729,9457,...\} \qquad (25)$$

$$fpm(0,\ln 2,15) = \\ = \{1,1,3,5,11,23,45,89,177,355,709,1419,2839,5679,11357,...\} \qquad (26)$$

$$fpm(1,\sqrt{2},15) = \\ = \{1,3,5,11,23,45,91,181,363,725,1449,2897,5793,11585,23171,...\} \qquad (27)$$

$$fpm(1,\phi,15) = \\ = \{1,3,7,13,25,51,103,207,415,829,1657,3313,6627,13255,26509,...\} \qquad (28)$$

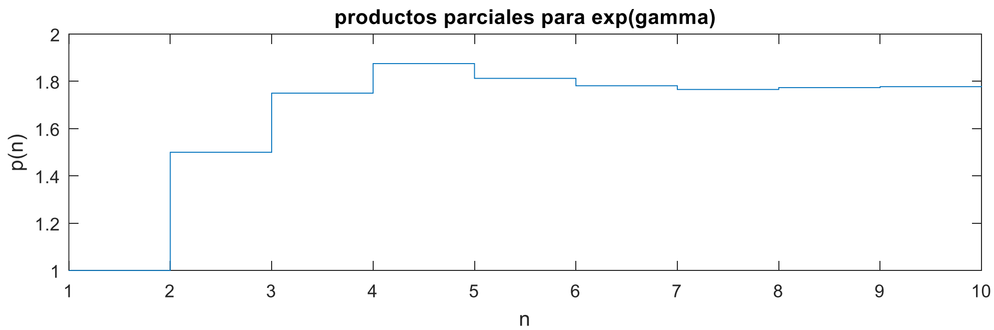
$$fpm(1,\zeta(3),15) = \\ = \{1,3,5,9,19,39,77,153,307,615,1231,2461,4923,9847,19695,...\} \qquad (29)$$

$$fpm(0,G,15) = \\ = \{1,1,3,7,15,29,59,117,235,469,937,1875,3751,7503,15007,...\} \qquad (30)$$

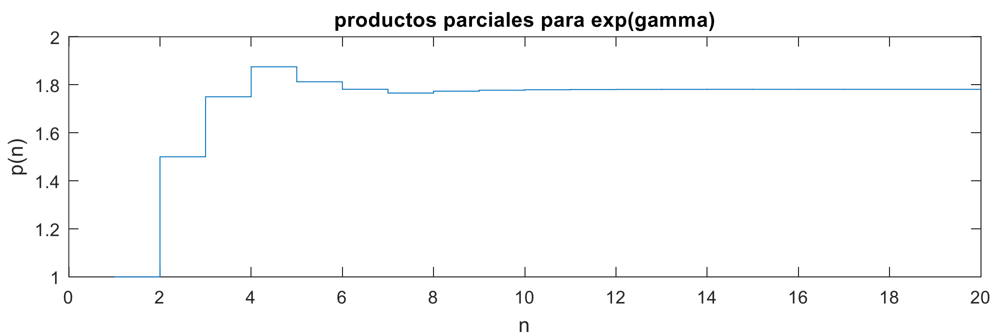
5. Ejemplos finales

$$e^{\gamma} = \frac{1}{1} \cdot \frac{3}{2} \cdot \frac{7}{6} \cdot \frac{15}{14} \cdot \frac{29}{30} \cdot \frac{57}{58} \cdot \frac{113}{114} \cdot \frac{227}{226} \cdot \frac{455}{254} \cdot \frac{911}{910} \cdots = 1.781072... \qquad (31)$$

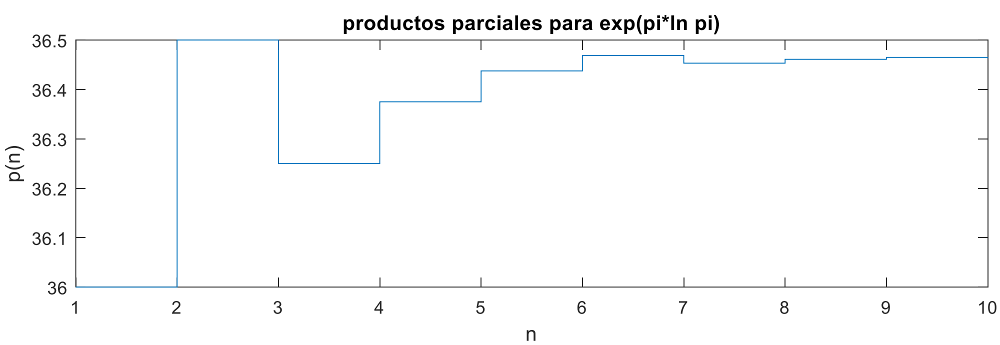
$$\pi^{\pi} = \frac{36}{1} \cdot \frac{73}{72} \cdot \frac{145}{146} \cdot \frac{291}{290} \cdot \frac{583}{582} \cdot \frac{1167}{1166} \cdot \frac{2333}{2334} \cdot \frac{4667}{4666} \cdots = 36.462159... \qquad (32)$$



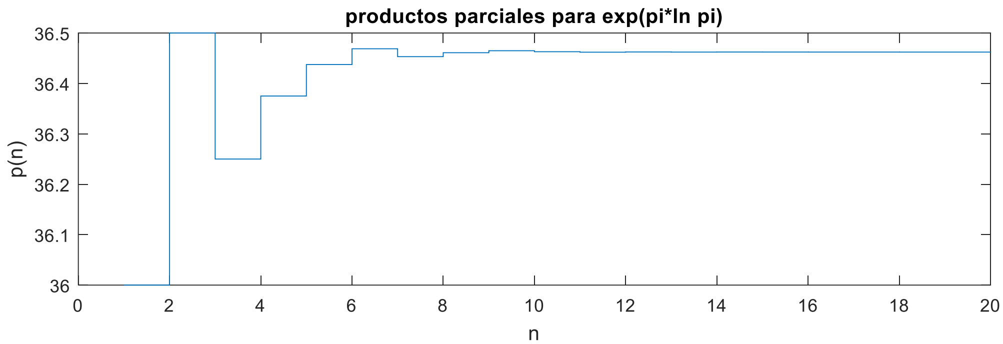
$$p(n) \to e^{\gamma} \quad , \quad n = 1 \ldots 10$$



$$p(n) \to e^{\gamma} \quad , \quad n = 1 \ldots 20$$



$$p(n) \to \pi^{\pi} \quad , \quad n = 1 \ldots 10$$



$$p(n) \to \pi^{\pi} \quad , \quad n = 1 \ldots 20$$

Referencias

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