Energy and Laplacian energy of a single value neutrosophic graph

Sumera Naz [*], Hossein Rashmanlou [†] and M. Aslam Malik [‡]
[*,‡] Department of Mathematics, University of the Punjab, Quaid-e-Azam Campus, Lahore-54590, Pakistan.
[†] Young Researchers and Elite club, Central Tehran Branch, Islamic Azad University, Tehran, Iran.

Abstract

Single valued neutrosophic model as an instance of a neutrosophic model provide an additional possibility to represent imprecise, uncertainty, inconsistent and incomplete information which exist in real situations. In this paper, the concept of energy of a graph is introduced in the context of a single valued neutrosophic environment, where for each element the truth-membership, indeterminacy-membership and falsity-membership degree, in $[0, 1]$, are independently assigned. Firstly, the novel concepts of energy of a single valued neutrosophic graph (SVNG) are proposed and their properties are investigated, then Laplacian energy of a SVNG is introduced. Between the properties of energy and Laplacian energy of a SVNG, there is a great deal of analogy, but also some significant differences.

Keywords: Neutrosophic set; neutrosophic graph; energy; Laplacian energy

1 Introduction

Smarandache [23] firstly proposed neutrosophy, a branch of philosophy which discusses the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. The prominent characteristic of neutrosophic set (NS) is that a truth-membership, an indeterminacy-membership and a falsity-membership degree, in non-standard unit interval $]0^-, 1^+[$, are independently assigned to each element in the set. NS as a powerful general formal framework extends the concept of classic set, fuzzy set [29], interval-valued fuzzy set [30], intuitionistic fuzzy set (IFS) [5], vague set [14],

[*] nsumeranaz@gmail.com
[†] rashmanlou.1987@gmail.com
[‡] malikpu@yahoo.com

interval-valued intuitionistic fuzzy set (IVIFS) [6], paraconsistent set [23], dialetheist set [23], paradoxist set [23], and tautological set [23]. IFSs and IVIFSs cannot deal with all types of uncertainty, such as indeterminate and inconsistent information, existing commonly in real situations. For instance, if during a voting process there are thirteen voters. In time t_1, five vote 'aye', four vote 'blackball' and four are undecided. According to neutrosophic notation, it can be represented as $x\langle 0.5, 0.4, 0.4\rangle$. In time t_2, three vote 'aye', four vote 'blackball' two give up and four are undecided, then according to neutrosophic notation, it can be represented as $x\langle 0.3, 0.4, 0.4\rangle$. This information is beyond the scope of IFS. That is why, the concept of NS is more extensive and overcomes the aforementioned issues. Gradually, it has been discovered that without a specific description, it is difficult to apply NSs in the real applications. After analyzing this difficulty, Wang et al. [25] initiated the notion of a single valued neutrosophic set (SVNS) from scientific or engineering point of view, as a subclass of the NS and an extension of IFS, and provide its various properties. NS, particularly SVNS has attracted significant interest from researchers in recent years. It has been widely applied in various fields, such as information fusion in which data are combined from different sensors [9], control theory [1], image processing [15], medical diagnosis [28], decision making [27], and graph theory [11, 24], etc.

Graph representations are generally used for dealing with structural information, in different domains such as operations research, networks, systems analysis, pattern recognition, economics and image interpretation. Gutman [16] introduced the notion of energy of a graph in chemistry, because of its relevance to the total π-electron energy of certain molecules and found upper and lower bounds for the energy of graphs [17]. In chemistry, the energy of a given molecular graph is interesting because of its relation to the total π-electron energy of the molecule represented by that graph. Later, Gutman and Zhou [18] defined the Laplacian energy of a graph as the sum of the absolute values of the differences of average vertex degree of G to the Laplacian eigenvalues of G.

Fuzzy graph theory is successfully used in many problems, to handle the uncertainty that occurs in graph theory. Fuzzy graphs are designed to represent structures of relationships between objects such that the existence of a concrete object (vertex) and relationship between two objects (edge) are matters of degree. The concept of fuzzy graphs was initiated by Kaufmann [19], based on Zadeh's fuzzy relations. Later, another elaborated definition of fuzzy graph with fuzzy vertex and fuzzy edges was introduced by Rosenfeld [21] and obtaining analogs of several graph theoretical concepts such as paths, cycles and connectedness etc, he developed the structure of fuzzy graphs. Energy of a fuzzy graph was investigated in [4] by Anjali and Mathew. Laplacian energy of a fuzzy graph was defined by Sharbaf and Fayazi [22]. Intuitionistic fuzzy graphs with vertex sets and edge sets as IFS were first introduced by Atanassov [7] and further discussed by Akram [2]. Praba et al. [20] defined the energy of intuitionistic fuzzy graphs as an extension of [4]. Basha and Kartheek [8] generalized the concept of the Laplacian energy of fuzzy graph to the Laplacian energy of an intuitionistic fuzzy graph. Borzooei and Rashmanlou [12] defined the energy of a vague graph. Broumi et al. [10, 11] put forward the concept of SVNGs. SVNGs [3, 11, 13] have attracted significant interest from researchers in recent years. However, to the best of our knowledge, no work addressing the energy and Laplacian energy in single valued neutrosophic setting is in literature. So, the main purpose of this paper is to introduce the energy and Laplacian energy of a SVNG.

The paper is structured as follows: section 2 contains a brief background about SVNSs and SVNGs. Section 3 proposes the concept of the energy of a SVNG, and investigates its properties. Section 4 puts forward Laplacian energy of SVNGs based on its Laplacian eigenvalues. In section 5, the proposed concepts of energy and Laplacian energy of SVNGs are illustrated with real time example and finally we draw conclusions in section 6.

Throughout this paper, V represents a crisp universe of generic elements, G stands for the crisp graph and $\mathcal{G}$ is the SVNG.

2 Preliminaries

In the following, some basic concepts on SVNSs and SVNGs are reviewed to facilitate next sections.

A graph is a pair of sets $G = (V, E)$, satisfying $E \subseteq V \times V$. The elements of V and E are, respectively, the vertices and edges of the graph G. Two vertices x and y of G are adjacent if $xy \in E$. The number of edges joined with a vertex x of a graph G is called the degree of x in G denoted by $\deg_G(x)$ or $\mathrm{d}_G(x)$. A graph with no loops and multiple edges is called simple graph. Throughout this paper we will consider only undirected, simple graphs. If G is a graph with n vertices and m edges, its adjacency matrix $A(G)$ is the $n \times n$ matrix whose ij-th entry is the number of edges joining vertices i and j. The degree matrix, $D(G)$, of G is a $n \times n$ diagonal matrix defined as:

$$d_{ij} = \begin{cases} d_G(x_i) & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then $L(G) = D(G) - A(G)$ is the Laplacian matrix of G with real and nonnegative eigenvalues. The eigenvalues $\lambda_i, i = 1, 2, \ldots, n$, of the adjacency matrix of G are the eigenvalues of G and the eigenvalues $\upsilon_i, i = 1, 2, \ldots, n$, of the Laplacian matrix of G are the Laplacian eigenvalues of G. The spectrum $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of the adjacency matrix of G is the spectrum, $\mathrm{Spec}(G)$, of G and the spectrum $\{\upsilon_1, \upsilon_2, \ldots, \upsilon_n\}$ of the Laplacian matrix of G is the Laplacian spectrum of G. The eigenvalues and Laplacian eigenvalues of a graph satisfy the following relations:

$$\sum_{i=1}^{n} \lambda_i = 0, \sum_{i=1}^{n} \lambda_i^2 = 2m, \sum_{i=1}^{n} \upsilon_i = 2m \text{ and } \sum_{i=1}^{n} \upsilon_i^2 = 2m + \sum_{i=1}^{n} d_G^2(x_i).$$

Definition 2.1. [16]. The energy of a graph G, denoted by $E(G)$, is defined as the sum of the absolute values of the eigenvalues of G, i.e., $E(G) = \sum\limits_{i=1}^{n} |\lambda_i|$. A graph with all isolated vertices K_n^c has zero energy while the complete graph K_n with n vertices has energy $2(n-1)$.

Definition 2.2. [18] Let G be a graph with n vertices, m edges and $\upsilon_i, i = 1, 2, \ldots, n$, Laplacian eigenvalues. Then the Laplacian energy of G is defined as $LE(G) = \sum_{i=1}^{n} |\gamma_i|$, where $\gamma_i = \upsilon_i - \frac{2m}{n}$. Moreover, $\sum_{i=1}^{n} \gamma_i = 0, \sum_{i=1}^{n} \gamma_i^2 = 2M$, where $M = m + \frac{1}{2} \sum_{i=1}^{n} \left(d_G(x_i) - \frac{2m}{n}\right)^2$.

In 1965, Zadeh [29] originally introduced the fuzzy set, characterized by a membership function in $[0, 1]$, which is very useful in dealing with uncertainty, imprecision and vagueness.

Definition 2.3. [21, 29] A fuzzy subset η of a set V is a function $\eta : V \to [0, 1]$. A mapping $\mu : V \times V \to [0, 1]$ such that $\mu(xy) \leq \min\{\eta(x), \eta(y)\}$ for all $x, y \in V$, is a fuzzy relation on V. A fuzzy graph $\mathcal{G} = (V, \eta, \mu)$ is a non-empty set V together with a pair of functions $\eta : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ such that $\mu(xy) \leq \min\{\eta(x), \eta(y)\}$ for all $x, y \in V$. Here μ is a symmetric fuzzy relation on η.

Atanassov [5] defined the concept of IFS as an extension of fuzzy set by introducing a non-membership and a hesitancy function as follows:

Definition 2.4. [5] An IFS X in V is an object having the form:

$$X = \{\langle x, \mu_X(x), \nu_X(x)\rangle \mid x \in V\},$$

where the functions $\mu_X : V \to [0, 1]$, $x \in V$, $\mu_X(x) \in [0, 1]$ and $\nu_X : V \to [0, 1]$, $x \in V$, $\nu_X(x) \in [0, 1]$ define the degree of membership and degree of non-membership of the element $x \in V$ to set X, respectively, such that

$$0 \leq \mu_X(x) + \nu_X(x) \leq 1 \quad \text{for all } x \in V$$

and

$$\pi_X(x) = 1 - \mu_X(x) - \nu_X(x) \quad \text{for all } x \in V$$

$\pi_X(x)$ is called a degree of indeterminacy or hesitancy of x to X.

Definition 2.5. [23] Let V be a space of points (objects), with a generic element in V denoted by x. A NS X in V is characterized by a truth-membership function T_X, an indeterminacy-membership function I_X and a falsity-membership function F_X. $T_X(x), I_X(x)$ and $F_X(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$. That is, $T_X : V \to]0^-, 1^+[$, $I_X : V \to]0^-, 1^+[$ and $F_X : V \to]0^-, 1^+[$.

There is no restriction on the sum of $T_X(x), I_X(x)$ and $F_X(x)$, therefore $0^- \leq \sup T_X(x) + \sup I_X(x) + \sup F_X(x) \leq 3^+$.

In fuzzy set the connector is defined according to membership only, and the information of non-membership and indeterminacy is lost. In IFS the connector is defined according to membership and non-membership only, and the indeterminacy is left from 1. While in SVNS, the connector can be defined according to any of them, i.e., without any restriction. Wang et al. [25] gave the following definition of SVNS.

