Measurement of Planarity in Product Bipolar Fuzzy Graphs

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Abstract

Bipolar fuzzy set theory provides a basis for bipolar cognitive modeling and multiagent decision analysis, where in some situations, the product operator may be preferred to the min operator, from theoretical and experimental aspects. In this paper, the definition of product bipolar fuzzy graphs (PBFGs) is modified. The concepts of product bipolar fuzzy multigraphs (PBFMGs), product bipolar fuzzy planar graphs (PBFPGs) and product bipolar fuzzy dual graphs (PBFDGs) are introduced and investigated. Product bipolar fuzzy planarity value of PBFPG is introduced. The relation between PBFPG and PBFDG is also established. Isomorphism between PBFPGs is discussed. Finally, an application of the proposed concepts is provided.

Keywords: Product bipolar fuzzy graph; Product bipolar fuzzy multigraph; Product bipolar fuzzy planar graph; Product bipolar fuzzy dual graph.

1 Introduction

Graph theory is now briskly moving into the core of mathematics because of its applications in different fields like biochemistry, computer science, operations research and electrical engineering. In 1736, Euler first introduced the theory of planar graphs, by

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finding an important formula relating the numbers of vertices, edges and faces of polyhedrons, that can be represented by planar graphs. In real-world applications, planar graphs arise quite naturally such as electric printed circuits, railway or road maps, chemical molecules, etc. In modern age, pipelines, subway tunnels, metro lines are essential in a city planning, where routes without crossing are perfect for safety. But, due to lack of space, crossing of such lines is allowed. Since crossing between congested (strong) route and non-congested (weak) route is more safe than the crossing between two congested (strong) routes. The terms 'strong route' and 'weak route' lead strong edge and weak edge of a fuzzy graph, respectively. And the permission of crossing between strong and weak edges leads to the concept of fuzzy planar graph.

In 1965, Zadeh [19] originally introduced the concept of fuzzy set, characterized by a membership function in $[0, 1]$, which is very useful in dealing with uncertainty and vagueness. Yager [17] initiated the concept of fuzzy multisets (fuzzy bag). Bipolar fuzzy sets (BFSs) as a generalization of fuzzy sets were first introduced by Zhang [20, 21]. The range of membership degree of BFSs is $[-1, 1]$. In a BFS, the membership degree (0, 1] of an element indicates that the element satisfies the property, the membership degree $[-1, 0)$ of an element indicates that the element satisfies the implicit counter-property and the membership degree 0 of an element means that the element is irrelevant to the corresponding property. BFSs have received great attention from researchers and have been applied to many fields, such as artificial intelligence, information science, decision science, economics, medical science, social science, computer science, and neural science.

Basically, graphs are the bonding of objects. To emphasis on a real life problem, the objects are being bonded by some relations, such as friendship is the bonding of people. But when the ambiguousness or uncertainty in bonding exists, then the corresponding graph can be modelled as fuzzy graph model. The concept of fuzzy graphs was initiated by Kaufmann [8], based on Zadeh's fuzzy relations. Later in 1975, considering fuzzy relations between fuzzy sets, Rosenfeld [15] developed the structure of fuzzy graphs, obtaining analogs of several basic graph theoretical concepts. Some remarks on fuzzy graphs were given by Bhattacharya [6]. The concept of fuzzy dual graph was initiated by Jabbar et al. [10]. Several properties of bipolar fuzzy graphs and its applications were discussed by Akram [1, 2, 3]. Rashmanlou [14] introduced the concept of product bipolar fuzzy graphs. Recently, fuzzy planar graph and its extensions [4, 5, 13, 16] have been studied.

Definition of PBFGs in [14] is valid only for the product of even number of negative-membership values. But for odd number of negative-membership values this definition is invalid. So, we give a modified definition of PBFGs.

The rest of the paper is organized as follows: Section 2 reviews basic concepts related to BFSs and PBFGs. Section 3 proposes the concepts of multigraphs, planar graphs and dual graphs under bipolar fuzzy environment based on the product operator. In Section 4, we define isomorphism between PBFPGs. In Section 5, an application of the proposed concepts is provided. Section 6 ends up the paper with some concluding remarks.

In this paper, we have used standard definitions and terminologies. For other terminologies and applications not mentioned here, the readers are referred to [7, 11, 18].

2 Preliminaries

In this section, we recall some basic concepts which are necessary for this paper.

A graph is a pair of sets $G = (V, E)$, satisfying $E \subseteq V \times V$. The elements of V are the vertices and the elements of E are the edges of the graph G. The (open) neighborhood $\mathcal{N}(x)$ of a vertex x in a graph G is the set of all the neighbors of x. The degree of a vertex x in G is the number of edges incident with x, denoted by $\deg_G(x)$. If every two distinct vertices of G are adjacent, then G is a complete graph. For a vertex set V of G, define an equivalence relation $\sim$ on $V \times V - \{xx \mid x \in V\}$ as follows: $x_1x_2 \sim x_1^{'}x_2^{'}$ if and only if either $x_1x_2 = x_1^{'}x_2^{'}$ or $x_1 = x_2^{'}$ and $x_2 = x_1^{'}$. The quotient set obtained in this way is denoted by $\widetilde{V^2}$. A graph with no loops but more than one edge can join two vertices is called multigraph. A graph can be drawn in many different ways. A planar graph is a particular diagram which can be drawn on the plane so that no two edges intersect geometrically except at a vertex at which they are both incident. Any plane drawing of a planar graph G divides the plane into a set of regions, called faces. In any planar graph, the unbounded face is called an infinite face. If a cycle in a planar graph is a boundary of a face, then it is a facial cycle. A graph is said to be a non-planar if it cannot be drawn without crossing. The minimum number of crossings that can occur when G is drawn in the plane is called the crossing number $cr(G)$ of a graph G.
Euler's Formula: Let G be a connected planar graph with order n, size m and f faces. Then $n - m + f = 2$.
A dual graph of a planar graph G is constructed as follows: place a vertex in each face of G and if two faces have an edge e in common, join the corresponding vertices by an edge $e^{'}$ crossing only e.

