The Numbers: $k1, k2, \pi$

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ABSTRACT

This note presents the numbers $k1$ and $k2$.

1. Function $f1(x) = x^2 + e^{-4x} - 1$

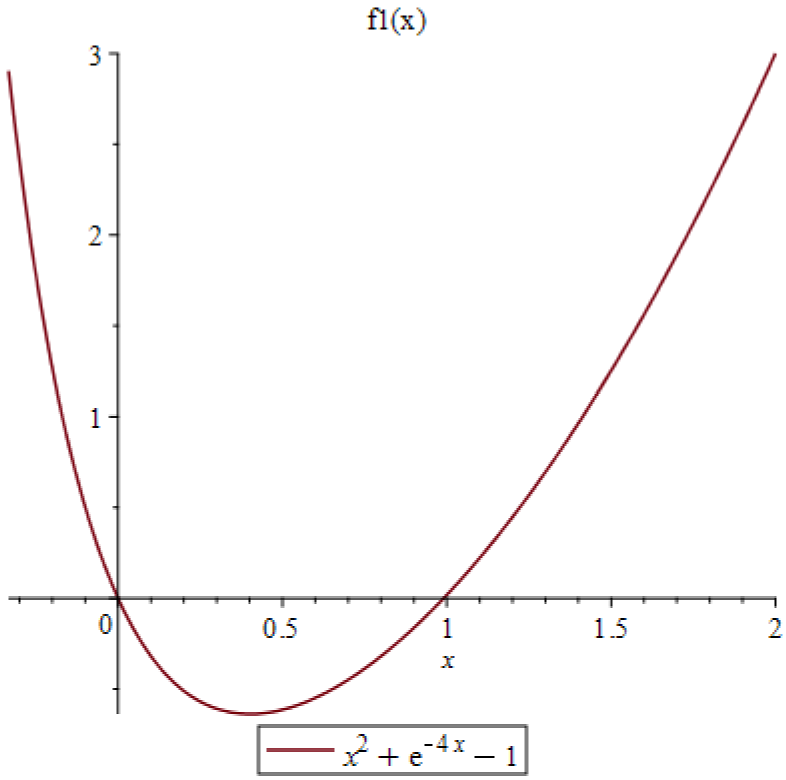

Figure 1.

2. Function $f2(x) = 1 + x - e^{2x}$

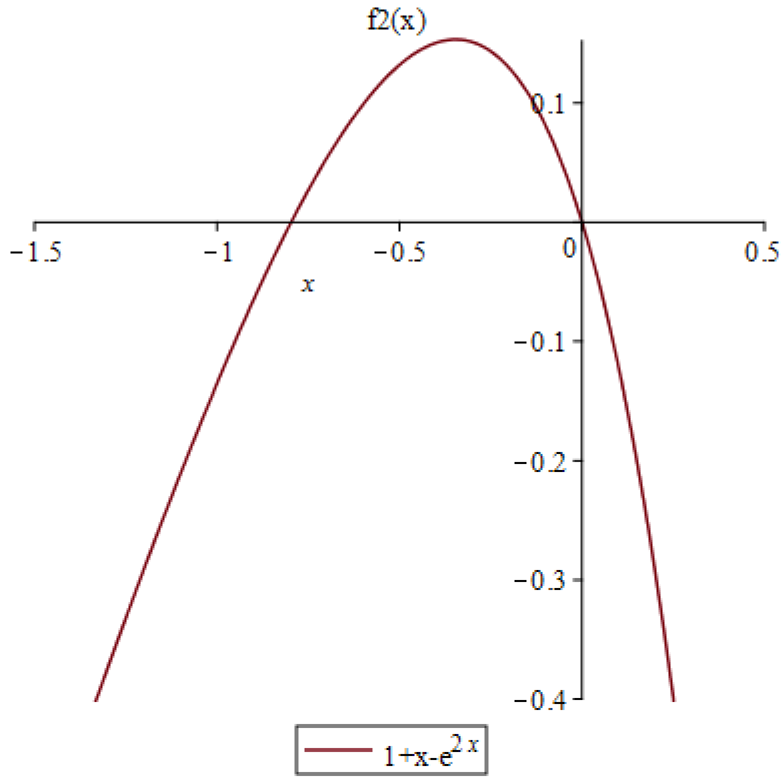

Figure 2.

