Contradiction of Infinite Bijections

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Abstract: Limits of sequences of sets required to define infinite bijections do not only raise paradoxes but cause self-contradictory results.

Bijections between the set of natural numbers $\mathbb{N}$ and other infinite sets belong to the pillars of set theory. Although it is often claimed that these bijections are non-constructive and somehow happen "in no time", this resembles rather religious belief than mathematics. Fact ist that the well-ordering of the sequence of natural numbers allows us to keep track of and to interrupt the process at every desired step. That proves that Cantor's original idea of *count*ability is correct and that every infinite bijection is a so-called super task the result of which has to be determined by means of the limit of the sequence of finite steps of the process. Without limit the bijection, even when defined by induction or recursion, will be restricted to a vanishing finite initial segment which is followed by nearly the whole infinite set, as can also be proven by induction. Therefore the following is based upon this premise: Limits of sequences of sets are needed to prove the completeness of bijections between infinite sets.

A sequence (S_n) of sets S_n has a limit if and only if

$$\text{LimSup } S_n = \text{LimInf } S_n = \text{Lim } S_n$$

where

$$\text{LimSup } S_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} S_k \quad \text{and} \quad \text{LimInf } S_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} S_k.$$

Fraenkel explained the bijection between the set $\mathbb{N}$ of natural numbers and the set $\mathbb{Q}$ of rational numbers by the story of Tristram Shandy: "Well known is the story of Tristram Shandy who undertakes to write his biography, in fact so pedantically, that the description of each day takes him a full year. Of course he will never get ready if continuing that way. But if he would live infinitely long (for instance a countable infinity of years), then his biography would get 'ready', because every day in his life, how late ever, finally would get its description. No part of his biography would remain unwritten, for to each day of his life a year devoted to that day's description would correspond." [1]

In natural order there are infinitely many rational numbers between two natural numbers. Since the described days of Tristram Shandy correspond to the rational numbers enumerated by natural numbers, Fraenkel reduced this ratio to about 365,25.

In order to further simplify the example let us consider Scrooge McDuck who per day earns 10 \$ and spends 1 \$ [2]. Let the dollar bills be enumerated by the natural numbers. McDuck receives and spends them in natural order. If he lived forever, which is possible for a comic character, he would go bankrupt. If however he would spend always the dollars received last then he would become infinitely rich. This is a very strange result, already excluding set theory from any scientific application because in science changing the label never must change the result.

Now we consider the sequence of singletons of natural numbers

$$\{2^0\}, \{2^1\}, \{2^2\}, \{2^3\}, ... \to \{\}$$

which has the empty set as its limit, i.e., the cardinal number of the limit is CardLim $S_n = 0$. In unary representation

$$\{|\}, \{||\}, \{||||\}, \{||||||||\}, ... \to \{\}$$

this sequence should have an empty limit too, although the continuously doubling strokes, like slipper animalcule (paramecium), cannot know that they are interpreted as natural numbers and eventually will have to disappear. This is not only sufficient to exclude set limits based on actual infinity from any reasonable science but it contradicts even set theory itself because the strokes can be assumed as ordered and indexed by initial segments of natural numbers. These initial segments don't disappear even in the set theoretic limit (their limit is $\mathbb{N}$) but exhibit in the limit a strange, unearthly picture:

$$\{|_1\}, \{|_1|_2\}, \{|_1|_2|_3|_4\}, \{|_1|_2|_3|_4|_5|_6|_7|_8\}, ... \to \{\ _1\ _2\ _3\ _4\ _5\ _6\ _7\ _8\ _{...}\}.$$

This result contradicts itself because indices cannot remain when the indexed elements have disappeared.

What remains is potential infinity. Instead of CardLim $S_n = 0$ we have to apply the (improper) analytical limit of the cardinalities of the sets, namely LimCard$S_n = \infty$, conveying an increase without bound and without end. The enumeration of the rational numbers is never completed and the set $\mathbb{Q}$ of all rational numbers cannot be proved equivalent or equinumerous to $\mathbb{N}$.

References

[1] A. Fraenkel, Einleitung in die Mengenlehre, Springer, Berlin (1928) 24
[2] W. Mückenheim, Transfinity - A Source Book, (2017) 251
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