Definition 2.6. [25] Let V be a space of points (objects), with a generic element in V denoted by x. A SVNS X in V is characterized by a truth-membership function T_X, an indeterminacy-membership function I_X and a falsity-membership function F_X. For each point $x \in X$, $T_X(x), I_X(x), F_X(x) \in [0,1]$. Therefore, a SVNS X in V can be written as

$$X = \{\langle x, T_X(x), I_X(x), F_X(x)\rangle \mid x \in V\},$$

Definition 2.7. [26] A SVNS Y in $V \times V$ is said to be a single valued neutrosophic relation in V, denoted by

$$Y = \{\langle xy, T_Y(xy), I_Y(xy), F_Y(xy)\rangle \mid xy \in V \times V\}$$

where $T_Y : V \times V \to [0,1]$, $I_Y : V \times V \to [0,1]$ and $F_Y : V \times V \to [0,1]$ represent the truth-membership, indeterminacy-membership and falsity-membership function of Y, respectively.

Definition 2.8. [11] A SVNG of a (crisp) graph $G = (V, E)$ is defined to be a pair $\mathcal{G} = (X, Y)$, where

(i) the functions $T_X : V \to [0,1]$, $I_X : V \to [0,1]$ and $F_X : V \to [0,1]$ denote the degree of truth-membership, indeterminacy-membership and falsity-membership of the element $x \in V$, respectively. The sum of $T_X(x), I_X(x)$ and $F_X(x)$ satisfies $0 \leq T_X(x) + I_X(x) + F_X(x) \leq 3$ for all $x \in V$,

(ii) the functions $T_Y : E \subseteq V \times V \to [0,1], I_Y : E \subseteq V \times V \to [0,1]$ and $F_Y : E \subseteq V \times V \to [0,1]$ are defined by

$T_Y(xy) \leq \min\{T_X(x), T_X(y)\}, I_Y(xy) \geq \max\{I_X(x), I_X(y)\}$ and $F_Y(xy) \geq \max\{F_X(x), F_X(y)\}$.

The sum of $T_Y(xy), I_Y(xy)$ and $F_Y(xy)$ satisfies $0 \leq T_Y(xy) + I_Y(xy) + F_Y(xy) \leq 3$ for all $xy \in E$.

We call X the single valued neutrosophic vertex set of $\mathcal{G}$ and Y the single valued neutrosophic edge set of $\mathcal{G}$.

Definition 2.9. [11] The degree of a vertex $u_i \in V$ in a SVNG $\mathcal{G}$ is defined as $d_{\mathcal{G}}(u_i) = \langle d_T(u_i), d_I(u_i), d_F(u_i)\rangle$, where

$$d_T(u_i) = \sum_{u_i, u_j \neq u_i \in V} T_Y(u_i u_j), d_I(u_i) = \sum_{u_i, u_j \neq u_i \in V} I_Y(u_i u_j) \text{ and } d_F(u_i) = \sum_{u_i, u_j \neq u_i \in V} F_Y(u_i u_j).$$

3 Energy of a single value neutrosophic graph

In this section, based on the extension of the energy of a fuzzy graph [4], we define the concept of energy of a SVNG, which is useful in real scientific applications.

Definition 3.1. The adjacency matrix $A(\mathcal{G})$ of a SVNG $\mathcal{G} = (X, Y)$ is defined as a square matrix $A(\mathcal{G}) = [a_{ij}]$, $a_{ij} = \langle T_Y(u_iu_j), I_Y(u_iu_j), F_Y(u_iu_j)\rangle$, where $T_Y(u_iu_j), I_Y(u_iu_j)$ and $F_Y(u_iu_j)$ represent the strength of relationship, strength of undecided relationship and strength of non-relationship between u_i and u_j, respectively.

The adjacency matrix of a SVNG can be expressed as three matrices, first matrix contains the entries as truth-membership values, second contains the entries as indeterminacy-membership values and the third contains the entries as falsity-membership values, i.e., $A(\mathcal{G}) = \langle A(T_Y(u_iu_j)), A(I_Y(u_iu_j)), A(F_Y(u_iu_j))\rangle$.

Example 3.1. Consider a graph $G = (V, E)$, where $V = \{u_1, u_2, u_3, u_4\}$ and $E = \{u_1u_2, u_2u_3, u_3u_1, u_2u_4, u_3u_4\}$. Let $\mathcal{G} = (X, Y)$ be a SVNG of a graph G defined by

$$\begin{aligned} X &= \left\langle \left(\frac{u_1}{0.4}, \frac{u_2}{0.5}, \frac{u_3}{0.6}, \frac{u_4}{0.3}\right), \left(\frac{u_1}{0.3}, \frac{u_2}{0.2}, \frac{u_3}{0.1}, \frac{u_4}{0.1}\right), \left(\frac{u_1}{0.1}, \frac{u_2}{0.3}, \frac{u_3}{0.4}, \frac{u_4}{0.5}\right)\right\rangle, \\ Y &= \left\langle \left(\frac{u_1u_2}{0.2}, \frac{u_2u_3}{0.4}, \frac{u_3u_1}{0.3}, \frac{u_4u_2}{0.1}, \frac{u_4u_3}{0.2}\right), \left(\frac{u_1u_2}{0.5}, \frac{u_2u_3}{0.3}, \frac{u_3u_1}{0.4}, \frac{u_4u_2}{0.4}, \frac{u_4u_3}{0.3}\right), \right. \\ &\quad \left. \left(\frac{u_1u_2}{0.6}, \frac{u_2u_3}{0.5}, \frac{u_3u_1}{0.5}, \frac{u_4u_2}{0.6}, \frac{u_4u_3}{0.7}\right)\right\rangle. \end{aligned}$$

The SVNG is given in Fig. 1. Tabular representation of a SVNG is given in Table 1.

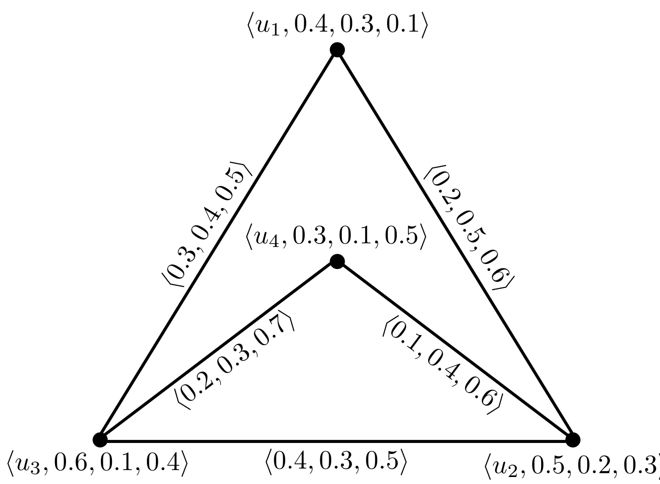

Figure 1: SVNG.

Table 1: Tabular representation of a SVNG.

	u_1	u_2	u_3	u_4
T_X	0.4	0.5	0.6	0.3
I_X	0.3	0.2	0.1	0.1
F_X	0.1	0.3	0.4	0.5

	u_1u_2	u_2u_3	u_3u_1	u_2u_4	u_3u_4
T_Y	0.2	0.4	0.3	0.1	0.2
I_Y	0.5	0.3	0.4	0.4	0.3
F_Y	0.6	0.5	0.5	0.6	0.7

The adjacency matrix of a SVNG given in Fig. 1, is

$$A(\mathcal{G}) = \begin{pmatrix} 0 & \langle 0.2, 0.5, 0.6\rangle & \langle 0.3, 0.4, 0.5\rangle & 0 \\ \langle 0.2, 0.5, 0.6\rangle & 0 & \langle 0.4, 0.3, 0.5\rangle & \langle 0.1, 0.4, 0.6\rangle \\ \langle 0.3, 0.4, 0.5\rangle & \langle 0.4, 0.3, 0.5\rangle & 0 & \langle 0.2, 0.3, 0.7\rangle \\ 0 & \langle 0.1, 0.4, 0.6\rangle & \langle 0.2, 0.3, 0.7\rangle & 0 \end{pmatrix}$$

Definition 3.2. The spectrum of adjacency matrix of a SVNG $A(\mathcal{G})$ is defined as $\langle P, Q, R\rangle$, where P, Q and R are the sets of eigenvalues of $A(T_Y(u_iu_j))$, $A(I_Y(u_iu_j))$ and $A(F_Y(u_iu_j))$, respectively.

Definition 3.3. The energy of a SVNG $\mathcal{G} = (X, Y)$ is defined as

$$E(\mathcal{G}) = \langle E(T_Y(u_iu_j)), E(I_Y(u_iu_j)), E(F_Y(u_iu_j))\rangle = \left\langle \sum_{\substack{i=1 \\ \lambda_i \in P}}^{n} |\lambda_i|, \sum_{\substack{i=1 \\ \delta_i \in Q}}^{n} |\delta_i|, \sum_{\substack{i=1 \\ \zeta_i \in R}}^{n} |\zeta_i| \right\rangle.$$

Definition 3.4. Two SVNGs with the same number of vertices and the same energy are called equienergetic.

Theorem 3.1. *Let $\mathcal{G} = (X, Y)$ be a SVNG and $A(\mathcal{G})$ be its adjacency matrix. If $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, $\delta_1 \geq \delta_2 \geq \ldots \geq \delta_n$ and $\zeta_1 \geq \zeta_2 \geq \ldots \geq \zeta_n$ are the eigenvalues of $A(T_Y(u_iu_j))$, $A(I_Y(u_iu_j))$ and $A(F_Y(u_iu_j))$, respectively. Then*

1. $\sum\limits_{\substack{i=1 \\ \lambda_i \in P}}^{n} \lambda_i = 0$, $\sum\limits_{\substack{i=1 \\ \delta_i \in Q}}^{n} \delta_i = 0$ and $\sum\limits_{\substack{i=1 \\ \zeta_i \in R}}^{n} \zeta_i = 0$

2. $\sum\limits_{\substack{i=1 \\ \lambda_i \in P}}^{n} \lambda_i^2 = 2 \sum\limits_{1 \leq i < j \leq n} (T_Y(u_iu_j))^2$, $\sum\limits_{\substack{i=1 \\ \delta_i \in Q}}^{n} \delta_i^2 = 2 \sum\limits_{1 \leq i < j \leq n} (I_Y(u_iu_j))^2$ and $\sum\limits_{\substack{i=1 \\ \zeta_i \in R}}^{n} \zeta_i^2 = 2 \sum\limits_{1 \leq i < j \leq n} (F_Y(u_iu_j))^2$.

Proof. 1. Since $A(\mathcal{G})$ is a symmetric matrix whose trace is zero, So its eigenvalues are real with zero sum.