3. The Number $k1$

- The sequence:

$$k1_1 = \sqrt{1-e^{-4}} \tag{1}$$

$$k1_2 = \sqrt{1-e^{-4\sqrt{1-e^{-4}}}} \tag{2}$$

$$k1_3 = \sqrt{1-e^{-4\sqrt{1-e^{-4\sqrt{1-e^{-4}}}}}} \tag{3}$$

$$k1_{n+1} = \sqrt{1-e^{-4k1_n}} \quad ,n \in \mathbb{N} \tag{4}$$

- Notation:

$$k1 = \lim_{n\to\infty} k1_n := \sqrt{1-e^{-4\sqrt{1-e^{-4\sqrt{1-e^{-4\ldots}}}}}} \tag{5}$$

$$k1 = 0.99043948218276239951\ldots \tag{6}$$

- The equation:

$$f1(k1) = 0 \tag{7}$$

4. The Number $k2$

- The sequence:

$$k2_1 = -1 + e^{-2} \tag{8}$$

$$k2_2 = -1 + e^{-2+2e^{-2}} \tag{9}$$

$$k2_3 = -1 + e^{-2+2e^{-2+2e^{-2}}} \tag{10}$$

$$k2_{n+1} = -1 + e^{2k2_n} \quad ,n \in \mathbb{N} \tag{11}$$

- Notation:

$$k2 = \lim_{n\to\infty} k2_n := -1 + e^{-2+2e^{-2+2e^{-2\ldots}}} \tag{12}$$

$$k2 = -0.79681213002002004616\ldots \tag{13}$$

- The equation:

$$f2(k2)=0 \tag{14}$$

5. Integrals for pi

$$\pi=\int_0^1 \frac{x\arcsin(k1x)}{\sqrt{(1-x^2)(1-k1^2x^2)}}dx \tag{15}$$

$$\pi=\int_0^1 \frac{x\arccos(k2x)}{\sqrt{(1-x^2)(1-k2^2x^2)}}dx \tag{16}$$

$$\pi\left(\frac{1}{4k1}\ln\left(\frac{1+k1}{1-k1}\right)-1\right)=\int_0^1 \frac{x\arccos(k1x)}{\sqrt{(1-x^2)(1-k1^2x^2)}}dx \tag{17}$$

$$\pi\left(1+\frac{1}{4k2}\ln\left(\frac{1-k2}{1+k2}\right)\right)=\int_0^1 \frac{x\arcsin(-k2x)}{\sqrt{(1-x^2)(1-k2^2x^2)}}dx \tag{18}$$

$$\pi\left(\frac{1}{2k2}\ln\left(\frac{1+k2}{1-k2}\right)-1\right)=\int_0^1 \frac{x\arccos(-k2x)}{\sqrt{(1-x^2)(1-k2^2x^2)}}dx \tag{19}$$

$$\pi=\frac{1}{k1}\int_0^{\arcsin k1} \frac{x\sin x}{\sqrt{k1^2-(\sin x)^2}}dx \tag{20}$$

$$\pi=\frac{1}{k2}\int_{\pi/2}^{\arccos k2} \frac{x\cos x}{\sqrt{k2^2-(\cos x)^2}}dx \tag{21}$$

$$\arccos k2=\pi-\arccos(-k2)=\frac{\pi}{2}+\arcsin(-k2)$$

$$\pi\left(\frac{1}{4k2}\ln\left(\frac{1+k2}{1-k2}\right)-1\right)=\frac{1}{k2}\int_0^{\arcsin(-k2)} \frac{x\sin x}{\sqrt{k2^2-(\sin x)^2}}dx \tag{22}$$