A fuzzy relation on a set V is a mapping $\mu : V \times V \to [0, 1]$. A fuzzy graph [12] $G = (V, \sigma, \mu)$ is a non-empty set V together with a pair of functions $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$, such that $\mu(xy) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$, where $x \wedge y = \min\{x, y\}$. We call σ the fuzzy vertex set of $\mathcal{G}$ and μ the fuzzy edge set of $\mathcal{G}$. Degree of a vertex $x \in V$ of a fuzzy graph $\mathcal{G}$ is defined by $\deg_G(x) = \sum\limits_{x,y \neq x} \mu(xy)$. An edge xy of a fuzzy graph $\mathcal{G}$ is strong if $2\mu(xy) \geq \sigma(x) \wedge \sigma(y)$ [13].

Definition 2.1. [17] A fuzzy multiset X drawn from nonempty set V is characterized by a function, 'count membership' of X denoted by CM_X such that $CM_X : V \to Q$, where Q is the set of all crisp multisets drawn from the unit interval $[0, 1]$. Then for any $x \in V$, the value $CM_X(x)$ is a crisp multiset drawn from $[0, 1]$.

Definition 2.2. [13] Let V be a non-empty set and $\sigma : V \to [0, 1]$ be a mapping. Let $\mu = \{(xy, \mu(xy)_k) \mid k = 1, 2, \ldots, m_{xy}, xy \in V \times V\}$ be a fuzzy multiset of $V \times V$ such that $\mu(xy)_k \leq \sigma(x) \wedge \sigma(y)$ for all $k = 1, 2, \ldots, m_{xy}$, where $m_{xy} = \max\{k \mid \mu(xy)_k \neq 0\}$. Then

$G = (V, \sigma, \mu)$ is called fuzzy multigraph, where $\sigma(x)$ and $\mu(xy)_k$ denote the membership value of the vertex x and the membership value of the edge xy in $\mathcal{G}$, respectively.

Definition 2.3. [21] A BFS X in a non-empty set V is an object having the form $X = \{(x, \mu_X^P(x), \mu_X^N(x)) \mid x \in V\}$, where $\mu_X^P : V \to [0, 1]$ and $\mu_X^N : V \to [-1, 0]$ are two mappings.

Definition 2.4. [21] A mapping $Y = (\mu_Y^P, \mu_Y^N) : V \times V \to [0, 1] \times [-1, 0]$ is said to be a bipolar fuzzy relation on a non-empty set V such that $\mu_Y^P(xy) \in [0, 1]$ and $\mu_Y^N(xy) \in [-1, 0]$.

Definition 2.5. [9, 14] A PBFG of a graph $G = (V, E)$ is a pair $\mathcal{G} = (X, Y)$, where $X = (\mu_X^P, \mu_X^N)$ is a BFS in V and $Y = (\mu_Y^P, \mu_Y^N)$ is a bipolar fuzzy relation on $\widetilde{V^2}$ such that $\mu_Y^P(xy) \leq \mu_X^P(x) \times \mu_X^P(y)$, $\mu_Y^N(xy) \geq -(\mu_X^N(x) \times \mu_X^N(y))$ for all $xy \in \widetilde{V^2}$ and $\mu_Y^P(xy) = \mu_Y^N(xy) = 0$ for all $xy \in \widetilde{V^2} - E$.

Definition 2.6. [4] A bipolar fuzzy multiset X drawn from nonempty set V is characterized by two functions: 'count positive membership' of X (CM_X^P) and 'count negative membership' of X (CM_X^N) given respectively by $CM_X^P : V \to Q_1$ and $CM_X^N : V \to Q_2$, where Q_1 and Q_2 are the sets of all crisp multisets drawn from the intervals $[0, 1]$ and $[-1, 0]$.

3 Product Bipolar Fuzzy Planar Graphs

Definition 3.1. A PBFG of a graph G is a pair $\mathcal{G} = (X, Y)$, where $X = (\tau_X^P, \tau_X^N)$ is a BFS in V and $Y = (\tau_Y^P, \tau_Y^N)$ is a BFS in $\widetilde{V^2}$ such that

$$\tau_Y^P(xy) \leq \tau_X^P(x)\tau_X^P(y),\ \tau_Y^N(xy) \geq -|\tau_X^N(x)||\tau_X^N(y)| \text{ for all } xy \in \widetilde{V^2} \text{ and}$$
$$\tau_Y^P(xy) = \tau_Y^N(xy) = 0 \text{ for all } xy \in \widetilde{V^2} - E.$$

Considering above modified definition of PBFGs, we define PBFMG using the concept of bipolar fuzzy multiset. Further based on these concepts, the concept of PBFPG is introduced.

Definition 3.2. A PBFMG with an underlying set V is defined to be a pair $\mathcal{G} = (X, Y)$, where $X = (\mu_X^P, \mu_X^N)$ is a BFS in V and $Y = \{(xy, \mu_Y^P(xy)_k, \mu_Y^N(xy)_k) \mid k = 1, 2, \ldots, m, xy \in \widetilde{V^2}\}$ is a bipolar fuzzy multiset in $\widetilde{V^2}$ such that

$$\mu_Y^P(xy)_k \leq \mu_X^P(x)\mu_X^P(y), \mu_Y^N(xy)_k \geq -|\mu_X^N(x)||\mu_X^N(y)| \text{ for all } xy \in \widetilde{V^2}$$
$$\text{and } \mu_Y^P(xy)_k = \mu_Y^N(xy)_k = 0 \text{ for all } xy \in \widetilde{V^2} - E, \text{ for all } k = 1, 2, \ldots, m.$$

In PBFMG $\mathcal{G}$, Y is called product bipolar fuzzy multiedge set.