2. By matrix trace properties, we have

$$tr((A(T_Y(u_iu_j)))^2) = \sum_{\substack{i=1\\ \lambda_i \in P}}^{n} \lambda_i^2$$

where

$$\begin{aligned} tr((A(T_Y(u_iu_j)))^2) &= (0 + T_Y^2(u_1u_2) + \ldots + T_Y^2(u_1u_n)) + (T_Y^2(u_2u_1) + 0 + \ldots + T_Y^2(u_2u_n)) \\ &+ \ldots + (T_Y^2(u_nu_1) + T_Y^2(u_nu_2) + \ldots + 0) \\ &= 2 \sum_{1 \le i < j \le n} (T_Y(u_iu_j))^2. \end{aligned}$$

Hence

$$\sum_{\substack{i=1\\ \lambda_i \in P}}^{n} \lambda_i^2 = 2 \sum_{1 \le i < j \le n} (T_Y(u_iu_j))^2.$$

Analogously, we can show that $\sum_{\substack{i=1\\ \delta_i \in Q}}^{n} \delta_i^2 = 2 \sum_{1 \le i < j \le n} (I_Y(u_iu_j))^2$ and $\sum_{\substack{i=1\\ \zeta_i \in R}}^{n} \zeta_i^2 = 2 \sum_{1 \le i < j \le n} (F_Y(u_iu_j))^2$.

□

Example 3.2. The spectrum and the energy of a SVNG $\mathcal{G}$, given in Fig. 1 are as follows:

$$\begin{aligned} Spec(T_Y(u_iu_j)) &= \{-0.4408, -0.2215, 0.0015, 0.6608\}, \\ Spec(I_Y(u_iu_j)) &= \{-0.6853, -0.2885, 0.0005, 0.9732\}, \\ Spec(F_Y(u_iu_j)) &= \{-0.9799, -0.5191, 0.0191, 1.4799\}. \end{aligned}$$

Therefore,

$$\begin{aligned} Spec(\mathcal{G}) = \{ &\langle -0.4408, -0.6853, -0.9799 \rangle, \langle -0.2215, -0.2885, -0.5191 \rangle, \langle 0.0015, 0.0005, 0.0191 \rangle, \\ &\langle 0.6608, 0.9732, 1.4799 \rangle \}. \end{aligned}$$

Now, $E(T_Y(u_iu_j)) = 1.3246$, $E(I_Y(u_iu_j)) = 1.9475$ and $E(F_Y(u_iu_j)) = 2.9980$
Therefore, $E(\mathcal{G}) = \langle 1.3246, 1.9475, 2.9980 \rangle$. Also we have

$$\sum_{\substack{i=1\\ \lambda_i \in P}}^{4} \lambda_i = -0.4408 - 0.2215 + 0.0015 + 0.6608 = 0,$$

$$\sum_{\substack{i=1\\ \delta_i \in Q}}^{4} \delta_i = -0.6853 - 0.2885 + 0.0005 + 0.9732 = 0,$$

$$\sum_{\substack{i=1\\ \zeta_i \in R}}^{4} \zeta_i = -0.9799 - 0.5191 + 0.0191 + 1.4799 = 0.$$

$$\sum_{\substack{i=1\\ \lambda_i\in P}}^{4} \lambda_i^2 = 0.6800 = 2(0.3400) = 2\sum_{1\le i<j\le 4}(T_Y(u_iu_j))^2,$$

$$\sum_{\substack{i=1\\ \delta_i\in Q}}^{4} \delta_i^2 = 1.5000 = 2(0.7500) = 2\sum_{1\le i<j\le 4}(I_Y(u_iu_j))^2,$$

$$\sum_{\substack{i=1\\ \zeta_i\in R}}^{4} \zeta_i^2 = 3.4200 = 2(1.7100) = 2\sum_{1\le i<j\le 4}(F_Y(u_iu_j))^2.$$

We now give upper and lower bounds on energy of a SVNG $\mathcal{G}$, in terms of the number of vertices and the sum of squares of truth-membership, indeterminacy-membership and falsity-membership values of edges.

Theorem 3.2. *Let $\mathcal{G} = (X, Y)$ be a SVNG on n vertices and the adjacency matrix $A(\mathcal{G}) = \langle A(T_Y(u_iu_j)), A(I_Y(u_iu_j)), A(F_Y(u_iu_j))\rangle$ of $\mathcal{G}$. Then*

(i) $\sqrt{2\sum_{1\le i<j\le n}(T_Y(u_iu_j))^2 + n(n-1)|T|^{\frac{2}{n}}} \le E(T_Y(u_iu_j)) \le \sqrt{2n\sum_{1\le i<j\le n}(T_Y(u_iu_j))^2}$

(ii) $\sqrt{2\sum_{1\le i<j\le n}(I_Y(u_iu_j))^2 + n(n-1)|I|^{\frac{2}{n}}} \le E(I_Y(u_iu_j)) \le \sqrt{2n\sum_{1\le i<j\le n}(I_Y(u_iu_j))^2}$

(iii) $\sqrt{2\sum_{1\le i<j\le n}(F_Y(u_iu_j))^2 + n(n-1)|F|^{\frac{2}{n}}} \le E(F_Y(u_iu_j)) \le \sqrt{2n\sum_{1\le i<j\le n}(F_Y(u_iu_j))^2}$

where $|T|, |I|$ and $|F|$ are the determinant of $A(T_Y(u_iu_j)), A(I_Y(u_iu_j))$ and $A(F_Y(u_iu_j))$, respectively.

Proof. (i) Upper bound:
Apply Cauchy-Schwarz inequality to the n numbers $1, 1, \ldots, 1$ and $|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|$, then

$$\sum_{i=1}^{n}|\lambda_i| \le \sqrt{n}\sqrt{\sum_{i=1}^{n}|\lambda_i|^2} \tag{3.1}$$

$$\left(\sum_{i=1}^{n}\lambda_i\right)^2 = \sum_{i=1}^{n}|\lambda_i|^2 + 2\sum_{1\le i<j\le n}\lambda_i\lambda_j \tag{3.2}$$

By comparing the coefficients of λ^{n-2} in the characteristic polynomial

$$\prod_{i=1}^{n}(\lambda-\lambda_i) = |A(\mathcal{G}) - \lambda I|,$$

we have

$$\sum_{1\leq i<j\leq n} \lambda_i\lambda_j = -\sum_{1\leq i<j\leq n} (T_Y(u_iu_j))^2. \tag{3.3}$$

Substituting (3.3) in (3.2), we obtain

$$\sum_{i=1}^{n} |\lambda_i|^2 = 2\sum_{1\leq i<j\leq n} (T_Y(u_iu_j))^2. \tag{3.4}$$

Substituting (3.4) in (3.1), we obtain

$$\sum_{i=1}^{n} |\lambda_i| \leq \sqrt{n}\sqrt{2\sum_{1\leq i<j\leq n} (T_Y(u_iu_j))^2} = \sqrt{2n\sum_{1\leq i<j\leq n} (T_Y(u_iu_j))^2}.$$

Therefore,

$$E(T_Y(u_iu_j)) \leq \sqrt{2n\sum_{1\leq i<j\leq n} (T_Y(u_iu_j))^2}$$

Lower bound:

$$\begin{aligned}(E(T_Y(u_iu_j)))^2 &= \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 = \sum_{i=1}^{n} |\lambda_i|^2 + 2\sum_{1\leq i<j\leq n} |\lambda_i\lambda_j| \\ &= 2\sum_{1\leq i<j\leq n} (T_Y(u_iu_j))^2 + \frac{2n(n-1)}{2} AM\{|\lambda_i\lambda_j|\}\end{aligned}$$

Since $AM\{|\lambda_i\lambda_j|\} \geq GM\{|\lambda_i\lambda_j|\}$, $1 \leq i < j \leq n$,
so,

$$E(T_Y(u_iu_j)) \geq \sqrt{2\sum_{1\leq i<j\leq n} (T_Y(u_iu_j))^2 + n(n-1)GM\{|\lambda_i\lambda_j|\}}$$

also since

$$GM\{|\lambda_i\lambda_j|\} = \left(\prod_{1\leq i<j\leq n} \{|\lambda_i\lambda_j|\}\right)^{\frac{2}{n(n-1)}} = \left(\prod_{i=1}^{n} \{|\lambda_i|^{n-1}\}\right)^{\frac{2}{n(n-1)}} = \left(\prod_{i=1}^{n} \{|\lambda_i|\}\right)^{\frac{2}{n}} = |T|^{\frac{2}{n}}$$

so,

$$E(T_Y(u_iu_j)) \geq \sqrt{2\sum_{1\leq i<j\leq n} (T_Y(u_iu_j))^2 + n(n-1)|T|^{\frac{2}{n}}}.$$

Thus,

$$\sqrt{2\sum_{1\le i<j\le n}(T_Y(u_iu_j))^2+n(n-1)|T|^{\frac{2}{n}}}\le E(T_Y(u_iu_j))\le\sqrt{2n\sum_{1\le i<j\le n}(T_Y(u_iu_j))^2}.$$

Analogously, we can show that

$$\sqrt{2\sum_{1\le i<j\le n}(I_Y(u_iu_j))^2+n(n-1)|I|^{\frac{2}{n}}}\le E(I_Y(u_iu_j))\le\sqrt{2n\sum_{1\le i<j\le n}(I_Y(u_iu_j))^2}\text{ and}$$

$$\sqrt{2\sum_{1\le i<j\le n}(F_Y(u_iu_j))^2+n(n-1)|F|^{\frac{2}{n}}}\le E(F_Y(u_iu_j))\le\sqrt{2n\sum_{1\le i<j\le n}(F_Y(u_iu_j))^2}. \qquad \square$$

Example 3.3. (Illustration to Theorem 3.2) For the SVNG $\mathcal{G}$, given in Fig. 1

$E(T_Y(u_iu_j)) = 1.3246$, lower bound=0.8944 and upper bound=1.6492,
therefore, $0.8944 \le 1.3246 \le 1.6492$
$E(I_Y(u_iu_j)) = 1.9475$, lower bound=1.2728 and upper bound=2.4495,
therefore, $1.2728 \le 1.9475 \le 2.4495$
$E(F_Y(u_iu_j)) = 2.9980$, lower bound=2.2045 and upper bound=3.6986,
therefore, $2.2045 \le 2.9980 \le 3.6986$.

The following result gives us upper bound of the energy of a SVNG, with the conditions $n \le 2\sum_{i=1}^{m}(T_Y(u_iu_j))^2$, $n \le 2\sum_{i=1}^{m}(I_Y(u_iu_j))^2$ and $n \le 2\sum_{i=1}^{m}(F_Y(u_iu_j))^2$.

