Elementary Proof of the Goldbach Conjecture

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Abstract

Christian Goldbach (March 18, 1690 – November 20, 1764) was a German mathematician. He is remembered today for Goldbach's conjecture.

Goldbach's conjecture is one of the oldest and best-known unsolved problems in number theory and all of mathematics. It states: Every even integer greater than 2 can be expressed as the sum of two primes.

On 7 June 1742, the German mathematician Christian Goldbach wrote a letter to Leonhard Euler (letter XLIII) in which he proposed the following conjecture:
Every even integer which is ≥ 4 can be written as the sum of two primes (the strong conjecture)
He then proposed a second conjecture in the margin of his letter:

Every odd integer greater than 5 can be written as the sum of three primes (the weak conjecture).

In number theory, Goldbach's weak conjecture, also known as the ternary Goldbach problem, states that every odd number greater than 5 can be expressed as the sum of three primes. (A prime may be used more than once in the same sum). In 2013, Harald Helfgott finally proved Goldbach's weak conjecture, a huge contribution to mathematics and number theory.

The "strong" conjecture has been shown to hold up through 4×10^{18}, but remains unproven for almost 300 years despite considerable effort by many mathematicians throughout history.

The author would like to give many thanks to Harald Helfgott for his proof of the weak conjecture, because this elementary proof of the strong conjecture is completely dependent on Helfgott's proof. Without Helfgott's proof, this elementary proof would not be possible.

Proof

Goldbach's conjecture is one of the oldest and best-known unsolved problems in number theory and all of mathematics. It states: Every even integer greater than 2 can be expressed as the sum of two primes.

The Goldbach Conjecture states that for every even integer N, and N > 2, then $N = P_1 + P_2$, where P_1, and P_2, are prime numbers.

For example, when N = 4, then 4 = 2 + 2, and since 2 is prime then the Goldbach Conjecture is satisfied. When N = 6, then 6 = 3 + 3, and since 3 is prime then the Goldbach Conjecture is satisfied again.

A proof of the strong Goldbach conjecture implies the ternary Goldbach conjecture, that is, all odd numbers greater than 5 are the sum of three primes. For example, in order to express an odd number $n > 5$ as the sum of three primes, subtract 3 and obtain an even number $n - 3 \geq 2$. If the strong conjecture is true, we can express $n - 3$ as a sum of two primes p_1, p_2; thus, since $n - 3$ is an even ≥ 2, then $n = (n - 3) + 3$ is the sum of the primes p_1, p_2 and 3, which is the sum of three prime numbers. Thus, proving the ternary Goldbach conjecture, if the strong conjecture is true. That is, for $n > 5$,

$$n = (n - 3) + 3$$

$$n = p_1 + p_2 + 3$$

While the weak Goldbach conjecture was finally proved, by Helfgott [1][2] in 2013, however the strong conjecture has remained unsolved. In this paper we shall use Helfgott's proof of the ternary Goldbach conjecture to prove the strong conjecture of even numbers.

Helfgott's proof of the ternary Goldbach conjecture does establish that every even number can be written as the sum of at most 4 primes. For example, subtract any odd prime number, p_4, from every even number, m, that is greater than the prime number being subtracted results with another odd number. That is, $m - p_4$ = an odd number. Now, to Helfgott's credit we can write the odd number, $m - p_4$, as the sum of three primes. This can be written as:

$$m - p_4 = p_1 + p_2 + p_3$$

$$m = p_1 + p_2 + p_3 + p_4$$

Thus, proving every even number can be written as the sum of at most 4 primes. However, to prove that the strong Goldbach conjecture we must reduce this improvement of sum of four primes down to the sum of two primes.

Let $n > 5$ be any odd number, then it is the sum of three primes p_1, p_2, p_3, then the ternary Goldbach conjecture can be written as follows:

$$n = p_1 + p_2 + p_3$$

Also, we assume p_1, p_2, and p_3, are all > 2

Subtract p_3 from both sides and the following even number is generated:

$$n - p_3 = p_1 + p_2$$

$$\text{and, } n - p_3 \geq 4$$

This proves that this even number is the sum of two primes, but it does not guarantee that every even number is the sum of two primes.

Now we will prove that every even number can be written as the sum of two prime numbers.

