The Kochen-Specker theorem with two results of finite-precision measurements

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We review non-classicality of quantum datum. We consider whether we can assign the predetermined "hidden" result to numbers 1 and -1 as in results of measurements in a thought experiment. We assume the number of measurements is two. If we detect $|\uparrow\rangle$ as 1 and detect $|\downarrow\rangle$ as -1, then we can derive the Kochen-Speker theorem. The same situation occurs when we use a finite-precision measurement theory that the results of measurements are either $1-\epsilon$ or $-1+\epsilon$.

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I. INTRODUCTION

The quantum theory (cf. [1–5]) is indeed successful physical theory. From the incompleteness argument of Einstein, Podolsky, and Rosen (EPR) [6], a hidden-variable interpretation of the quantum theory has been as an attractive topic of research [2, 3]. The no-hidden-variables theorem of Kochen and Specker (KS theorem) [7] is very famous. In general, the quantum theory does not accept the KS type of hidden-variable theory. Greenberger, Horne, and Zeilinger discover [8, 9] the so-called GHZ theorem for four-partite GHZ state. And, the KS theorem becomes very simple form (see also Refs. [10–14]). For the KS theorem, it is begun to research the validity of the KS theorem by using inequalities (see Refs. [15–18]). To find such inequalities to test the validity of the KS theorem is particularly useful for experimental investigation [19].

Many researches address non-classicality of observables. And non-classicality of quantum state itself is not investigated at all (however see [20]). Further, non-classicality of quantum datum is not investigated very well. Does finite-precision measurement nullify the Kochen-Specker theorem? Meyer discusses that finite precision measurement nullifies the Kochen-Specker theorem [21]. Cabello discusses that finite-precision measurement does not nullify the Kochen-Specker theorem [17]. We address the problem.

Here we ask: Can we assign definite value into each quantum datum? We cannot assign definite value into each quantum datum. This gives the very simple reason why Kochen-Specker inequalities are violated in real experiments. Further, our discussion says that we cannot assign definite value to each quantum datum even though the number of measurements is two. This gives the Kochen-Specker theorem in two trials of measurements. These argumentations would provide supporting evidence of the statement by Cabello.

In this paper, we review non-classicality of quantum datum. We consider whether we can assign the predetermined "hidden" result to numbers 1 and -1 as in results of measurements in a thought experiment. We assume the number of measurements is two. If we detect $|\uparrow\rangle$ as 1 and detect $|\downarrow\rangle$ as -1, then we can derive the Kochen-Speker theorem. The same situation occurs when we use a finite-precision measurement theory that the results of measurements are either $1-\epsilon$ or $-1+\epsilon$.

II. THE KS THEOREM WITH PRECISION MEASUREMENTS

We consider a value V which is the sum of data in some experiments. The measured results of trials are either 1 or -1. We assume the number of -1 is equal to the number of 1. The number of trials is 2. Then we have

$$V = -1 + 1 = 0. \tag{1}$$

First, we assign definite value into each experimental datum. In the case, we consider the Kochen-Specker realism. By using r_1, r_2, $r_{1'}$ and $r_{2'}$, we can define experimental data as follows $r_1 = 1$, $r_2 = -1$, $r_{1'} = 1$ and $r_{2'} = -1$. Let us write V as follows

$$V = (\sum_{l=1}^{2} r_l). \tag{2}$$

The possible values of the measured results r_l are either 1 or -1. The same value is given by

$$V = (\sum_{l'=1}^{2} r_{l'}). \tag{3}$$

We change the label as $l \to l'$. The possible values of the measured results $r_{l'}$ are either 1 or -1.

In the following, we evaluate a value $(V \times V)$ and derive a necessary condition under an assumption that we assign definite value into each experimental datum.

We introduce an assumption that Sum rule and Prodct rule commute [22]. We have

$$\begin{aligned}
&V \times V \\
&= (\sum_{l=1}^{2} r_l) \times (\sum_{l'=1}^{2} r_{l'}) \\
&= \sum_{l=1}^{2} \cdot \sum_{l'=1}^{2} r_l r_{l'} \\
&\leq \sum_{l=1}^{2} \cdot \sum_{l'=1}^{2} |r_l r_{l'}| \\
&= \sum_{l=1}^{2} \cdot \sum_{l'=1}^{2} (r_l)^2 \\
&= 2((1)^2 + (-1)^2) \\
&= 4.
\end{aligned} \tag{4}$$

The inequality (4) can be saturated because the following case is possible

$$\begin{aligned}
&\|\{l|r_l = 1\}\| = \|\{l'|r_{l'} = 1\}\| \\
&\|\{l|r_l = -1\}\| = \|\{l'|r_{l'} = -1\}\|.
\end{aligned} \tag{5}$$

Thus,

$$(V \times V)_{\max} = 4. \tag{6}$$

Therefore we have the following assumption concerning the Kochen-Specker realism

$$(V \times V)_{\max} = 4. \tag{7}$$

Next, we derive another possible value of the product $V \times V$ of the value V under an assumption that we do not assign definite value into each experimental datum. This is quantum mechanical case.