$$\pi=\int_0^\infty \frac{x}{(1+x^2)\sqrt{1+(1-k1^2)x^2}}\arcsin\left(\frac{k1x}{\sqrt{1+x^2}}\right)dx \tag{23}$$

$$\pi=\int_0^\infty \frac{x}{(1+x^2)\sqrt{1+(1-k2^2)x^2}}\arccos\left(\frac{k2x}{\sqrt{1+x^2}}\right)dx \tag{24}$$

$$\pi = \frac{1}{k1}\int_0^{k1/\sqrt{1-k1^2}} \frac{x}{\left(1+x^2\right)\sqrt{k1^2-\left(1-k1^2\right)x^2}} \arcsin\left(\frac{x}{\sqrt{1+x^2}}\right)dx \tag{25}$$

$$\pi = -\frac{1}{k2}\int_0^{k2/\sqrt{1-k2^2}} \frac{x}{\left(1+x^2\right)\sqrt{k2^2-\left(1-k2^2\right)x^2}} \arccos\left(\frac{x}{\sqrt{1+x^2}}\right)dx \tag{26}$$

$$\pi = \frac{k1}{4}\int_0^1\int_0^1 \sqrt{\frac{y}{\left(1-y\right)\left(1-k1^2 y\right)\left(1-k1^2 xy\right)x}}dxdy \tag{27}$$

6. Recurrences for $k1$

$$x_{n+1} = \frac{1+x_n^2-\left(1+4x_n\right)e^{-4x_n}}{2x_n-4e^{-4x_n}} \quad , x_1 = 1 \Rightarrow \lim_{n\to\infty} x_n = k1 \tag{28}$$

$$x_{n+1} = 1-\frac{e^{-4x_n}}{1+x_n} \quad , x_1 = 1 \Rightarrow \lim_{n\to\infty} x_n = k1 \tag{29}$$

$$x_{n+1} = \sqrt{2}e^{-x_n}\sqrt{\sinh\left(2x_n\right)} \quad , x_1 = 1 \Rightarrow \lim_{n\to\infty} x_n = k1 \tag{30}$$

$$x_{n+1} = \frac{1+2x_n-x_n^2-e^{-4x_n}}{2} \quad , x_1 = 1 \Rightarrow \lim_{n\to\infty} x_n = k1 \tag{31}$$

$$x_{n+1} = \frac{6x_n-\ln\left(1+x_n^2 e^{4x_n}\right)}{2} \quad , x_1 = 1 \Rightarrow \lim_{n\to\infty} x_n = k1 \tag{32}$$

7. Recurrences for $k2$

$$x_{n+1} = \frac{\left(1-2x_n\right)e^{2x_n}-1}{1-2e^{2x_n}} \quad , x_1 = -4/5 \Rightarrow \lim_{n\to\infty} x_n = k2 \tag{33}$$

$$x_{n+1} = -2-x_n+2e^{2x_n} \quad , x_1 = -4/5 \Rightarrow \lim_{n\to\infty} x_n = k2 \tag{34}$$

$$x_{n+1} = 2e^{x_n}\sinh x_n \quad , x_1 = -4/5 \Rightarrow \lim_{n\to\infty} x_n = k2 \tag{35}$$

$$x_{n+1} = \frac{7}{4}x_n - \frac{3}{8}\ln(1+x_n) \quad , x_1 = -4/5 \Rightarrow \lim_{n\to\infty} x_n = k2 \tag{36}$$

8. Formula for $k1$

$$k1 = -\frac{1}{4} + \frac{1}{4}\sqrt{17 + \frac{4\pi}{I}} \tag{37}$$

$$I = \int_0^{2\pi} \frac{e^{xi}}{4e^{xi} + e^{2xi} + 4e^{-4-2e^{xi}}} dx \tag{38}$$

9. Formula for $k2$

$$k2 = -1 - \frac{1}{2} LambertW\left(-2e^{-2}\right) \tag{39}$$

$LambertW(x)$: Lambert Function.