Theorem 3.3. *Let $\mathcal{G} = (X, Y)$ be a SVNG on n vertices and the adjacency matrix $A(\mathcal{G}) = \langle A(T_Y(u_iu_j)), A(I_Y(u_iu_j)), A(F_Y(u_iu_j))\rangle$ of $\mathcal{G}$. If $n \le 2\sum_{i=1}^{m}(T_Y(u_iu_j))^2$, $n \le 2\sum_{i=1}^{m}(I_Y(u_iu_j))^2$ and $n \le 2\sum_{i=1}^{m}(F_Y(u_iu_j))^2$. Then*

(i) $$E(T_Y(u_iu_j)) \le \frac{2\sum_{i=1}^{m}(T_Y(u_iu_j))^2}{n} + \sqrt{(n-1)\left\{2\sum_{i=1}^{m}(T_Y(u_iu_j))^2 - \left(\frac{2\sum_{i=1}^{m}(T_Y(u_iu_j))^2}{n}\right)^2\right\}}$$

(ii) $$E(I_Y(u_iu_j)) \le \frac{2\sum_{i=1}^{m}(I_Y(u_iu_j))^2}{n} + \sqrt{(n-1)\left\{2\sum_{i=1}^{m}(I_Y(u_iu_j))^2 - \left(\frac{2\sum_{i=1}^{m}(I_Y(u_iu_j))^2}{n}\right)^2\right\}}$$

(iii) $$E(F_Y(u_iu_j)) \le \frac{2\sum_{i=1}^{m}(F_Y(u_iu_j))^2}{n} + \sqrt{(n-1)\left\{2\sum_{i=1}^{m}(F_Y(u_iu_j))^2 - \left(\frac{2\sum_{i=1}^{m}(F_Y(u_iu_j))^2}{n}\right)^2\right\}}.$$

Proof. If $A=[a_{ij}]_{n\times n}$ is a symmetric matrix whose trace is zero. Then $\lambda_{\max}\geq \frac{2\sum\limits_{1\leq i<j\leq n} a_{ij}}{n}$, where, $\lambda_{\max}$ is the maximum eigenvalue of A. If $A(\mathcal{G})$ is the adjacency matrix of a SVNG $\mathcal{G}$, then $\lambda_1 \geq \frac{2\sum\limits_{i=1}^{m} T_Y(u_iu_j)}{n}$, where $\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_n$. Moreover, since

$$\sum_{i=1}^{n}\lambda_i^2 = 2\sum_{i=1}^{m}(T_Y(u_iu_j))^2$$

$$\sum_{i=2}^{n}\lambda_i^2 = 2\sum_{i=1}^{m}(T_Y(u_iu_j))^2-\lambda_1^2 \tag{3.5}$$

Using Cauchy-Schwarz inequality, we get

$$E(T_Y(u_iu_j))-\lambda_1=\sum_{i=2}^{n}|\lambda_i|\leq\sqrt{(n-1)\sum_{i=2}^{n}|\lambda_i|^2}. \tag{3.6}$$

Substituting (3.5) in (3.6), we must have

$$E(T_Y(u_iu_j))-\lambda_1\leq\sqrt{(n-1)\left(2\sum_{i=1}^{m}(T_Y(u_iu_j))^2-\lambda_1^2\right)}$$

$$E(T_Y(u_iu_j))\leq\lambda_1+\sqrt{(n-1)\left(2\sum_{i=1}^{m}(T_Y(u_iu_j))^2-\lambda_1^2\right)}. \tag{3.7}$$

Now, since the function

$$F(x)=x+\sqrt{(n-1)\left(2\sum_{i=1}^{m}(T_Y(u_iu_j))^2-x^2\right)}$$

decreases on the interval

$$\left(\sqrt{\frac{2\sum\limits_{i=1}^{m}(T_Y(u_iu_j))^2}{n}},\sqrt{2\sum_{i=1}^{m}(T_Y(u_iu_j))^2}\right),$$

also $n\leq 2\sum\limits_{i=1}^{m}(T_Y(u_iu_j))^2$, $1\leq\frac{2\sum\limits_{i=1}^{m}(T_Y(u_iu_j))^2}{n}$. So,

$$\sqrt{\frac{2\sum\limits_{i=1}^{m}(T_Y(u_iu_j))^2}{n}}\leq\frac{2\sum\limits_{i=1}^{m}(T_Y(u_iu_j))^2}{n}\leq\frac{2\sum\limits_{i=1}^{m}T_Y(u_iu_j)}{n}\leq\lambda_1\leq\sqrt{2\sum_{i=1}^{m}(T_Y(u_iu_j))^2}.$$

Therefore, (3.7) implies

$$E(T_Y(u_iu_j)) \leq \frac{2\sum_{i=1}^{m}(T_Y(u_iu_j))^2}{n} + \sqrt{(n-1)\left\{2\sum_{i=1}^{m}(T_Y(u_iu_j))^2 - \left(\frac{2\sum_{i=1}^{m}(T_Y(u_iu_j))^2}{n}\right)^2\right\}}.$$

Analogously, we can show that

$$E(I_Y(u_iu_j)) \leq \frac{2\sum_{i=1}^{m}(I_Y(u_iu_j))^2}{n} + \sqrt{(n-1)\left\{2\sum_{i=1}^{m}(I_Y(u_iu_j))^2 - \left(\frac{2\sum_{i=1}^{m}(I_Y(u_iu_j))^2}{n}\right)^2\right\}}$$

and

$$E(F_Y(u_iu_j)) \leq \frac{2\sum_{i=1}^{m}(F_Y(u_iu_j))^2}{n} + \sqrt{(n-1)\left\{2\sum_{i=1}^{m}(F_Y(u_iu_j))^2 - \left(\frac{2\sum_{i=1}^{m}(F_Y(u_iu_j))^2}{n}\right)^2\right\}}.$$

□

Theorem 3.4. *Let $\mathcal{G} = (X, Y)$ be a SVNG on n vertices. If $2\sum_{i=1}^{m}(T_Y(u_iu_j))^2 \leq n$, $2\sum_{i=1}^{m}(I_Y(u_iu_j))^2 \leq n$ and $2\sum_{i=1}^{m}(F_Y(u_iu_j))^2 \leq n$. Then*

(i) $E(T_Y(u_iu_j)) \leq 2\sum_{i=1}^{m}(T_Y(u_iu_j))^2$,

(ii) $E(I_Y(u_iu_j)) \leq 2\sum_{i=1}^{m}(I_Y(u_iu_j))^2$,

(iii) $E(F_Y(u_iu_j)) \leq 2\sum_{i=1}^{m}(F_Y(u_iu_j))^2$.

Proof. Since $2\sum_{i=1}^{m}(T_Y(u_iu_j))^2 \leq n$, where $2\sum_{i=1}^{m}(T_Y(u_iu_j))^2$ is the sum of the vertex degrees, so, $\mathcal{G}$ has $n - 2\sum_{i=1}^{m}(T_Y(u_iu_j))^2$ isolated vertices. Let $\mathcal{G}'$ be the graph obtained by removing the isolated vertices $n - 2\sum_{i=1}^{m}(T_Y(u_iu_j))^2$ from $\mathcal{G}$, then $\mathcal{G}'$ has $2\sum_{i=1}^{m}(T_Y(u_iu_j))^2$ vertices and

$\sum_{i=1}^{m}(T_Y(u_iu_j))^2$ edges. Using Theorem 3.3, we get

$$\begin{aligned} E(T_Y(u_iu_j)) &\leq 1+\sqrt{\left(2\sum_{i=1}^{m}(T_Y(u_iu_j))^2-1\right)\left(2\sum_{i=1}^{m}(T_Y(u_iu_j))^2-1\right)} \\ &= 1+(2\sum_{i=1}^{m}(T_Y(u_iu_j))^2-1) \\ &= 2\sum_{i=1}^{m}(T_Y(u_iu_j))^2. \end{aligned}$$

Therefore, $E(T_Y(u_iu_j)) \leq 2\sum_{i=1}^{m}(T_Y(u_iu_j))^2$. Similarly, it is easy to show that $E(I_Y(u_iu_j)) \leq 2\sum_{i=1}^{m}(I_Y(u_iu_j))^2$ and $E(F_Y(u_iu_j)) \leq 2\sum_{i=1}^{m}(F_Y(u_iu_j))^2$. □

Theorem 3.5. *Let $\mathcal{G}=(X,Y)$ be a SVNG on n vertices. Then $E(\mathcal{G}) \leq \frac{n}{2}(1+\sqrt{n})$.*

Proof. Suppose that $\mathcal{G}=(X,Y)$ is a SVNG with n vertices. If $n \leq 2\sum_{i=1}^{m}(T_Y(u_iu_j))^2 = 2y$, then by routine calculus, it is easy to show that $f(y)=\frac{2y}{n}+\sqrt{(n-1)(2y-(\frac{2y}{n})^2)}$ is maximized when $y=\frac{n^2+n\sqrt{n}}{4}$. Substituting this value of y in place of $y=\sum_{i=1}^{m}(T_Y(u_iu_j))^2$ in Theorem 3.3, we must have $E(T_Y(u_iu_j)) \leq \frac{n}{2}(1+\sqrt{n})$. Similarly, it is easy to show that $E(I_Y(u_iu_j)) \leq \frac{n}{2}(1+\sqrt{n})$ and $E(F_Y(u_iu_j)) \leq \frac{n}{2}(1+\sqrt{n})$. Hence $E(\mathcal{G}) \leq \frac{n}{2}(1+\sqrt{n})$. □

4 Laplacian energy of a single value neutrosophic graph

In this section, we define and investigate the Laplacian energy of a graph under single valued neutrosophic environment and investigate its properties.

Definition 4.1. Let $\mathcal{G}=(X,Y)$ be a SVNG on n vertices. The degree matrix, $D(\mathcal{G}) = \langle D(T_Y(u_iu_j)), D(I_Y(u_iu_j)), D(F_Y(u_iu_j))\rangle = [d_{ij}]$, of $\mathcal{G}$ is a $n\times n$ diagonal matrix defined as:

$$d_{ij}=\begin{cases} d_{\mathcal{G}}(u_i) & \text{if } i=j, \\ 0 & \text{otherwise} \end{cases}$$

Definition 4.2. The Laplacian matrix of a SVNG $\mathcal{G}=(X,Y)$ is defined as $L(\mathcal{G}) = \langle L(T_Y(u_iu_j)), L(I_Y(u_iu_j)), L(F_Y(u_iu_j))\rangle = D(\mathcal{G})-A(\mathcal{G})$, where $A(\mathcal{G})$ is an adjacency matrix and $D(\mathcal{G})$ is a degree matrix of a SVNG $\mathcal{G}$.

Example 4.1. The Laplacian matrix of the SVNG shown in Fig. 1 is

$$L(\mathcal{G}) = \begin{pmatrix} \langle 0.5, 0.9, 1.1\rangle & \langle -0.2, -0.5, -0.6\rangle & \langle -0.3, -0.4, -0.5\rangle & 0 \\ \langle -0.2, -0.5, -0.6\rangle & \langle 0.7, 1.2, 1.7\rangle & \langle -0.4, -0.3, -0.5\rangle & \langle -0.1, -0.4, -0.6\rangle \\ \langle -0.3, -0.4, -0.5\rangle & \langle -0.4, -0.3, -0.5\rangle & \langle 0.9, 1.0, 1.7\rangle & \langle -0.2, -0.3, -0.7\rangle \\ 0 & \langle -0.1, -0.4, -0.6\rangle & \langle -0.2, -0.3, -0.7\rangle & \langle 0.3, 0.7, 1.3\rangle \end{pmatrix}.$$

Definition 4.3. The spectrum of Laplacian matrix of a SVNG $L(\mathcal{G})$ is defined as $\langle P_L, Q_L, R_L\rangle$, where P_L, Q_L and R_L are the sets of Laplacian eigenvalues of $L(T_Y(u_iu_j))$, $L(I_Y(u_iu_j))$ and $L(F_Y(u_iu_j))$, respectively.