In this case, we have

$$V \times V = 0. \tag{8}$$

We have the following assumption concerning quantum mechanics

$$(V \times V)_{\max} = 0. \tag{9}$$

We cannot assign the truth value "1" for the two assumptions (7) and (9), simultaneously. We derive the KS paradox. Thus we cannot assign definite value into each experimental datum. The number of data is two.

III. THE KS THEOREM WITH FINITE-PRECISION MEASUREMENTS

Next, we consider a value V which is the sum of data in some experiments. The measured results of trials are either $1-\epsilon$ or $-1+\epsilon$. We assume the number of $-1+\epsilon$ is equal to the number of $1-\epsilon$. The number of trials is 2. Then we have

$$V = -1 + \epsilon + 1 - \epsilon = 0. \tag{10}$$

First, we assign definite value into each experimental datum. In the case, we consider the Kochen-Specker realism. By using r_1, r_2, $r_{1'}$ and $r_{2'}$, we can define experimental data as follows $r_1 = 1-\epsilon$, $r_2 = -1+\epsilon$, $r_{1'} = 1-\epsilon$ and $r_{2'} = -1+\epsilon$. Let us write V as follows

$$V = (\sum_{l=1}^{2} r_l). \tag{11}$$

The possible values of the measured results r_l are either $1-\epsilon$ or $-1+\epsilon$. The same value is given by

$$V = (\sum_{l'=1}^{2} r_{l'}). \tag{12}$$

We change the label as $l \to l'$. The possible values of the measured results $r_{l'}$ are either $1-\epsilon$ or $-1+\epsilon$.

In the following, we evaluate a value $(V \times V)$ and derive a necessary condition under an assumption that we assign definite value into each experimental datum.

We introduce an assumption that Sum rule and Prodct rule commute [22]. We have

$$\begin{aligned}
&V \times V \\
&= (\sum_{l=1}^{2} r_l) \times (\sum_{l'=1}^{2} r_{l'}) \\
&= \sum_{l=1}^{2} \cdot \sum_{l'=1}^{2} r_l r_{l'} \\
&\leq \sum_{l=1}^{2} \cdot \sum_{l'=1}^{2} |r_l r_{l'}| \\
&= \sum_{l=1}^{2} \cdot \sum_{l'=1}^{2} (r_l)^2 \\
&= 2((1-\epsilon)^2 + (-1+\epsilon)^2) \\
&= 4(1-\epsilon)^2.
\end{aligned} \tag{13}$$

The inequality (13) can be saturated because the following case is possible

$$\begin{aligned}
&\|\{l|r_l = 1-\epsilon\}\| = \|\{l'|r_{l'} = 1-\epsilon\}\| \\
&\|\{l|r_l = -1+\epsilon\}\| = \|\{l'|r_{l'} = -1+\epsilon\}\|.
\end{aligned} \tag{14}$$

Thus,

$$(V \times V)_{\max} = 4(1-\epsilon)^2. \tag{15}$$

Therefore we have the following assumption concerning the Kochen-Specker realism

$$(V \times V)_{\max} = 4(1-\epsilon)^2. \tag{16}$$

Next, we derive another possible value of the product $V \times V$ of the value V under an assumption that we do not assign definite value into each experimental datum. This is quantum mechanical case.

In this case, we have

$$V \times V = 0. \tag{17}$$

We have the following assumption concerning quantum mechanics

$$(V \times V)_{\max} = 0. \quad (18)$$

We cannot assign the truth value "1" for the two assumptions (16) and (18), simultaneously. We derive the KS paradox. Thus we cannot assign definite value into each experimental datum. The number of data is two.

IV. CONCLUSIONS

In conclusions, non-classicality of quantum datum has been investigated. We have considered whether we can assign the predetermined "hidden" result to natural numbers 1 and -1 as in results of measurement in a thought experiment. The number of trials has been twice. If we detect $|\uparrow\rangle$ as 1 and detect $|\downarrow\rangle$ as -1, then we can have derived the Kochen-Speker theorem. The same situation has occurred when we use a finite-precision measurement theory that the results of measurements are either $1-\epsilon$ or $-1+\epsilon$.

Generally Multiplication is completed by Addition. Therefore, we think that Addition of the starting point may be superior to any other case.

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