Example 3.1. Consider a multigraph $G = (V, E)$, where $V = \{v_1,\ v_2,\ v_3\}$ and $E = \{v_1v_2,\ v_1v_2,\ v_2v_3,\ v_1v_3\}$. Let X be a BFS of V and Y be a bipolar fuzzy multiedge set of $\widetilde{V^2}$ defined by

	v_1	v_2	v_3
μ_X^P	0.6	0.5	0.4
μ_X^N	−0.4	− 0.7	− 0.3

	v_1v_2	v_1v_2	v_2v_3	v_3v_1
μ_Y^P	0.3	0.2	0.2	0.1
μ_Y^N	−0.1	− 0.1	− 0.2	− 0.1

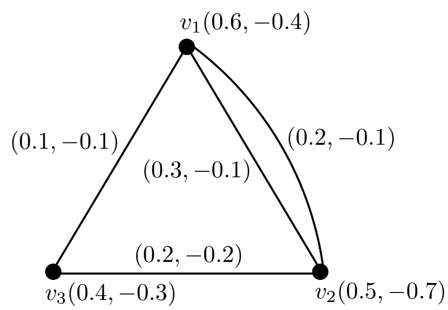

Figure 1: PBFMG.

Definition 3.3. Let $\mathcal{G} = (X, Y)$ be a PBFMG, where $Y = \{(xy, \mu_Y^P(xy)_k, \mu_Y^N(xy)_k) \mid k = 1, 2, \ldots, m, xy \in \widetilde{V^2}\}$. The degree of a vertex $x \in V$ in $\mathcal{G}$, is denoted by $\deg_G(x)$ and is defined as $\deg_G(x) = (\sum\limits_{k=1}^{m} \mu_Y^P(xy)_k, \sum\limits_{k=1}^{m} \mu_Y^N(xy)_k)$ for all $y \in V$.

Example 3.2. In Example 3.1, $\deg_G(v_1) = (0.6, -0.3)$, $\deg_G(v_2) = (0.7, -0.4)$ and $\deg_G(v_3) = (0.3, -0.3)$.

Definition 3.4. Let $Y = \{(xy, \mu_Y^P(xy)_k, \mu_Y^N(xy)_k) \mid k = 1, 2, \ldots, m, xy \in \widetilde{V^2}\}$ be a bipolar fuzzy multiedge set in PBFMG $\mathcal{G}$. A multiedge xy of $\mathcal{G}$ is strong if $\mu_X^P(x)\mu_X^P(y) \leq 2\mu_Y^P(xy)_k$ and $-|\mu_X^N(x)||\mu_X^N(y)| \geq 2\mu_Y^N(xy)_k$, k is fixed integer.

Example 3.3. In Example 3.1, $(\mu_Y^P(v_2v_3), \mu_Y^N(v_2v_3))$ is a strong edge as $(0.4)(0.5) < 2(0.2)$ and $-|-0.3||-0.7| > 2(-0.2)$.

Definition 3.5. Let $\mathcal{G} = (X, Y)$ be a PBFMG, where $Y = \{(xy, \mu_Y^P(xy)_k, \mu_Y^N(xy)_k) \mid k = 1, 2, \ldots, m, xy \in \widetilde{V^2}\}$ is a bipolar fuzzy multiedge set. An edge xy of $\mathcal{G}$ is effective if $\mu_X^P(x)\mu_X^P(y) = \mu_Y^P(xy)_k$ and $-|\mu_X^N(x)||\mu_X^N(y)| = \mu_Y^N(xy)_k$, k is fixed integer.

Definition 3.6. A PBFMG $\mathcal{G} = (X, Y)$, where $Y = \{(xy, \mu_Y^P(xy)_k, \mu_Y^N(xy)_k) \mid k = 1, 2, \ldots, m, xy \in \widetilde{V^2}\}$, is said to be complete if $\mu_Y^P(xy)_k = \mu_X^P(x)\mu_X^P(y)$ and $\mu_Y^N(xy)_k = -|\mu_X^N(x)||\mu_X^N(y)|$ for all $x, y \in V$ and for all $k = 1, 2, \ldots, m$.

Example 3.4. Consider a multigraph $G = (V, E)$, where $V = \{v_1, v_2\}$ and $E = \{v_1v_2, v_1v_2, v_1v_2\}$. Let X be a BFS of V and Y be a bipolar fuzzy multiedge set of $\widetilde{V^2}$ defined by

	v_1	v_2
μ_X^P	0.5	0.2
μ_X^N	-0.6	-0.5

	v_1v_2	v_1v_2	v_1v_2
μ_Y^P	0.1	0.1	0.1
μ_Y^N	-0.3	-0.3	-0.3

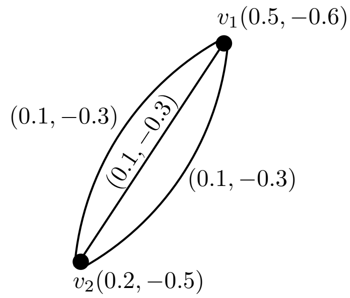

Figure 2: complete PBFMG

By routine computations, it is easy to see that it is a complete PBFMG.

Assume that geometric insight for PBFG has only one crossing between bipolar fuzzy edges $(uv, \mu_Y^P(uv)_k, \mu_Y^N(uv)_k)$ and $(wx, \mu_Y^P(wx)_k, \mu_Y^N(wx)_k)$. If $(\mu_Y^P(uv)_k, \mu_Y^N(uv)_k) = (1, -1)$ and $(\mu_Y^P(wx)_k, \mu_Y^N(wx)_k) = (0, 0)$ or $(\mu_Y^P(uv)_k, \mu_Y^N(uv)_k) = (0, 0)$ and $(\mu_Y^P(wx)_k, \mu_Y^N(wx)_k) = (1, -1)$, the PBFG has no crossing, while if $(\mu_Y^P(uv)_k, \mu_Y^N(uv)_k) = (1, -1)$ and $(\mu_Y^P(wx)_k, \mu_Y^N(wx)_k) = (1, -1)$, then there exists a crossing for the representation of the graph.

Definition 3.7. The strength of the bipolar fuzzy edge uv is defined as

$$I_{uv} = (I_{uv}^P, I_{uv}^N) = \left(\frac{\mu_Y^P(uv)_k}{\mu_X^P(u)\mu_X^P(v)}, \frac{-\mu_Y^N(uv)_k}{-|\mu_X^N(u)||\mu_X^N(v)|} \right).$$

An edge uv of a PBFMG is strong if $I_{uv}^P \geq 0.5$ and $I_{uv}^N \leq -0.5$. An edge of a PBFMG which is not strong is called weak.