Theorem 4.1. *Let $\mathcal{G} = (X, Y)$ be a SVNG and let $L(\mathcal{G})$ be the Laplacian matrix of $\mathcal{G}$. If $\upsilon_1 \geq \upsilon_2 \geq \ldots \geq \upsilon_n$, $\psi_1 \geq \psi_2 \geq \ldots \geq \psi_n$ and $\omega_1 \geq \omega_2 \geq \ldots \geq \omega_n$ are the eigenvalues of $L(T_Y(u_iu_j))$, $L(I_Y(u_iu_j))$ and $L(F_Y(u_iu_j))$, respectively. Then*

1. $\sum\limits_{\substack{i=1\\ \upsilon_i\in P_L}}^{n} \upsilon_i = 2\sum\limits_{1\leq i<j\leq n} T_Y(u_iu_j)$, $\sum\limits_{\substack{i=1\\ \psi_i\in Q_L}}^{n} \psi_i = 2\sum\limits_{1\leq i<j\leq n} I_Y(u_iu_j)$ and $\sum\limits_{\substack{i=1\\ \omega_i\in R_L}}^{n} \omega_i = 2\sum\limits_{1\leq i<j\leq n} F_Y(u_iu_j)$

2. $\sum\limits_{\substack{i=1\\ \upsilon_i\in P_L}}^{n} \upsilon_i^2 = 2\sum\limits_{1\leq i<j\leq n} (T_Y(u_iu_j))^2 + \sum\limits_{i=1}^{n} d^2_{T_Y(u_iu_j)}(u_i)$, $\sum\limits_{\substack{i=1\\ \psi_i\in Q_L}}^{n} \psi_i^2 = 2\sum\limits_{1\leq i<j\leq n} (I_Y(u_iu_j))^2 + \sum\limits_{i=1}^{n} d^2_{I_Y(u_iu_j)}(u_i)$ and $\sum\limits_{\substack{i=1\\ \omega_i\in R_L}}^{n} \omega_i^2 = 2\sum\limits_{1\leq i<j\leq n} (F_Y(u_iu_j))^2 + \sum\limits_{i=1}^{n} d^2_{F_Y(u_iu_j)}(u_i)$.

Proof. 1. Since $L(\mathcal{G})$ is a symmetric matrix with non negative Laplacian eigenvalues, such that

$$\sum_{\substack{i=1\\ \upsilon_i\in P_L}}^{n} \upsilon_i = tr(L(\mathcal{G})) = \sum_{i=1}^{n} d_{T_Y(u_iu_j)}(u_i) = 2\sum_{1\leq i<j\leq n} T_Y(u_iu_j).$$

Similarly, it is easy to show that, $\sum\limits_{\substack{i=1\\ \psi_i\in Q_L}}^{n} \psi_i = 2\sum\limits_{1\leq i<j\leq n} I_Y(u_iu_j)$ and $\sum\limits_{\substack{i=1\\ \omega_i\in R_L}}^{n} \omega_i = 2\sum\limits_{1\leq i<j\leq n} F_Y(u_iu_j)$.

2. By definition of Laplacian matrix, we have

$$L(T_Y(u_iu_j)) = \begin{pmatrix} d_{T_Y(u_iu_j)}(u_1) & -T_Y(u_1u_2) & \ldots & -T_Y(u_1u_n) \\ -T_Y(u_2u_1) & d_{T_Y(u_iu_j)}(u_2) & \ldots & -T_Y(u_2u_n) \\ \vdots & \vdots & \ddots & \vdots \\ -T_Y(u_nu_1) & -T_Y(u_nu_2) & \ldots & d_{T_Y(u_iu_j)}(u_n) \end{pmatrix}.$$

By trace properties of a matrix, we have

$$tr((L(T_Y(u_iu_j)))^2) = \sum_{\substack{i=1 \\ \upsilon_i \in P_L}}^{n} \upsilon_i^2$$

where

$$\begin{aligned} tr((L(T_Y(u_iu_j)))^2) &= (d^2_{T_Y(u_iu_j)}(u_1) + T_Y^2(u_1u_2) + \ldots + T_Y^2(u_1u_n)) \\ &\quad +(T_Y^2(u_2u_1) + d^2_{T_Y(u_iu_j)}(u_2) + \ldots + T_Y^2(u_2u_n)) \\ &\quad + \ldots + (T_Y^2(u_nu_1) + T_Y^2(u_nu_2) + \ldots + d^2_{T_Y(u_iu_j)}(u_n)) \\ &= 2\sum_{1 \le i < j \le n} (T_Y(u_iu_j))^2 + \sum_{i=1}^{n} d^2_{T_Y(u_iu_j)}(u_i). \end{aligned}$$

Therefore,

$$\sum_{\substack{i=1 \\ \upsilon_i \in P_L}}^{n} \upsilon_i^2 = 2\sum_{1 \le i < j \le n} (T_Y(u_iu_j))^2 + \sum_{i=1}^{n} d^2_{T_Y(u_iu_j)}(u_i).$$

Analogously, we can show that $\sum\limits_{\substack{i=1 \\ \psi_i \in Q_L}}^{n} \psi_i^2 = 2\sum\limits_{1 \le i < j \le n} (I_Y(u_iu_j))^2 + \sum\limits_{i=1}^{n} d^2_{I_Y(u_iu_j)}(u_i)$ and $\sum\limits_{\substack{i=1 \\ \omega_i \in R_L}}^{n} \omega_i^2 = 2\sum\limits_{1 \le i < j \le n} (F_Y(u_iu_j))^2 + \sum\limits_{i=1}^{n} d^2_{F_Y(u_iu_j)}(u_i)$. □

Definition 4.4. The Laplacian energy of a SVNG $\mathcal{G} = (X, Y)$ is defined as

$$LE(\mathcal{G}) = \langle LE(T_Y(u_iu_j)), LE(I_Y(u_iu_j)), LE(F_Y(u_iu_j)) \rangle = \left\langle \sum_{i=1}^{n} |\gamma_i|, \sum_{i=1}^{n} |\rho_i|, \sum_{i=1}^{n} |\tau_i| \right\rangle$$

where

$$\begin{aligned} \gamma_i &= \upsilon_i - \frac{2\sum\limits_{1 \le i < j \le n} T_Y(u_iu_j)}{n}, \\ \rho_i &= \psi_i - \frac{2\sum\limits_{1 \le i < j \le n} I_Y(u_iu_j)}{n}, \\ \tau_i &= \omega_i - \frac{2\sum\limits_{1 \le i < j \le n} F_Y(u_iu_j)}{n}. \end{aligned}$$

Theorem 4.2. *Let $\mathcal{G} = (X, Y)$ be a SVNG and let $L(\mathcal{G})$ be the Laplacian matrix of $\mathcal{G}$. If $\upsilon_1 \ge \upsilon_2 \ge \ldots \ge \upsilon_n$, $\psi_1 \ge \psi_2 \ge \ldots \ge \psi_n$ and $\omega_1 \ge \omega_2 \ge \ldots \ge \omega_n$ are the* *eigenvalues of* $L(T_Y(u_iu_j))$, $L(I_Y(u_iu_j))$ *and* $L(F_Y(u_iu_j))$, *respectively, and* $\gamma_i = \upsilon_i - \frac{2\sum\limits_{1\le i<j\le n} T_Y(u_iu_j)}{n}$, $\rho_i = \psi_i - \frac{2\sum\limits_{1\le i<j\le n} I_Y(u_iu_j)}{n}$, $\tau_i = \omega_i - \frac{2\sum\limits_{1\le i<j\le n} F_Y(u_iu_j)}{n}$. *Then*

$$\sum_{i=1}^{n}\gamma_i = 0, \sum_{i=1}^{n}\rho_i = 0, \sum_{i=1}^{n}\tau_i = 0,$$
$$\sum_{i=1}^{n}\gamma_i^2 = 2M_T, \sum_{i=1}^{n}\rho_i^2 = 2M_I, \sum_{i=1}^{n}\tau_i^2 = 2M_F,$$

where

$$M_T = \sum_{1\le i<j\le n}(T_Y(u_iu_j))^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{T_Y(u_iu_j)}(u_i) - \frac{2\sum\limits_{1\le i<j\le n} T_Y(u_iu_j)}{n}\right)^2,$$
$$M_I = \sum_{1\le i<j\le n}(I_Y(u_iu_j))^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{I_Y(u_iu_j)}(u_i) - \frac{2\sum\limits_{1\le i<j\le n} I_Y(u_iu_j)}{n}\right)^2,$$
$$M_F = \sum_{1\le i<j\le n}(F_Y(u_iu_j))^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{F_Y(u_iu_j)}(u_i) - \frac{2\sum\limits_{1\le i<j\le n} F_Y(u_iu_j)}{n}\right)^2.$$

Example 4.2. The Laplacian spectrum and the Laplacian energy of a SVNG $\mathcal{G}$, given in Fig. 1 are as follows:

$$\text{Laplacian Spec}(T_Y(u_iu_j)) = \{0, 0.3528, 0.8000, 1.2472\},$$
$$\text{Laplacian Spec}(I_Y(u_iu_j)) = \{0, 0.7748, 1.3341, 1.6911\},$$
$$\text{Laplacian Spec}(F_Y(u_iu_j)) = \{0, 1.1739, 2.2090, 2.4171\}.$$

Therefore,

$$\text{Laplacian Spec}(\mathcal{G}) = \{\langle 0,0,0\rangle, \langle 0.3528, 0.7748, 1.1739\rangle, \langle 0.8000, 1.3341, 2.2090\rangle, \langle 1.2472, 1.6911, 2.4171\rangle\}.$$

Now,
$LE(T_Y(u_iu_j)) = \left|0 - \frac{2(1.2)}{4}\right| + \left|0.3528 - \frac{2(1.2)}{4}\right| + \left|0.8000 - \frac{2(1.2)}{4}\right| + \left|1.2472 - \frac{2(1.2)}{4}\right| = 1.6944$,
$LE(I_Y(u_iu_j)) = \left|0 - \frac{2(1.9)}{4}\right| + \left|0.7748 - \frac{2(1.9)}{4}\right| + \left|1.3341 - \frac{2(1.9)}{4}\right| + \left|1.6911 - \frac{2(1.9)}{4}\right| = 2.2504$,
$LE(F_Y(u_iu_j)) = \left|0 - \frac{2(2.9)}{4}\right| + \left|1.1739 - \frac{2(2.9)}{4}\right| + \left|2.2090 - \frac{2(2.9)}{4}\right| + \left|2.4171 - \frac{2(2.9)}{4}\right| = 3.4522$.
Therefore, $LE(\mathcal{G}) = \langle 1.6944, 2.2504, 3.4522\rangle$.
Also we have

$$\sum_{i=1}^{4}\gamma_i = 0, \sum_{i=1}^{4}\rho_i = 0, \sum_{i=1}^{4}\tau_i = 0.$$

$$\sum_{i=1}^{4} \gamma_i^2 = 0.8800 = 2(0.4400) = 2M_T,$$

$$\sum_{i=1}^{4} \rho_i^2 = 1.6300 = 2(0.8150) = 2M_I,$$

$$\sum_{i=1}^{4} \tau_i^2 = 3.6900 = 2(1.8450) = 2M_F.$$

Theorem 4.3. *Let $\mathcal{G} = (X, Y)$ be a SVNG on n vertices and the Laplacian matrix $L(\mathcal{G}) = \langle L(T_Y(u_iu_j)), L(I_Y(u_iu_j)), L(F_Y(u_iu_j))\rangle$ of $\mathcal{G}$. Then*

