Elementary Proof of the Goldbach Conjecture

Stephen Michael Marshall

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Abstract

Christian Goldbach (March 18, 1690 – November 20, 1764) was a German mathematician. He is remembered today for Goldbach's conjecture.

Goldbach's conjecture is one of the oldest and best-known unsolved problems in number theory and all of mathematics. It states: Every even integer greater than 2 can be expressed as the sum of two primes.

On 7 June 1742, the German mathematician Christian Goldbach wrote a letter to Leonhard Euler (letter XLIII) in which he proposed the following conjecture:
Every even integer which is ≥ 4 can be written as the sum of two primes (the strong conjecture)
He then proposed a second conjecture in the margin of his letter:

Every odd integer greater than 5 can be written as the sum of three primes (the weak conjecture).

In number theory, Goldbach's weak conjecture, also known as the ternary Goldbach problem, states that every odd number greater than 5 can be expressed as the sum of three primes. (A prime may be used more than once in the same sum). In 2013, Harald Helfgott finally proved Goldbach's weak conjecture, a huge contribution to mathematics and number theory.

The “strong” conjecture has been shown to hold up through 4×10^{18}, but remains unproven for almost 300 years despite considerable effort by many mathematicians throughout history.

The author would like to give many thanks to Harald Helfgott for his proof of the weak conjecture, because this elementary proof of the strong conjecture is completely dependent on Helfgott’s proof. Without Helfgott’s proof, this elementary proof would not be possible.

Proof

Goldbach's conjecture is one of the oldest and best-known unsolved problems in number theory and all of mathematics. It states: Every even integer greater than 2 can be expressed as the sum of two primes.

The Goldbach Conjecture states that for every even integer N, and N > 2, then $N = P_1 + P_2$, where P_1, and P_2, are prime numbers.

For example, when N = 4, then 4 = 2 + 2, and since 2 is prime then the Goldbach Conjecture is satisfied. When N = 6, then 6 = 3 + 3, and since 3 is prime then the Goldbach Conjecture is satisfied again.

A proof of the strong Goldbach conjecture implies the ternary Goldbach conjecture, that is, all odd numbers greater than 5 are the sum of three primes. For example, in order to express an odd number $n > 5$ as the sum of three primes, subtract 3 and obtain an even number $n - 3 \geq 2$. If the strong conjecture is true, we can express $n – 3$ as a sum of two primes p_1, p_2; thus, since $n - 3$ is an even ≥ 2, then $n = (n - 3) + 3$ is the sum of the primes p_1, p_2 and 3, which is the sum of three prime numbers. Thus, proving the ternary Goldbach conjecture, if the strong conjecture is true. That is, for $n > 5$,

$$n = (n - 3) + 3$$

$$n = p_1 + p_2 + 3$$

While the weak Goldbach conjecture was finally proved, by Helfgott [1][2] in 2013, however the strong conjecture has remained unsolved. In this paper we shall use Helfgott’s proof of the ternary Goldbach conjecture to prove the strong conjecture of even numbers.

Helfgott’s proof of the ternary Goldbach conjecture does establish that every even number can be written as the sum of at most 4 primes. For example, subtract any odd prime number, p_4, from every even number, m, that is greater than the prime number being subtracted results with another odd number. That is, $m - p_4$ = an odd number. Now, to Helfgott’s credit we can write the odd number, $m - p_4$, as the sum of three primes. This can be written as:

$$m - p_4 = p_1 + p_2 + p_3$$

$$m = p_1 + p_2 + p_3 + p_4$$

Thus, proving every even number can be written as the sum of at most 4 primes. However, to prove that the strong Goldbach conjecture we must reduce this improvement of sum of four primes down to the sum of two primes.

Let $n > 5$ be any odd number, then it is the sum of three primes p_1, p_2, p_3, then the ternary Goldbach conjecture can be written as follows:

$$n = p_1 + p_2 + p_3$$

Subtract p_3 from both sides and the following even number is generated:

$$n - p_3 = p_1 + p_2$$

$$\text{and, } n - p_3 \geq 4$$

This proves that this even number is the sum of two primes, but it does not guarantee that every even number is the sum of two primes.

Now we will prove that every even number can be written as a prime number subtracted from an odd number.