Definition 3.8. Let $\mathcal{G} = (X, Y)$ be a PBFMG, where Y contains two edges $(uv, \mu_Y^P(uv)_r, \mu_Y^N(uv)_r)$ and $(wx, \mu_Y^P(wx)_s, \mu_Y^N(wx)_s)$ intersecting at a point C (r and s are fixed integers). The intersecting value at the point C is defined as

$$\mathcal{I}_C = (\mathcal{I}_C^P, \mathcal{I}_C^N) = \left(\frac{I_{uv}^P + I_{wx}^P}{2}, \frac{I_{uv}^N + I_{wx}^N}{2} \right).$$

In a PBFMG, if the number of point of intersections increases, planarity decreases. That is, $\mathcal{I}_C$ is inversely proportional to the planarity.

Definition 3.9. Let $\mathcal{G}$ be a PBFMG and let $C_1, C_2, \ldots, C_z$ be the intersecting points between the edges for geometric insight. $\mathcal{G}$ is called a PBFPG with product bipolar fuzzy planarity value f, where

$$f = (f^P, f^N) = \left(\frac{1}{1 + \{\mathcal{I}^P_{C_1} + \mathcal{I}^P_{C_2} + \ldots + \mathcal{I}^P_{C_z}\}}, \frac{-1}{1 - \{\mathcal{I}^N_{C_1} + \mathcal{I}^N_{C_2} + \ldots + \mathcal{I}^N_{C_z}\}} \right).$$

Obviously, f is bounded and $0 < f^P \leq 1$, $-1 \leq f^N < 0$. If geometric insight of a PBFPG has no point of intersection, then its product bipolar fuzzy planarity value is $(1, -1)$ and this PBFPG has underlying crisp graph as the crisp planar graph.

Example 3.5. Consider a PBFMG $\mathcal{G}$, such that $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1, v_2v_5, v_2v_5, v_3v_6, v_1v_4\}$.

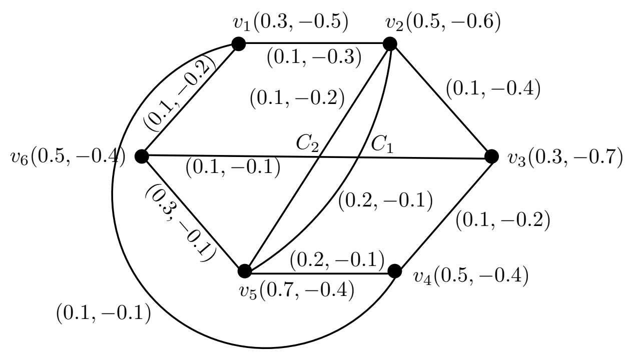

Figure 3: PBFMG.

There are two point of intersections C_1 and C_2 in PBFMG. C_1 is the point of intersection between the edges $(v_2v_5, 0.2, -0.1)$ and $(v_3v_6, 0.1, -0.1)$ and C_2 is the point of intersection between the edges $(v_2v_5, 0.1, -0.2)$ and $(v_3v_6, 0.1, -0.1)$. For the edges $(v_2v_5, 0.2, -0.1)$, $(v_2v_5, 0.1, -0.2)$ and $(v_3v_6, 0.1, -0.1)$, $I_{v_2v_5} = (0.57, -0.42)$, $I_{v_2v_5} = (0.29, -0.83)$ and $I_{v_3v_6} = (0.67, -0.36)$, respectively. For the first point of intersection C_1, $\mathcal{I}_{C_1} = (0.62, -0.39)$. For the second point of intersection C_2, $\mathcal{I}_{C_2} = (0.48, -0.60)$. Therefore, the product bipolar fuzzy planarity value for PBFMG is $f = (f^P, f^N) = (0.48, -0.50)$.

Theorem 3.1. *Let $\mathcal{G}$ be a PBFMG such that each intersecting edge is effective. Then product bipolar fuzzy planarity value f of $\mathcal{G}$ is given by*

$$f = (f^P, f^N) = \left(\frac{1}{1 + n_C}, \frac{-1}{1 + n_C} \right)$$

where n_C is the number of point of intersections between the edges in $\mathcal{G}$.

Proof. Let $\mathcal{G}$ be a PBFMG such that each intersecting edge is effective. Let $C_1, C_2, \ldots, C_z$, $z \in Z$ be the point of intersections between the edges in $\mathcal{G}$. For each intersecting edge

uv in $\mathcal{G}$, $I_{uv} = (I^P_{uv}, I^N_{uv}) = \left(\frac{\mu^P_Y(uv)_k}{\mu^P_X(u)\mu^P_X(v)}, \frac{\mu^N_Y(uv)_k}{|\mu^N_X(u)||\mu^N_X(v)|}\right) = (1, -1)$. Thus for C_1, the point of intersection between the edges uv and wx, $\mathcal{I}_{C_1} = (\mathcal{I}^P_{C_1}, \mathcal{I}^N_{C_1}) = (\frac{1+1}{2}, \frac{-1-1}{2}) = (1, -1)$. Hence $\mathcal{I}_{C_\eta} = (1, -1)$ for all $\eta = 1, 2, \ldots, z$.

Now $f = (f^P, f^N) = (\frac{1}{1+\{\mathcal{I}^P_{C_1}+\mathcal{I}^P_{C_2}+\ldots+\mathcal{I}^P_{C_z}\}}, \frac{-1}{1-\{\mathcal{I}^N_{C_1}+\mathcal{I}^N_{C_2}+\ldots+\mathcal{I}^N_{C_z}\}}) = (\frac{1}{1+\{1+1+\ldots+1\}}, \frac{-1}{1-\{-1-1,\ldots,-1\}}) = (\frac{1}{1+n_C}, \frac{-1}{1+n_C})$, where n_C is the number of point of intersections between the edges in $\mathcal{G}$. $\square$

Definition 3.10. A PBFPG $\mathcal{G}$ is said to be strong if the product bipolar fuzzy planarity value $f = (f^P, f^N)$ of $\mathcal{G}$ is such that $f^P \geq 0.5$ and $f^N \leq -0.5$.