(i) $LE(T_Y(u_iu_j)) \leq \sqrt{2n \sum\limits_{1\leq i<j\leq n} (T_Y(u_iu_j))^2 + n\sum\limits_{i=1}^{n} \left(d_{T_Y(u_iu_j)}(u_i) - \frac{2\sum\limits_{1\leq i<j\leq n} T_Y(u_iu_j)}{n} \right)^2}$

(ii) $LE(I_Y(u_iu_j)) \leq \sqrt{2n \sum\limits_{1\leq i<j\leq n} (I_Y(u_iu_j))^2 + n\sum\limits_{i=1}^{n} \left(d_{I_Y(u_iu_j)}(u_i) - \frac{2\sum\limits_{1\leq i<j\leq n} I_Y(u_iu_j)}{n} \right)^2}$

(iii) $LE(F_Y(u_iu_j)) \leq \sqrt{2n \sum\limits_{1\leq i<j\leq n} (F_Y(u_iu_j))^2 + n\sum\limits_{i=1}^{n} \left(d_{F_Y(u_iu_j)}(u_i) - \frac{2\sum\limits_{1\leq i<j\leq n} F_Y(u_iu_j)}{n} \right)^2}.$

Proof. Apply Cauchy-Schwarz inequality to the n numbers $1, 1, \ldots, 1$ and $|\gamma_1|, |\gamma_2|, \ldots, |\gamma_n|$, we have

$$\sum_{i=1}^{n} |\gamma_i| \leq \sqrt{n}\sqrt{\sum_{i=1}^{n} |\gamma_i|^2}$$

$$LE(T_Y(u_iu_j)) \leq \sqrt{n}\sqrt{2M_T} = \sqrt{2nM_T}.$$

Since $M_T = \sum\limits_{1\leq i<j\leq n} (T_Y(u_iu_j))^2 + \frac{1}{2}\sum\limits_{i=1}^{n} \left(d_{T_Y(u_iu_j)}(u_i) - \frac{2\sum\limits_{1\leq i<j\leq n} T_Y(u_iu_j)}{n} \right)^2$,

therefore, $LE(T_Y(u_iu_j)) \leq \sqrt{2n \sum\limits_{1\leq i<j\leq n} (T_Y(u_iu_j))^2 + n\sum\limits_{i=1}^{n} \left(d_{T_Y(u_iu_j)}(u_i) - \frac{2\sum\limits_{1\leq i<j\leq n} T_Y(u_iu_j)}{n} \right)^2}.$

Analogously, it is easy to show that

$$LE(I_Y(u_iu_j)) \leq \sqrt{2n \sum_{1\leq i<j\leq n} (I_Y(u_iu_j))^2 + n\sum_{i=1}^{n} \left(d_{I_Y(u_iu_j)}(u_i) - \frac{2\sum\limits_{1\leq i<j\leq n} I_Y(u_iu_j)}{n} \right)^2}$$

and $LE(F_Y(u_iu_j)) \leq \sqrt{2n \sum\limits_{1\leq i<j\leq n} (F_Y(u_iu_j))^2 + n\sum\limits_{i=1}^{n} \left(d_{F_Y(u_iu_j)}(u_i) - \frac{2\sum\limits_{1\leq i<j\leq n} F_Y(u_iu_j)}{n} \right)^2}.$

□

Theorem 4.4. *Let $\mathcal{G} = (X, Y)$ be a SVNG on n vertices and the Laplacian matrix $L(\mathcal{G}) = \langle L(T_Y(u_iu_j)), L(I_Y(u_iu_j)), L(F_Y(u_iu_j))\rangle$ of $\mathcal{G}$. Then*

(i) $LE(T_Y(u_iu_j)) \geq 2\sqrt{\sum_{1\leq i<j\leq n}(T_Y(u_iu_j))^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{T_Y(u_iu_j)}(u_i) - \frac{2\sum_{1\leq i<j\leq n} T_Y(u_iu_j)}{n}\right)^2}$

(ii) $LE(I_Y(u_iu_j)) \geq 2\sqrt{\sum_{1\leq i<j\leq n}(I_Y(u_iu_j))^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{I_Y(u_iu_j)}(u_i) - \frac{2\sum_{1\leq i<j\leq n} I_Y(u_iu_j)}{n}\right)^2}$

(iii) $LE(F_Y(u_iu_j)) \geq 2\sqrt{\sum_{1\leq i<j\leq n}(F_Y(u_iu_j))^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{F_Y(u_iu_j)}(u_i) - \frac{2\sum_{1\leq i<j\leq n} F_Y(u_iu_j)}{n}\right)^2}$.

Proof.

$$\left(\sum_{i=1}^{n}|\gamma_i|\right)^2 = \sum_{i=1}^{n}|\gamma_i|^2 + 2\sum_{1\leq i<j\leq n}|\gamma_i\gamma_j| \geq 4M_T$$

$$LE(T_Y(u_iu_j)) \geq 2\sqrt{M_T}$$

Since $M_T = \sum_{1\leq i<j\leq n}(T_Y(u_iu_j))^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{T_Y(u_iu_j)}(u_i) - \frac{2\sum_{1\leq i<j\leq n} T_Y(u_iu_j)}{n}\right)^2$,

therefore, $LE(T_Y(u_iu_j)) \geq 2\sqrt{\sum_{1\leq i<j\leq n}(T_Y(u_iu_j))^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{T_Y(u_iu_j)}(u_i) - \frac{2\sum_{1\leq i<j\leq n} T_Y(u_iu_j)}{n}\right)^2}$.

Similarly, it is easy to show that

$LE(I_Y(u_iu_j)) \geq 2\sqrt{\sum_{1\leq i<j\leq n}(I_Y(u_iu_j))^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{I_Y(u_iu_j)}(u_i) - \frac{2\sum_{1\leq i<j\leq n} I_Y(u_iu_j)}{n}\right)^2}$

and $LE(F_Y(u_iu_j)) \geq 2\sqrt{\sum_{1\leq i<j\leq n}(F_Y(u_iu_j))^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d_{F_Y(u_iu_j)}(u_i) - \frac{2\sum_{1\leq i<j\leq n} F_Y(u_iu_j)}{n}\right)^2}$. □

Theorem 4.5. *Let $\mathcal{G} = (X, Y)$ be a SVNG on n vertices and the Laplacian matrix $L(\mathcal{G}) = \langle L(T_Y(u_iu_j)), L(I_Y(u_iu_j)), L(F_Y(u_iu_j))\rangle$ of $\mathcal{G}$. Then*

(i) $LE(T_Y(u_iu_j)) \leq \gamma_1$

$$+\sqrt{(n-1)\left(2\sum_{1\leq i<j\leq n}(T_Y(u_iu_j))^2 + \sum_{i=1}^{n}\left(d_{T_Y(u_iu_j)}(u_i) - \frac{2\sum_{1\leq i<j\leq n} T_Y(u_iu_j)}{n}\right)^2 - \gamma_1^2\right)}$$

(ii) $LE(I_Y(u_iu_j)) \leq \rho_1$

$$+\sqrt{(n-1)\left(2\sum_{1\leq i<j\leq n}(I_Y(u_iu_j))^2+\sum_{i=1}^{n}\left(d_{I_Y(u_iu_j)}(u_i)-\frac{2\sum\limits_{1\leq i<j\leq n}I_Y(u_iu_j)}{n}\right)^2-\rho_1^2\right)}$$

(iii) $LE(F_Y(u_iu_j)) \leq \tau_1$

$$+\sqrt{(n-1)\left(2\sum_{1\leq i<j\leq n}(F_Y(u_iu_j))^2+\sum_{i=1}^{n}\left(d_{F_Y(u_iu_j)}(u_i)-\frac{2\sum\limits_{1\leq i<j\leq n}F_Y(u_iu_j)}{n}\right)^2-\tau_1^2\right)}.$$

Proof. Using Cauchy-Schwarz inequality, we get

$$\sum_{i=1}^{n}|\gamma_i| \leq \sqrt{n\sum_{i=1}^{n}|\gamma_i|^2}$$

$$\sum_{i=2}^{n}|\gamma_i| \leq \sqrt{(n-1)\sum_{i=2}^{n}|\gamma_i|^2}$$

$$LE(T_Y(u_iu_j)) - \gamma_1 \leq \sqrt{(n-1)(2M_T-\gamma_1^2)}$$

$$LE(T_Y(u_iu_j)) \leq \gamma_1 + \sqrt{(n-1)(2M_T-\gamma_1^2)}$$

Since $M_T = \sum\limits_{1\leq i<j\leq n}(T_Y(u_iu_j))^2+\frac{1}{2}\sum\limits_{i=1}^{n}\left(d_{T_Y(u_iu_j)}(u_i)-\frac{2\sum\limits_{1\leq i<j\leq n}T_Y(u_iu_j)}{n}\right)^2$,
therefore, $LE(T_Y(u_iu_j)) \leq \gamma_1$

$$+\sqrt{(n-1)\left(2\sum_{1\leq i<j\leq n}(T_Y(u_iu_j))^2+\sum_{i=1}^{n}\left(d_{T_Y(u_iu_j)}(u_i)-\frac{2\sum\limits_{1\leq i<j\leq n}T_Y(u_iu_j)}{n}\right)^2-\gamma_1^2\right)}. \tag{4.2}$$

Similarly, we can show that $LE(I_Y(u_iu_j)) \leq \rho_1$

$$+\sqrt{(n-1)\left(2\sum_{1\leq i<j\leq n}(I_Y(u_iu_j))^2+\sum_{i=1}^{n}\left(d_{I_Y(u_iu_j)}(u_i)-\frac{2\sum\limits_{1\leq i<j\leq n}I_Y(u_iu_j)}{n}\right)^2-\rho_1^2\right)}$$

and $LE(F_Y(u_iu_j)) \leq \tau_1$

$$+\sqrt{(n-1)\left(2\sum_{1\leq i<j\leq n}(F_Y(u_iu_j))^2+\sum_{i=1}^{n}\left(d_{F_Y(u_iu_j)}(u_i)-\frac{2\sum\limits_{1\leq i<j\leq n}F_Y(u_iu_j)}{n}\right)^2-\tau_1^2\right)}. \qquad \square$$

Theorem 4.6. *If the SVNG $\mathcal{G} = (X, Y)$ is regular, then*

(i) $LE(T_Y(u_iu_j)) \leq \gamma_1 + \sqrt{(n-1)\left(2\sum\limits_{1\leq i<j\leq n}(T_Y(u_iu_j))^2 - \gamma_1^2\right)}$

(ii) $LE(I_Y(u_iu_j)) \leq \rho_1 + \sqrt{(n-1)\left(2\sum\limits_{1\leq i<j\leq n}(I_Y(u_iu_j))^2 - \rho_1^2\right)}$

(iii) $LE(F_Y(u_iu_j)) \leq \tau_1 + \sqrt{(n-1)\left(2\sum\limits_{1\leq i<j\leq n}(F_Y(u_iu_j))^2 - \tau_1^2\right)}$.