$n - p_3$ = even number, where, $n > 5$ and n is odd and p_3 = prime

If $n = 7$, and $p_3 = 3$, then $n - p_3 = 4$ (the 2nd even number)

Then continuing again, if $n = 9$, and $p_3 = 3$, then $n - p_3 = 6$ (the 3rd even number)

Then continuing again, if $n = 11$, and $p_3 = 3$, then $n - p_3 = 8$ (the 4th even number)

Then continuing again, if $n = 13$, and $p_3 = 3$, then $n - p_3 = 10$ (the 5th even number)

From the above series, it can be easily seen that if $p_3 = 3$ is held constant and starting with the odd number 7 and increasing incrementally to the next odd number (i.e., 9) the next incrementally larger even number is increased by 2 (i.e., 6), etc.

We have demonstrated our logic above, but have not proven it yet. Now we proceed with the proof.

The formal definition of an odd number is that it is an integer of the form $n = 2k + 1$, where k is an integer. Consider the following where every odd number, n can be represented in the following form:

$$n = 2k + 1, \text{ for } k \geq 1$$

However, for the Ternary Goldbach conjecture, $n > 5$, therefore the smallest odd number used is $n = 7$.

Subtracting the prime number 3 from both sides' yields:

$$n - 3 = 2k - 2$$

$$n - 3 = 2(k - 1)$$

This implies that every even number ≥ 4 (for the Ternary Goldbach conjecture) can be generated from the above equation for integers $k = 3, 4, 5, \ldots. k_i, k_{i+1}, \ldots. k_{\infty.}$ As $k_{\mapsto\infty}$, approaches infinity all even numbers ≥ 4 are generated, this completes the proof.

This proves that every even number can be written as a prime number subtracted from an odd number. Specifically, in our case above $p_3 = 3$ was held constant. The above logic, would work for any prime, p_3, when it is held constant, the odd number n would just need to be increased accordingly.

Additionally, we will prove every even number is represented by using Mathematical Induction as a proof. Since $n > 5$, then for the Ternary Goldbach conjecture the baseline for n is 7 rather than 1 that is typically used as the baseline for Mathematical Induction. Therefore, for the baseline we must show $n - 3 = 2(k - 1)$ holds for $n = 7$,

$$n - 3 = 2(k - 1)$$

$$7 - 3 = 2(k - 1)$$

$$4 = 2(k - 1) = 2k - 2$$

$$2k = 6$$

$$k = 3$$

Therefore, 3 is the baseline for k

Thus, for the induction baseline, we must show $n - 3 = 2(k - 1)$ is even for $k = 3$

$$n - 3 = 2(k - 1)$$

$$n - 3 = 2(3 - 1)$$

$$n - 3 = 4$$

This is correct for the baseline, as 4 is the first even number included in the Goldbach's strong conjecture.

Then, we must suppose it holds for any even number k. This supposition is known as the induction hypothesis. We assume it is true, and aim to show that our assumption is true:

$$n - 3 = 2(k - 1), \text{ is true for } k$$

Using the above induction hypothesis, we must show it is true for $k + 1$, that is:

$$n - 3 = 2((k + 1) - 1)$$

$$n - 3 = 2k + 2 - 2$$

$$n - 3 = 2k$$

Then, we substitute the baseline for $k = 3$, and $n - 3 = 6$, which is the next even number. Since $n - 3 = 2k$, as k is incremented by 1, every even number will be generated by $2k$. Thus we have proven using Mathematical Induction that every even number is generated by $n - p_3$.

Summarizing, the Ternary Goldbach conjecture can be written as follows, for an odd number $n > 5$:

$$n = p_1 + p_2 + p_3$$

Subtract p_3 from both sides and the following even number is generated:

$$n - p_3 = p_1 + p_2$$

Since we have already proven that $n - p_3$, when we hold p_3 constant at 3, generates every even number > 4, then we have proven every even number > 4 can be written as the sum of two primes in the following form:

$$n - p_3 = p_1 + p_2$$

Based on our proof, the above is identical to the following form for e = all even numbers ≥ 4:

$$e = p_1 + p_2$$

Again, the Goldbach Conjecture states that for every even integer n, and $n > 2$, then $n = p_1 + p_2$, where p_1, and p_2, are prime numbers. Therefore, our proof above, thoroughly proves the Goldbach Conjecture.

Again, the author expresses his eternal gratefulness to Harald Helfgott for his outstanding proof of the of the ternary Goldbach conjecture. Without Helfgott proof, the author's elementary proof would not have been possible, it is totally dependent on Helfgott's proof.

The author is not aware of an attempt having previously been made to approach the Goldbach conjecture in this way. If that is so, it would be remarkable that such a simple argument has hitherto been overlooked.

References:

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