In Example 3.5, the product bipolar fuzzy planarity value $f = (f^P, f^N)$ of the PBFPG $\mathcal{G}$ is such that $f^P \leq 0.5$ and $f^N \geq -0.5$. So, $\mathcal{G}$ is not strong.

Corollary 3.1. *Let $\mathcal{G}$ be a complete PBFMG. Then the product bipolar fuzzy planarity value f of $\mathcal{G}$ is given by*

$$f = (f^P, f^N) = \left(\frac{1}{1+n_C}, \frac{-1}{1+n_C}\right)$$

where n_C is the number of point of intersections between the edges in $\mathcal{G}$.

Theorem 3.2. *Let $\mathcal{G}$ be a strong PBFPG. Then the number of point of intersections between strong edges in $\mathcal{G}$ is at most one.*

Proof. Let $\mathcal{G}$ be a strong PBFPG. Suppose $\mathcal{G}$ has at least two point of intersections C_1 and C_2 between two strong edges in $\mathcal{G}$. For any strong edge $(uv, \mu^P_Y(uv)_k, \mu^N_Y(uv)_k)$, $\mu^P_X(u)\mu^P_X(v) \leq 2\mu^P_Y(uv)_k$ and $-|\mu^N_X(u)||\mu^N_X(v)| \geq 2\mu^N_Y(uv)_k$, that is, $I^P_{uv} \geq 0.5$ and $I^N_{uv} \leq -0.5$. Therefore, if two strong edges $(uv, \mu^P_Y(uv)_k, \mu^N_Y(uv)_k)$ and $(wx, \mu^P_Y(wx)_k, \mu^N_Y(wx)_k)$ intersect, then $\mathcal{I}^P_{C_1} = \frac{I^P_{uv}+I^P_{wx}}{2} \geq 0.5$ and $\mathcal{I}^N_{C_1} = \frac{I^N_{uv}+I^N_{wx}}{2} \leq -0.5$. Similarly, $\mathcal{I}^P_{C_2} \geq 0.5$ and $\mathcal{I}^N_{C_2} \leq -0.5$. This implies that, $1 + \mathcal{I}^P_{C_1} + \mathcal{I}^P_{C_2} \geq 2$ and $1 - (\mathcal{I}^N_{C_1} + \mathcal{I}^N_{C_2}) \geq 2$. Therefore, $f^P = \frac{1}{1+\{\mathcal{I}^P_{C_1}+\mathcal{I}^P_{C_2}\}} \leq 0.5$ and $f^N = \frac{-1}{1-\{\mathcal{I}^N_{C_1}+\mathcal{I}^N_{C_2}\}} \geq -0.5$, a contradiction, as $\mathcal{G}$ is a strong PBFPG. $\square$

A fundamental theorem of PBFPG is given below.

Theorem 3.3. *A PBFPG $\mathcal{G}$ does not contain any point of intersection between two strong edges, if $\mathcal{G}$ has product bipolar fuzzy planarity value $f = (f^P, f^N)$ such that $f^P > 0.67$ and $f^N < -0.67$.*

Proof. Let $\mathcal{G}$ be a PBFPG with product bipolar fuzzy planarity value f, such that $f^P > 0.67$ and $f^N < -0.67$. Let C be the point of intersection between two strong bipolar fuzzy edges $(uv, \mu^P_Y(uv)_k, \mu^N_Y(uv)_k)$ and $(wx, \mu^P_Y(wx)_k, \mu^N_Y(wx)_k)$. For any strong edge $(uv, \mu^P_Y(uv)_k, \mu^N_Y(uv)_k)$, $\mu^P_X(u)\mu^P_X(v) \leq 2\mu^P_Y(uv)_k$ and $-|\mu^P_X(u)||\mu^P_X(v)| \geq 2\mu^P_Y(uv)_k$. That is, $I^P_{uv} \geq 0.5$ and $I^N_{uv} \leq -0.5$. Similarly $I^P_{wx} \geq 0.5$ and $I^N_{wx} \leq -0.5$. For the minimum value of I^P_{uv} and I^P_{wx}, $\mathcal{I}^P_C = \frac{0.5+0.5}{2} = 0.5$ and for the maximum value of I^N_{uv} and I^N_{wx}, $\mathcal{I}^N_C = \frac{-0.5-0.5}{2} = -0.5$. So, $f^P = \frac{1}{1+\mathcal{I}^P_C} \leq 0.67$ and $f^N = \frac{-1}{1-\mathcal{I}^N_C} \geq -0.67$, a contradiction. Hence, $\mathcal{G}$ does not contain any point of intersection between strong edges. $\square$

We denote a PBFPG with product bipolar fuzzy planarity value $f = (f^P, f^N)$ such that $f^P > 0.67$ and $f^N < -0.67$, as $(0.67, -0.67)$-PBFPG.

Face of a PBFPG is an important feature. Face of a PBFPG is a region bounded by bipolar fuzzy edges. If all the edges in the boundary of a product bipolar fuzzy face have membership value $(1, -1)$, then it is a crisp face. If one of such edges is removed or has membership value $(0, 0)$, the product bipolar fuzzy face does not exist. A product bipolar fuzzy face and its positive membership and negative membership values are defined below.

Definition 3.11. Let $\mathcal{G} = (X, Y)$ be a PBFPG, with product bipolar fuzzy planarity value $(1, -1)$. A region bounded by the set of bipolar fuzzy edges $E^* \subset E$ of a geometric representation of $\mathcal{G}$ is said to be a product bipolar fuzzy face of $\mathcal{G}$. The positive and negative membership values of the product bipolar fuzzy face are $\prod\{I^P_{uv} \mid uv \in E^*\}$ and $-\prod\{|I^N_{uv}| \mid uv \in E^*\}$.

Definition 3.12. A product bipolar fuzzy face is said to be strong if its positive membership value is greater than and equal to 0.5 and negative membership value is less than and equal to -0.5. A product bipolar fuzzy face which is not strong is called weak. Every PBFPG has an infinite region called outer product bipolar fuzzy face. Other faces are called inner product bipolar fuzzy faces.

Remark 3.1. Every edge of a strong product bipolar fuzzy face is a strong bipolar fuzzy edge.