Proof. Let $\mathcal{G}$ be a regular SVNG, then

$$d_{T_Y(u_iu_j)}(u_i) = \frac{2\sum\limits_{1\leq i<j\leq n} T_Y(u_iu_j)}{n} \tag{4.1}$$

Substituting (4.1) in (4.2), we get

$LE(T_Y(u_iu_j)) \leq \gamma_1 + \sqrt{(n-1)\left(2\sum\limits_{1\leq i<j\leq n}(T_Y(u_iu_j))^2 - \gamma_1^2\right)}$.

Similarly, it is easy to show that $LE(I_Y(u_iu_j)) \leq \rho_1 + \sqrt{(n-1)\left(2\sum\limits_{1\leq i<j\leq n}(I_Y(u_iu_j))^2 - \rho_1^2\right)}$

and $LE(F_Y(u_iu_j)) \leq \tau_1 + \sqrt{(n-1)\left(2\sum\limits_{1\leq i<j\leq n}(F_Y(u_iu_j))^2 - \tau_1^2\right)}$. □

5 Application

In this section, the concept of energy and Laplacian energy of a SVNG is explained through a real time example. We have taken the website http://www.pantechsolutions.net modeled as a SVNG by considering the navigation of the customer. We have taken the four links: 1. microcontroller-boards, 2./log-in html, 3./ and 4./ project kits for our calculation. A SVNG of this site for four different time periods is considered. The energy and Laplacian energy of a SVNG is calculated for each of these periods. The energy and Laplacian energy is also represented in terms of bar diagram. In the website http://www.pantechsolutions.net the above 4 links are considered for the period January 16, 2016 to February 15, 2016 and for this graph, we have

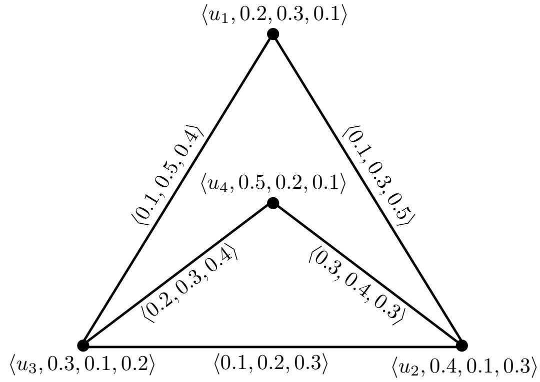

Figure 2: SVNG $\mathcal{G}_1$.

$\text{Spec}(T_Y(u_iu_j)) = \{-0.3442, -0.1000, 0.0066, 0.4376\}$,
$\text{Spec}(I_Y(u_iu_j)) = \{-0.6630, -0.2742, 0.0774, 0.8598\}$,
$\text{Spec}(F_Y(u_iu_j)) = \{-0.6703, -0.3296, 0.0299, 0.9701\}$,
$E(T_Y(u_iu_j)) = 0.8884$, $E(I_Y(u_iu_j)) = 1.8744$, $E(F_Y(u_iu_j)) = 1.9999$.
Therefore, $E(\mathcal{G}_1) = \langle 0.8884, 1.8744, 1.9999 \rangle$.

Laplasian $\text{Spec}(T_Y(u_iu_j)) = \{0, 0.2492, 0.5244, 0.8264\}$,
Laplasian $\text{Spec}(I_Y(u_iu_j)) = \{0, 0.6975, 1.1757, 1.5269\}$,
Laplasian $\text{Spec}(F_Y(u_iu_j)) = \{0, 0.7605, 1.4139, 1.6256\}$,
$LE(T_Y(u_iu_j)) = 1.1016$,$LE(I_Y(u_iu_j)) = 2.0051$, $LE(F_Y(u_iu_j)) = 2.2790$.
Therefore, $LE(\mathcal{G}_1) = \langle 1.1016, 2.0051, 2.2790 \rangle$.

For the period February 16, 2016 to March 15, 2016, we have

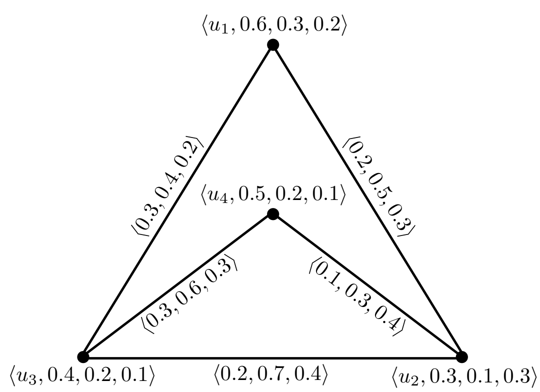

Figure 3: SVNG $\mathcal{G}_2$.

$\text{Spec}(T_Y(u_iu_j)) = \{-0.4245, -0.1714, 0.0215, 0.5744\}$,
$\text{Spec}(I_Y(u_iu_j)) = \{-0.7909, -0.5799, 0.0536, 1.3173\}$,

$\text{Spec}(F_Y(u_iu_j)) = \{-0.5037, -0.3400, 0.0007, 0.8430\}$,
$E(T_Y(u_iu_j)) = 1.1919$, $E(I_Y(u_iu_j)) = 2.7418$,$E(F_Y(u_iu_j)) = 1.6874$.
Therefore,$E(\mathcal{G}_2) = \langle 1.1919, 2.7418, 1.6874\rangle$.

Laplasian $\text{Spec}(T_Y(u_iu_j)) = \{0, 0.4200, 0.6908, 1.0892\}$,
laplasian $\text{Spec}(I_Y(u_iu_j)) = \{0, 0.8716, 1.7656, 2.3629\}$,
Laplasian $\text{Spec}(F_Y(u_iu_j)) = \{0, 0.5672, 1.1546, 1.4783\}$,
$LE(T_Y(u_iu_j)) = 1.36$, $LE(I_Y(u_iu_j)) = 3.2569$, $LE(F_Y(u_iu_j)) = 2.0657$.
Therefore, $LE(\mathcal{G}_2) = \langle 1.36, 3.2569, 2.0657\rangle$.

For the period March 16, 2016 to April 15, 2016, we have

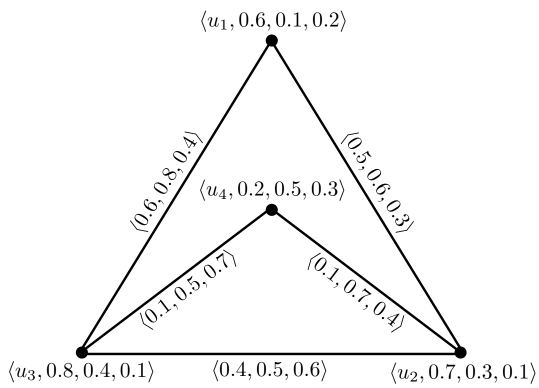

Figure 4: SVNG $\mathcal{G}_3$.

$\text{Spec}(T_Y(u_iu_j)) = \{-0.6287, -0.3884, 0.0004, 1.0168\}$,
$\text{Spec}(I_Y(u_iu_j)) = \{-1.0779, -0.5696, 0.0698, 1.5776\}$,
$\text{Spec}(F_Y(u_iu_j)) = \{-0.8184, -0.4650, 0.0051, 1.2783\}$,
$E(T_Y(u_iu_j)) = 2.0343$, $E(I_Y(u_iu_j)) = 3.2949$, $E(F_Y(u_iu_j)) = 2.5668$.
Therefore, $E(\mathcal{G}_3) = \langle 2.0343, 3.2949, 2.5668\rangle$.

Laplasian $\text{Spec}(T_Y(u_iu_j)) = \{0, 0.2604, 1.4221, 1.7175\}$,
Laplasian $\text{Spec}(I_Y(u_iu_j)) = \{0, 1.2472, 2.3360, 2.6168\}$,
Laplasian $\text{Spec}(F_Y(u_iu_j)) = \{0, 0.8182, 1.6721, 2.3097\}$,
$LE(T_Y(u_iu_j)) = 2.8792$, $LE(I_Y(u_iu_j)) = 3.7056$,$LE(F_Y(u_iu_j)) = 3.1636$.
Therefore, $LE(\mathcal{G}_3) = \langle 2.8792, 3.7056, 3.1636\rangle$.

Finally, for the period April 16, 2016 to May 15, 2016, we have

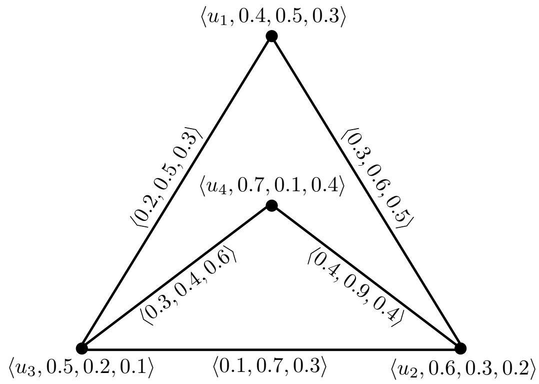

Figure 5: SVNG $\mathcal{G}_4$.

$\text{Spec}(T_Y(u_iu_j)) = \{-0.5716, -0.0973, 0.0027, 0.6662\}$,
$\text{Spec}(I_Y(u_iu_j)) = \{-1.0878, -0.5755, 0.0435, 1.6198\}$,
$\text{Spec}(F_Y(u_iu_j)) = \{-0.7686, -0.3985, 0.0990, 1.0680\}$,
$E(T_Y(u_iu_j)) = 1.3378$,$E(I_Y(u_iu_j)) = 3.3265$,$E(F_Y(u_iu_j)) = 2.3342$.
Therefore, $E(\mathcal{G}_4) = \langle 1.3378, 3.3265, 2.3342\rangle$.

Laplasian $\text{Spec}(T_Y(u_iu_j)) = \{0, 0.5637, 0.7641, 1.2721\}$,
Laplasian $\text{Spec}(I_Y(u_iu_j)) = \{0, 1.1660, 2.0643, 2.9697\}$,
Laplasian $\text{Spec}(F_Y(u_iu_j)) = \{0, 0.8207, 1.5544, 1.8249\}$,
$LE(T_Y(u_iu_j)) = 1.4725$, $LE(I_Y(u_iu_j)) = 3.868$, $LE(F_Y(u_iu_j)) = 2.5586$.
Therefore, $LE(\mathcal{G}_4) = \langle 1.4725, 3.8680, 2.5586\rangle$.