The formal definition of an odd number is that it is an integer of the form $n = 2k + 1$, where k is an integer. Consider the following where every odd number, n can be represented in the following form:

$$n = 2k + 1, \text{ for } k \geq 1$$

The formal definition of an even number is that it is an integer of the form $m = 2k$, where k is an integer. Consider the following where every even number, m can be represented in the following form:

$$m = 2k, \text{ for } k \geq 1$$

Reviewing, Harald Helfgott has proven the Ternary Goldbach conjecture, which can be written as follows, for an odd number $n > 5$:

$$n = p_1 + p_2 + p_3$$

$$2k + 1 = n = p_1 + p_2 + p_3$$

Reducing,

$$2k = p_1 + p_2 + p_3 – 1$$

Therefore, since $2k$ is every even number, we have drawn a step closer to proving the Goldbach conjecture, as follows:

$$2k = p_1 + p_2 + (p_3 – 1)$$

Rearranging,

$$2k + (1 - p_3) = p_1 + p_2$$

Now, we have proven that $2k + (1 - p_3) = p_1 + p_2$, however, we must still prove that every even integer greater than 2 can be expressed as the sum of two primes ($p_1 + p_2$). To accomplish this, we will take an approach to prove that $2k + (1 - p_3)$ includes, at a minimum, every even positive integer greater than 2. First let us review that every even number greater than 2, m can be represented in the following form:

$$m = 2k, \text{ for } k \geq 2$$

$$\text{Or, in other words, } m = 2k, \text{ for } k = 2, 3, 4 \ldots \infty$$

First, we know that $2k + (1 - p_3)$ is always even since p_3 is always odd (unless p_3 is equal to 2, however, by definition we will exclude 2 from being equal to p_3 since we are only interested in $2k + (1 - p_3)$ being even. Therefore, we observe that the minimum vale for p_3 is 3, since it is prime. When p_3 is 3 and $k = 2$, then $2k + (1 - p_3) = 2*2 + (1 - 3) = 4 - 2 = 2$, however, for Goldbach's Conjecture we are only interested in even integer greater than 2. Therefore, when p_3 is 3 and $k = 3$, then $2*3 + (1 - 3) = 6 – 2 = 4$, which is the first even number we are interested in. We should notice here that when p_3 is large, there will be times when $2k + (1 - p_3)$ will be negative even numbers, this does not matter if negative even numbers are included as long as all positive even numbers greater than 2 are included.

Now we will examine $2k + (1 - p_3)$ more closely. First, we know that $2k + (1 - p_3)$ represents 4 the first even number of interest, when p_3 is 3 and $k = 3$. Since $k = 2, 3, 4 \ldots \infty$, then $2k + (1 - p_3)$ will continue to increase in steps of 2 until k reaches infinity, and we know by definition infinity will never be reached, however, because of this all even numbers greater than 2 will be reached in increments of 2. We will demonstrate this with the following example, by showing that the largest even number is reached using both methods:

Recall, all even numbers greater than 2 are defined as, $m = 2k$, for $k = 2, 3, 4 \ldots \infty$
Therefore, the largest even number is, $m = 2k$, for $k = \infty$

Now examining, $2k + (1 - p_3)$, the largest even number that can be reached is when $k = \infty$
Therefore, the largest even number is, $m = 2k + (1 - p_3)$, for $k = \infty$

However, since ∞ is indeterminant because it has no end, we will demonstrate that both methods are the largest even number because the largest even number is indeterminant because it is infinite. First, we will examine the first form,

$m = (2)(\infty) = \infty$, this is because for any integer a, $(a)(\infty) = \infty$

Second,

$(2)(\infty) + (1 - p_3) = \infty + (1 - p_3) = \infty, \infty$, this is because for any integer a, where $a \neq \infty$, then $\infty + a = \infty$

Therefore, when $k = \infty$, then $m = 2k = \infty$, and $2k + (1 - p_3) = \infty$
Therefore, we can conclude that, $2k = 2k + (1 - p_3)$, when $k = \infty$

Thus, we have proven that both methods include all even numbers from 4 to infinity, therefore all even numbers of the Goldbach Conjecture have been included in our proof.

We recall, that when p_3 is large, there will be times when $2k + (1 - p_3)$ will be negative even numbers, this does not matter if negative even numbers are included, this just implies that $2k + (1 - p_3)$ is a larger set of infinity than $2k$, because it covers some of the negative even numbers, while $2k$ only covers the positive even numbers. For the Goldbach Conjecture, all that matters is that the positive even integers are covered by both methods.

Thus, we have proven that $2k + (1 - p_3)$ = every even integer greater than 2.

Therefore, since, $2k + (1 - p_3) = p_1 + p_2$ we have proven the Goldbach Conjecture.

Again, the author expresses his eternal gratitude to Harald Helfgott for his outstanding proof of the of the ternary Goldbach conjecture. Without Helfgott proof, the author's elementary proof would not have been possible, it is totally dependent on Helfgott's proof.

The author is not aware of an attempt having previously been made to approach the Goldbach conjecture in this way. If that is so, it would be remarkable that such a simple argument has hitherto been overlooked.

References:

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