Example 3.6. Consider a PBFPG $\mathcal{G}$. Let f_1, f_2 and f_3 be the product bipolar fuzzy faces,

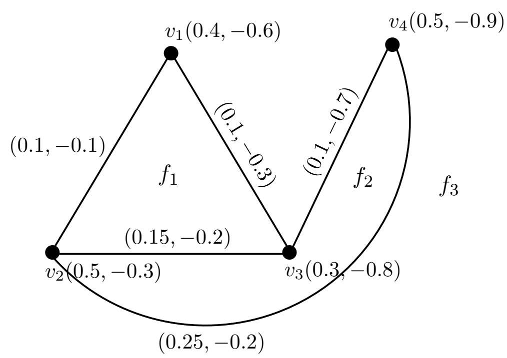

Figure 4: Faces in PBFPG.

f_1 is bounded by the edges $(v_1v_2, (0.1, -0.1)), (v_2v_3, (0.15, -0.2)), (v_3v_1, (0.1, -0.3))$, f_2 is bounded by the edges $(v_2v_3, (0.15, -0.2)), (v_3v_4, (0.1, -0.7), (v_2v_4, (0.25, -0.2))$ and f_3 is bounded by the edges $(v_1v_2, (0.1, -0.1)), (v_1v_3, (0.1, -0.3)), (v_3v_4, (0.1, -0.7)), (v_2v_4, (0.25, -0.2)$. The positive membership and negative membership values of product bipolar fuzzy faces f_1, f_2 and f_3 are $(0.42, -0.29), (0.67, -0.6)$ and $(0.28, -0.25)$, respectively. Here f_2 is a strong product bipolar fuzzy face and f_1, f_3 are weak product bipolar fuzzy faces.

Now we define dual of $(0.67, -0.67)$-PBFPG.

Definition 3.13. Let $\mathcal{G}$ be a $(0.67, -0.67)$-PBFPG, where $Y = \{(xy, \mu_Y^P(xy)_k, \mu_Y^N(xy)_k) \mid k = 1, 2, \ldots, m, xy \in \widetilde{V^2}\}$ is a bipolar fuzzy multiset on $\widetilde{V^2}$. Let $f_1, f_2, \ldots, f_t$ be the strong product bipolar fuzzy faces of $\mathcal{G}$. A PBFPG $\mathcal{G}' = (X', Y')$ such that $V' = \{x_i, i = 1, 2, \ldots, t\}$, and the vertex x_i of $\mathcal{G}'$ is considered for the face f_i of $\mathcal{G}$ is said to be a PBFDG of $\mathcal{G}$. The positive membership values of vertices are given by the mapping $\mu_{X'}^P : V' \to [0,1]$ such that $\mu_{X'}^P(x_i) = \prod\{\mu_Y^P(uv)_k, k = 1, 2, \ldots, m \mid uv$ is an edge of the boundary of the strong product bipolar fuzzy face $f_i\}$ and negative membership values of vertices are given by the mapping $\mu_{X'}^N : V' \to [-1, 0]$ such that $\mu_{X'}^N(x_i) = -\prod\{|\mu_Y^N(uv)_k|, k = 1, 2, \ldots, m \mid uv$ is an edge of the boundary of the strong product bipolar fuzzy face $f_i\}$.

There may exist at least two common edges between two faces f_i and f_j of $\mathcal{G}$. So, there may exist at least two edges between two vertices x_i and x_j in PBFDG $\mathcal{G}'$. The positive membership values and negative membership values of the bipolar fuzzy edges of the PBFDG are $\mu_Y^P(uv)_k^l = \mu_{Y'}^P(x_ix_j)_l$ and $\mu_Y^N(uv)_k^l = \mu_{Y'}^N(x_ix_j)_l$, where $(uv)^l$ is an edge in the boundary between two strong product bipolar fuzzy faces f_i and f_j, $l = 1, 2, \ldots, s$, where s is the number of edges between x_i and x_j or the number of common edges in the boundary between f_i and f_j.

Example 3.7. Consider a planar graph $G = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1, v_1v_4\}$. Let X be a BFS of V and Y be a bipolar fuzzy multiedge set of $\widetilde{V^2}$ defined by

	v_1	v_2	v_3	v_4	v_5
μ_X^P	0.8	0.9	0.7	0.5	0.3
μ_X^N	-0.7	-0.5	-0.6	-0.9	-0.3

	v_1v_2	v_2v_3	v_3v_4	v_4v_5	v_5v_1	v_1v_4
μ_Y^P	0.7	0.6	0.3	0.15	0.2	0.4
μ_Y^N	-0.3	-0.3	-0.5	-0.2	-0.2	-0.6

Clearly, it is a $(0.67, -0.67)$-PBFPG (as shown in Fig. 5) with three strong product bipolar fuzzy faces f_1, f_2 and f_3. Product bipolar fuzzy face f_1 is bounded by edges $(v_1v_2, 0.7, -0.3)$, $(v_2v_3, 0.6, -0.3)$, $(v_3v_4, 0.3, -0.5)$ and $(v_1v_4, 0.4, -0.6)$. f_2 is bounded by edges $(v_1v_4, 0.4, -0.6)$, $(v_4v_5, 0.15, -0.2)$ and $(v_5v_1, 0.2, -0.2)$. f_3 is bounded by edges $(v_1v_2, 0.7, -0.3)$, $(v_2v_3, 0.6, -0.3)$, $(v_3v_4, 0.3, -0.5)$, $(v_4v_5, 0.15, -0.2)$ and $(v_5v_1, 0.2, -0.2)$.

We represent the vertices of PBFDG by small white circles and the edges by dashed lines. For each strong product bipolar fuzzy face, we consider a vertex for the PBFDG. Thus the vertex set $V' = \{x_1, x_2, x_3\}$, where the vertex x_i is taken corresponding to the strong product bipolar fuzzy face f_i, $i = 1, 2, 3$. Therefore

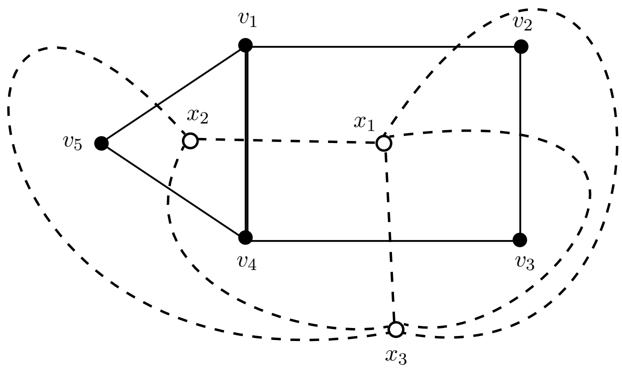

Figure 5: PBFDG.