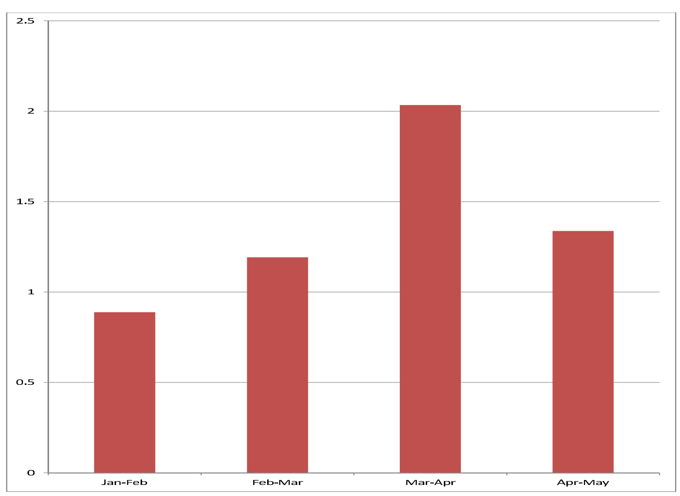

Figure 6: Table for energy of truth-membership values

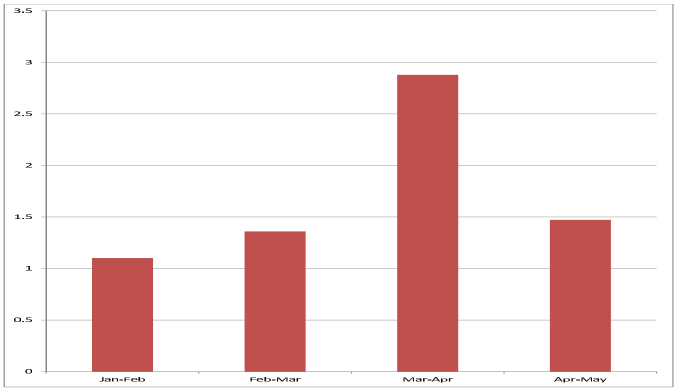

Figure 7: Table for Laplacian energy of truth-membership values

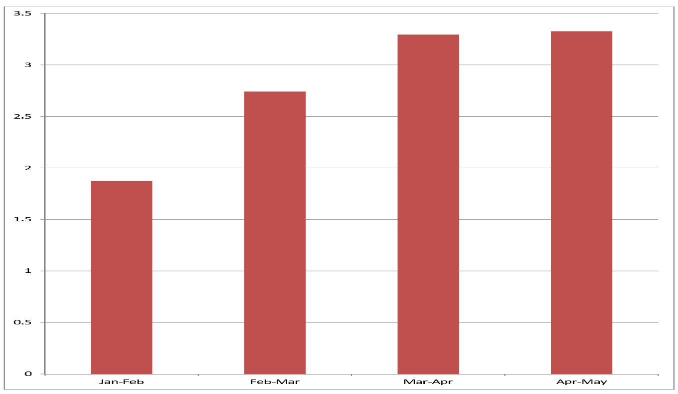

Figure 8: Table for energy of indeterminacy-membership values

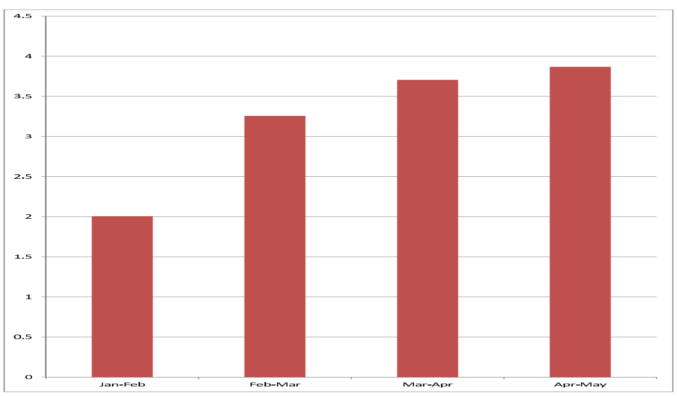

Figure 9: Table for Laplacian energy of indeterminacy-membership values

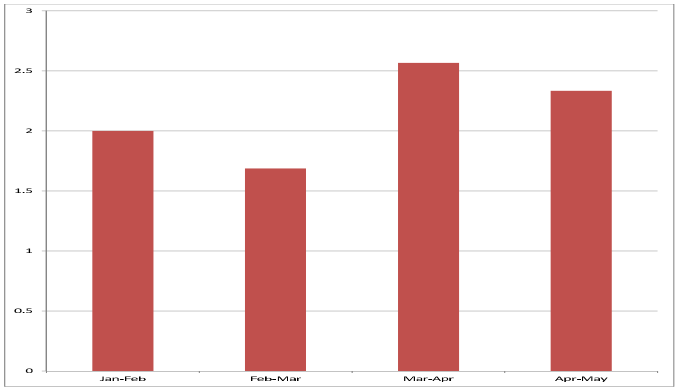

Figure 10: Table for energy of falsity-membership values

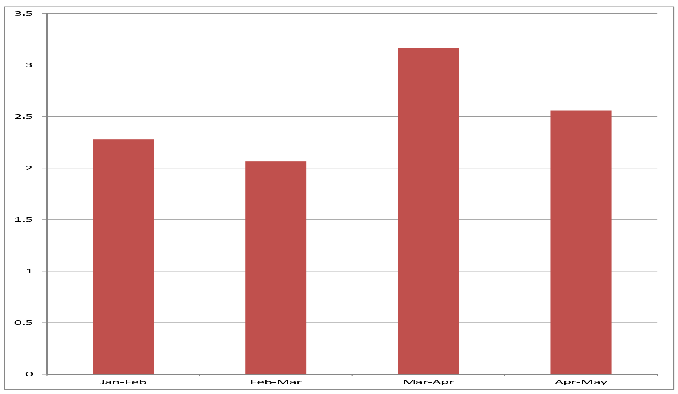

Figure 11: Table for Laplacian energy of falsity-membership values

The following bar diagrams represent the energy and Laplacian energy of four links for the above four periods corresponding to the truth-membership, indeterminacy-membership and falsity-membership values, respectively. From the above bar diagrams, the energy and Laplacian energy of truth-membership for the period March to April is high as compared to other periods, the energy and Laplacian energy of indeterminacy-membership for the period April to May is high and, the energy and Laplacian energy of falsity-membership for the period March to April is high.

6 Conclusions

Single valued neutrosophic models are more flexible and practical than fuzzy, intuitionistic fuzzy, vague and interval-valued intuitionistic fuzzy models. SVNGs can be used in computer technology, networking, communication, economics, genetics, linguistics, sociology etc, when the concept of indeterminacy is present. In this paper, we have introduced the concepts of energy and Laplacian energy of graphs under single valued neutrosophic environment and investigated their properties. We have derived the lower and upper bounds for the energy and Laplacian energy of a SVNG. These concepts are also illustrated with real time example. The proposed concepts of the energy and Laplacian energy of a SVNG can handle incomplete, imprecise, indeterminate and inconsistent information existing in different real situations.

References

[1] S. Aggarwal, R. Biswas, A. Q. Ansari, Neutrosophic modeling and control, Computer and Communication Technology (2010) 718-723.

[2] M. Akram, B. Davvaz, Strong intuitionistic fuzzy graphs, Filomat 26(2012) 177-196.

[3] Akram, M. and Shahzadi, S., (2016), Neutrosophic soft graphs with application, J. Intell. Fuzzy Syst., pp. 1-18. DOI: 10.3233/JIFS-16090.

[4] N. Anjali, S. Mathew, Energy of a fuzzy graph, Annals Fuzzy Maths and Informatics 6 (2013): 455-65.

[5] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets Syst. 20(1)(1986)87-96.

[6] K. Atanassov, G. Gargov, Interval-valued intuitionistic fuzzy sets, Fuzzy Sets Syst. 31(3)(1989) 343-349.

[7] K. T. Atanassov, Intuitionistic Fuzzy Sets, Physica-Verlag, Heidelberg, New York, 1999.

[8] S. S. Basha, E. Kartheek, Laplacian energy of an intuitionistic fuzzy graph, Indian Journal of Science and Technology, 8 (2015).

[9] S. Bhattacharya, Neutrosophic information fusion applied to the options market, Investment Management and Financial Innovations 1 (2005) 139-145.

[10] Broumi, S., Smarandache, F., Talea, M. and Bakali, A., (2016), Single valued neutrosophic graphs: degree, order and size, IEEE International Conference on Fuzzy Systems, pp. 2444-2451.

[11] S. Broumi, M. Talea, A. Bakali, F. Smarandache, Single valued neutrosophic graphs, Journal of New Theory (2016) 86-101.

[12] R. A. Borzooei, H. Rashmanlou, New concepts of vague graphs, International Journal of Machine Learning and Cybernetics (2015) 1-12.

[13] Dhavaseelan, R., Vikramaprasad, R. and Krishnaraj. V., (2015), Certain types of neutrosophic graphs, Int. J. of Mathematical Sciences and Applications, 5 (2), pp. 333-339.

[14] W. L. Gau and D. J. Buehrer, Vague sets, IEEE Transactions on Systems, Man and Cybernetics 23(2)(1993) 610-614.

[15] Y. Guo, H. D. Cheng, New neutrosophic approach to image segmentation, Pattern Recognition 42 (2009) 587-595.

[16] I. Gutman, The energy of a graph, Ber Math Stat Sekt Forsch Graz 103 (1978) 1-22.

[17] I. Gutman, The energy of a graph: old and new results, Algebraic Combinatorics and Applications, Springer Berlin Heidelberg (2001) 196-211.

[18] I. Gutman, B. Zhou, Laplacian energy of a graph, Linear Algebra and its Applications 414 (2006) 29-37.

[19] A. Kaufmann, Introduction a la Theorie des Sour-ensembles Flous, Masson et Cie 1 (1973).

[20] B. Praba, V. M. Chandrasekaran, G. Deepa, Energy of an intutionistic fuzzy graph, Ital J Pure Appl Math 32 (2014) 431-444.

[21] A. Rosenfeld, Fuzzy graphs, Fuzzy Sets and their Applications (L. A. Zadeh, K. S. Fu, M. Shimura, Eds.) Academic Press, New York (1975) 77-95.

[22] S. R. Sharbaf, F. Fayazi, Laplacian energy of a fuzzy graph, Iranian Journal of Mathematical Chemistry 5 (2014) 1-10.

[23] F. Smarandache, A unifying field in logics. Neutrosophy: neutrosophic probability, set and logic, American Research Press, Rehoboth, 1999.

[24] W. B. Vasantha Kandasamy, K. Ilanthenral, F. Smarandache, Neutrosophic Graphs: A New Dimension to Graph Theory, kindle Edition (2015).

[25] H. Wang, F. Smarandache, Y.Q. Zhang, R. Sunderraman, Single valued neutrosophic sets, Multispace and Multistructure 4 (2010) 410-413.

[26] H. L. Yang, Z.L. Guo, Y. She, X. Liao, On single valued neutrosophic relations, J. Intell. Fuzzy Syst. 30 (2016) 1045-1056.

[27] J. Ye, Multicriteria decision-making method using the correlation coefficient under single valued neutrosophic environment, International Journal of General Systems, 42(4) (2013) 386-394.

[28] J. Ye, J. Fu, Multi-period medical diagnosis method using a single valued neutrosophic similarity measure based on tangent function, Computer methods and programs in biomedicine 123 (2016) 142-149.

[29] L. A. Zadeh, Fuzzy sets, Information and Control 8(3) (1965) 338-353.

[30] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-I, Inf. Sci. 8(3)(1975) 199-249.