10. The number kr

$$kr = \frac{1}{3}\left(19 + 3\sqrt{33}\right)^{1/3} + \frac{4}{3\left(19 + 3\sqrt{33}\right)^{1/3}} - \frac{2}{3} \tag{40}$$

$$\pi \frac{\ln(1+kr)}{2kr} = \int_0^1 \frac{x \arcsin(kr\, x)}{\sqrt{(1-x^2)(1-kr^2x^2)}} dx = \int_0^1 \frac{x \arccos(kr\, x)}{\sqrt{(1-x^2)(1-kr^2x^2)}} dx \tag{41}$$

11. Fractal for $f1(x)$

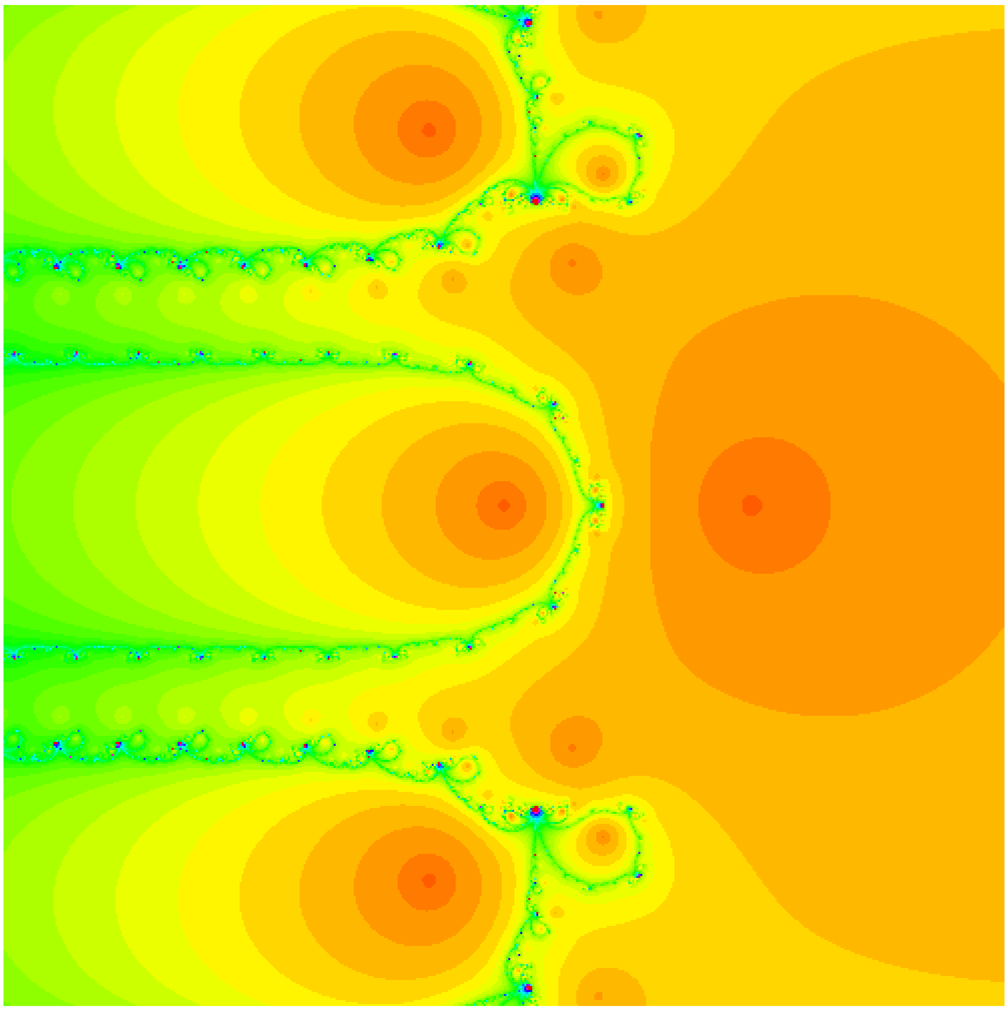

Figure 3. Newton-Julia set for: $f1(z) = z^2 + e^{-4z} - 1$

12. Fractal for $f2(x)$

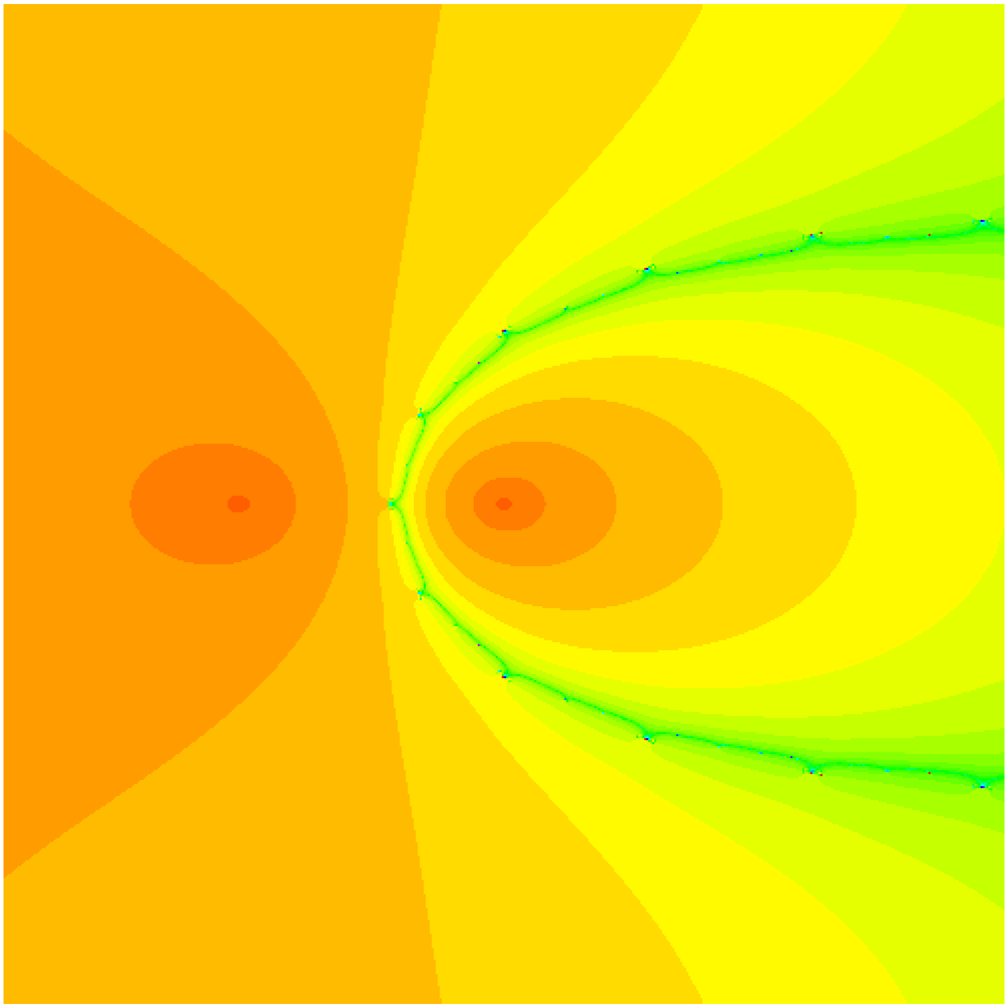

Figure 4. Newton-Julia set for: $f2(z) = 1 + z - e^{2z}$

References

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