$\mu_{X'}^P(x_1) = (0.7)(0.6)(0.3)(0.4) = 0.05$,
$\mu_{X'}^N(x_1) = -|-0.3||-0.3||-0.5||-0.6| = -0.03$,
$\mu_{X'}^P(x_2) = (0.4)(0.2)(0.15) = 0.01$,
$\mu_{X'}^N(x_2) = -|-0.6||-0.2||-0.2| = -0.02$,
$\mu_{X'}^P(x_3) = (0.7)(0.6)(0.3)(0.15)(0.2) = 0.004$,
$\mu_{X'}^N(x_3) = -|-0.3||-0.3||-0.5||-0.2||-0.2| = -0.002$.

There are three common edges v_1v_2, v_2v_3 and v_3v_4 between the faces f_1 and f_3 in $\mathcal{G}$. Therefore, there exist three edges between the vertices x_1 and x_3, in PBFDG of $\mathcal{G}$. Positive and negative membership values of the edges of PBFDG are given by

$\mu_{Y'}^P(x_1x_2) = \mu_Y^P(v_1v_4) = 0.4$, $\mu_{Y'}^N(x_1x_2) = \mu_Y^N(v_1v_4) = -0.6$,
$\mu_{Y'}^P(x_2x_3) = \mu_Y^P(v_1v_5) = 0.2$, $\mu_{Y'}^N(x_2x_3) = \mu_Y^N(v_1v_5) = -0.2$,
$\mu_{Y'}^P(x_2x_3) = \mu_Y^P(v_4v_5) = 0.15$, $\mu_{Y'}^N(x_2x_3) = \mu_Y^N(v_4v_5) = -0.2$,
$\mu_{Y'}^P(x_1x_3) = \mu_Y^P(v_1v_2) = 0.7$, $\mu_{Y'}^N(x_1x_3) = \mu_Y^N(v_1v_2) = -0.3$,
$\mu_{Y'}^P(x_1x_3) = \mu_Y^P(v_2v_3) = 0.6$, $\mu_{Y'}^N(x_1x_3) = \mu_Y^N(v_2v_3) = -0.3$,
$\mu_{Y'}^P(x_1x_3) = \mu_Y^P(v_3v_4) = 0.3$, $\mu_{Y'}^N(x_1x_3) = \mu_Y^N(v_3v_4) = -0.5$.

So, the edge set of PBFDG is

$Y' = \{(x_1x_2, 0.4, -0.6), (x_2x_3, 0.2, -0.2), (x_2x_3, 0.15, -0.2), (x_1x_3, 0.7, -0.3), (x_1x_3, 0.6, -0.3),$

$(x_1x_3, 0.3, -0.5)\}$.

We state the following theorems without their proofs.

Theorem 3.4. *Let $\mathcal{G}$ be a $(0.67, -0.67)$-PBFPG without weak edges, with n vertices, m bipolar fuzzy edges and f strong faces, and let $\mathcal{G}'$ be a PBFDG of $\mathcal{G}$ with n' vertices, m' edges and f' faces, then $n' = f$, $m' = m$ and $f' = n$.*

Theorem 3.5. *Let $\mathcal{G}'$ be a PBFDG of a $(0.67, -0.67)$-PBFPG $\mathcal{G}$. The number of strong product bipolar fuzzy faces in $\mathcal{G}'$ is less than or equal to the number of vertices of $\mathcal{G}$.*

4 Isomorphism between PBFPGs

Definition 4.1. A homomorphism $h : \mathcal{G}_1 \to \mathcal{G}_2$ of two PBFPGs $\mathcal{G}_1$ and $\mathcal{G}_2$ is a mapping $h : V_1 \to V_2$ which satisfies

(a) $\mu^P_{X_1}(x_1) \le \mu^P_{X_2}(h(x_1))$, $\mu^N_{X_1}(x_1) \ge \mu^N_{X_2}(h(x_1))$,

(b) $\mu^P_{Y_1}(x_1y_1) \le \mu^P_{Y_2}(h(x_1)h(y_1))$, $\mu^N_{Y_1}(x_1y_1) \ge \mu^N_{Y_2}(h(x_1)h(y_1))$
for all $x_1 \in V_1$, $x_1y_1 \in \widetilde{V_1^2}$.

Definition 4.2. An isomorphism $h : \mathcal{G}_1 \to \mathcal{G}_2$ of two PBFPGs $\mathcal{G}_1$ and $\mathcal{G}_2$ is a bijective mapping $h : V_1 \to V_2$ which satisfies

(c) $\mu^P_{X_1}(x_1) = \mu^P_{X_2}(h(x_1))$, $\mu^N_{X_1}(x_1) = \mu^N_{X_2}(h(x_1))$,

(d) $\mu^P_{Y_1}(x_1y_1) = \mu^P_{Y_2}(h(x_1)h(y_1))$, $\mu^N_{Y_1}(x_1y_1) = \mu^N_{Y_2}(h(x_1)h(y_1))$
for all $x_1 \in V_1$, $x_1y_1 \in \widetilde{V_1^2}$.

Definition 4.3. A weak isomorphism $h : \mathcal{G}_1 \to \mathcal{G}_2$ of two PBFPGs and G_2 is a bijective mapping $h : V_1 \to V_2$ which $h : V_1 \to V_2$ which satisfies

(e) h is homomorphism,

(f) $\mu^P_{X_1}(x_1) = \mu^P_{X_2}(h(x_1))$, $\mu^N_{X_1}(x_1) = \mu^N_{X_2}(h(x_1))$
for all $x_1 \in V_1$.

Definition 4.4. A co-weak isomorphism $h : \mathcal{G}_1 \to \mathcal{G}_2$ of two PBFPGs and G_2 is a bijective mapping $h : V_1 \to V_2$ which $h : V_1 \to V_2$ which satisfies

(g) h is homomorphism,

(h) $\mu^P_{Y_1}(x_1y_1) = \mu^P_{Y_2}(h(x_1)h(y_1))$, $\mu^N_{Y_1}(x_1y_1) = \mu^N_{Y_2}(h(x_1)h(y_1))$
for all $x_1y_1 \in \widetilde{V_1^2}$.

As isomorphism between PBFGs is an equivalence relation, if there exits an isomorphism between two PBFGs and one is PBFPG, then the other will also be PBFPG. This result is given below.

Theorem 4.1. *Let $h : \mathcal{G} \to \mathcal{H}$ be an isomorphism from a PBFPG $\mathcal{G}$ to a PBFG $\mathcal{H}$. Then $\mathcal{H}$ can be drawn as PBFPG with same planarity value of $\mathcal{G}$.*

Proof. Let $h : \mathcal{G} \to \mathcal{H}$ be an isomorphism. Since isomorphism preserves size and order of PBFGs. So, the size and order of $\mathcal{H}$ will be equal to size and order of $\mathcal{G}$ and drawing of $\mathcal{G}$ and $\mathcal{H}$ is same. Thus the number of intersections between edges and product bipolar fuzzy planarity value of $\mathcal{H}$ will be same as $\mathcal{G}$. Hence $\mathcal{H}$ can be drawn as PBFPG with same product bipolar fuzzy planarity value as of $\mathcal{G}$. □

Using above Theorem, we can immediately prove the following results.

Theorem 4.2. *Two isomorphic PBFGs have the same product bipolar fuzzy planarity values.*

Theorem 4.3. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two weak isomorphic PBFGs with product bipolar fuzzy planarity values $f_1 = (f_1^P, f_1^N)$ and $f_2 = (f_2^P, f_2^N)$, respectively. If the edge positive membership and negative membership values of corresponding intersecting edges are same. Then $(f_1^P, f_1^N) = (f_2^P, f_2^N)$.*

Theorem 4.4. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two co-weak isomorphic PBFGs with product bipolar fuzzy planarity values $f_1 = (f_1^P, f_1^N)$ and $f_2 = (f_2^P, f_2^N)$, respectively. If the minimum of positive membership values and maximum of negative membership values of the end vertices of corresponding intersecting edges are same. Then $(f_1^P, f_1^N) = (f_2^P, f_2^N)$.*

5 Application of PBFPGs

We consider a road network as shown in Fig.5. Each city in the network may be referred as vertex and each road between any two cities may be called as edge. This graph does not contain loops and multiple edges. It is well known that, the length of a road between any two cities is a crisp quantity but vehicle travel time or vehicle capacity on a road network is fuzzy.

In this network, the membership values of vertices are representing the degree of capacity of vehicles of a city belongs to a network of 5 cities. The degree of capacity of vehicles of a city is defined in terms of its positive membership and negative membership. Positive membership degree can be depicted as how much capacity, vehicles of a city posses and negative membership can be depicted as how much capacity is lost by the vehicles of a city. The membership values of edges of this graph show the capacity of vehicles on the road joining any two cities. The positive and negative membership degree of edges can

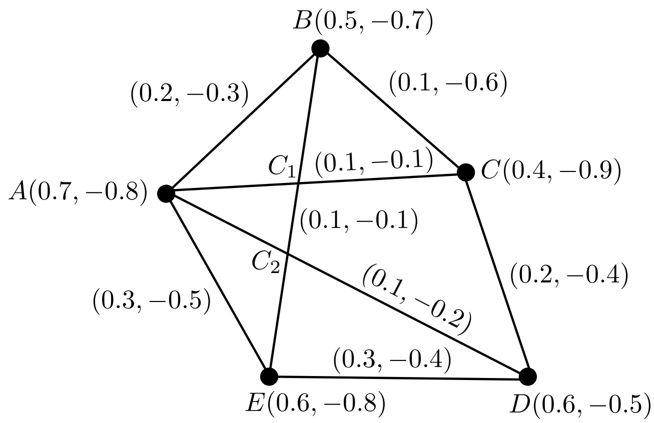

Figure 6: Product bipolar fuzzy planar graph of a road network.

be interpreted as the percentage of increasing and decreasing capacity of vehicles on the road between any two cities.

Due to crossing, the vehicle capacity or vehicle travel time decreases, in order to reach from one city to another city and so traveling cost decreases. But we construct the roads in such a way that the number of crossing decreases, that is, the planarity value increases, because generally, as the crossing road increases, vehicle capacity increases, due to the increase of crowdedness on roads. That's why, the measurement of product bipolar fuzzy planarity value is important.

There are only two crossings, at the points C_1 and C_2, between the roads AC and BE, and AD and BE, respectively, in this road network of Fig.5. For the roads AC, BE and AD, $I_{AC} = (0.36, -0.14)$, $I_{BE} = (0.33, -0.18)$ and $I_{AD} = (0.24, -0.50)$, respectively. For first crossing at C_1, $\mathcal{I}_{C_1} = (0.35, -0.16)$ and for second crossing at C_2, $\mathcal{I}_{C_2} = (0.29, -0.34)$. Therefore, the product bipolar fuzzy planarity value f of above road network is $(0.61, -0.67)$. So, the roads are not to much crowded and due to road crossing between cities vehicle travel time saves and traveling cost decreases.

6 Conclusion

Fuzzy graph theory is highly exploited in computer science applications. Particularly, in research areas of computer science including image capturing, data mining, clustering, image segmentation, networking etc. In this paper, we have initiated the concepts of multigraphs, planar graphs and dual graphs under bipolar fuzzy environment based on the product operator. We have also introduced the product bipolar fuzzy planarity value of PBFPGs. If the product bipolar fuzzy planarity value of a PBFPG is $(1, -1)$, that is, if there exists no crossing between edges, then the PBFPG is same as crisp planar graph. Therefore, the product bipolar fuzzy planarity value measures the amount of planarity in a PBFPG.

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