The infinitesimal error

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Abstract

Unfortunately, Cantor was wrong. His notion of transfinite bijection is flawed. Cantor introduced this notion of transfinite bijection as the additional axiom, even though without even realising this. From this error, other errors sprung into the existence. He did all this in the heroic effort to justify the death of infinitesimals, even though he wasn't aware of this either. Cantor went bravely on to defend the established error in higher mathematics before his mentors and peers who banished infinitesimals. Instead, he demonstrated the error of it. He never realised this as well. This paper elucidates this link between Cantor's errors and infinitesimals.

1 Introduction

Our ancestors introduced infinitesimals, thus creating higher mathematics. What is an infinitesimal? Infinitesimals are infinitely small numbers, still different from zero. They are simply beyond any measuring. One cannot possibly measure their magnitude. However, infinitesimals are not identically zero.

Then came other mathematicians who abandoned infinitesimals in favour of the "for every epsilon there is delta" prescription. But our ancestors who invented infinitesimals had to die first, because they would not agree with this prescription. Unfortunately, our ancestors were not around to argue with it any more. This paper argues in favour of infinitesimals on our ancestors behalf, by simply demonstrating explicitly the errors in subsequent development of higher mathematics, and also by constructing infinitesimals explicitly!

The mathematician who laid logical foundations for the exclusion of infinitesimals was Cantor, even though he was probably not aware of this fact. He invented his beautiful set theory.

The basic notion in his theory, as in any other set theory, is the notion of bijection. It's hardly necessary to define what a bijection is. What Cantor did, was to define bijection between infinitely large sets. In particular, if one can count members of an infinite set, then there is a bijection between that set and $\mathbb{N}$. The set is said to be countable. This is demonstrated in this paper to be the fundamental error of Cantor's set theory.

Two very popular and lovely proofs from his theory are the diagonal argument and the book argument. We describe these arguments here in this Introduction for the same of completeness, because we shall argue about them later on in this paper, and also because they truly are beautiful. Later on in this paper we shall demonstrate they are also faulty as well, beside being beautiful.

The diagonal argument runs as follows. The aim is to demonstrate that there are more real numbers than natural numbers. We list real numbers in a list with each row containing only one real number from the segment $(0, 1)$ in a decimal form in base 10. The list may look like this:

$$\begin{array}{c} 0.95016081... \\ 0.00259103... \\ 0.50268716... \\ 0.10850692... \\ \vdots \end{array}$$

Notice the diagonal running through numbers 9, 0, 2, 5 and so on in this list, starting with the first decimal of the first digit, then continuing through the second decimal of the second digit, and so on, through nth decimal of the nth digit.

Create the new number now, such that its first decimal is different than the first decimal of the first number, its second decimal is different than the second decimal of the second number, and so on, its nth decimal is different than the nth decimal of the nth number. In other words, use numbers on a diagonal, and change them all into something else, in order to form new decimal number.

In our example, one may construct the number 0.8136... this way.

The argument goes on to conclude that, since the new number created this way, by changing the numbers on the diagonal, is not equal to any of the listed numbers, is not listed. This way, one finds that reals cannot be listed. In other words, one cannot count real numbers. In conclusion, there are more reals than naturals.

The book argument goes as follows. Imagine we have a book with infinitely many infinitely large pages. There's a different subset of $\mathbb{N}$ written down on each different page.

Let us create our own subset now. Make number 1 member of this new subset of ours if it's not listed on page 1. Then, include number 2 in our new subset if number 2 is not written on page 2. And so on. Include number n only if it doesn't appear on nth page. The subset created this way is different than any subset from the book. Therefore, one cannot list all the subsets of $\mathbb{N}$. In other words, there is no bijection between $\mathbb{N}$ and its power set $\mathcal{P}\{\mathbb{N}\}$.

This paper demonstrates that these proofs are faulty and not correct.

2 Discrete reals

Consider the interval $[0, 1]$ of real numbers on a number line. There are c real numbers on this interval.

If we were to calculate the minimal distance $d_{\mathbb{R}}$ between any pair of these c numbers, we would simply divide the length $l_{[0,1]}$ of the entire interval $[0, 1]$ on the number line, which is $l_{[0,1]} = 1$, by the number $n_{[0,1]}$ of reals on this interval, which is $n_{[0,1]} = c$. This way, we would find that the minimal distance between reals is $d_{\mathbb{R}} = l_{[0,1]}/n_{[0,1]} = 1/c = 0$. In other words, there is no minimal distance: the set of reals $\mathbb{R}$ is dense, one can always find another real number between any other two distinct real numbers.

If we apply this procedure to find the minimal distance $d_{\mathbb{N}}$ between any two natural numbers, we'd find $d_{\mathbb{N}} = l_{\mathbb{N}}/n_{\mathbb{N}} = \aleph_0/\aleph_0 = 1$. This is because the length of $\mathbb{N}$ on the number line is $\aleph_0$, and this is also the number of naturals on it.

So, one can calculate the minimal distance in a fairly obvious and simple way.

Let's magnify the interval $[0, 1]$ now. Multiply all reals on $[0, 1]$ by c. This way, the origin 0 remains where it was, and the number 1 on the number line is located at the distance c from the origin now after the magnification by factor c. All the reals from the interval $[0, 1]$ are on the interval $[0, c]$ now after the stretching by the factor c.

What is the minimal distance $d_{[0,c]}$ between two reals now on stretched $[0, 1]$? The interval $[0, 1]$ is stretched and has become the interval $[0, c]$. The length $l_{[0,1]}$ is also stretched by factor c. The length $l_{[0,c]}$ of the interval $[0, c]$ is obviously $l_{[0,c]} = c$. The number $n_{[0,c]}$ of reals on interval $[0, c]$ is obviously $n_{[0,c]} = c$ still. Thus, the minimal distance is $d_{[0,c]} = l_{[0,c]}/n_{[0,c]} = c/c = 1$!

So, there is the minimal distance between two reals, once one magnifies $\mathbb{R}$ enough. In our example, we magnified $\mathbb{R}$ by factor c, and the minimal distance became $d_{[0,c]} = 1$.

If we now shrink the stretched interval $[0, c]$ by factor $1/c$, all the distances d become d/c. The interval $[0, c]$ becomes the unit interval $[0, 1]$ again. The minimal distance $d_{[0,c]} = 1$ becomes shrinked to $d_{[0,1]} = 1/c$.

What is $1/c$? What is this minimal distance? Can it be equal to zero?

If it were equal to zero identically, then after multiplying it by any factor, it would remain zero. So, $1/c$ cannot be zero identically. This is obviously the number that is not zero, but is beyond any measuring. This is an infinitesimal.

3 Discussion about discreteness of reals

The only objection to the arguments of the section 2 that comes to mind is that the number of reals on the unit interval changes in some way during magnification. However, the original number of reals c when multiplied by any other number, ordinal or cardinal, remains c. So, even $c^2 = c$. So, the argument of section 2 seems simple and obvious.

4 Cantor bijection

The existence of infinitesimals is in a sharp contrast with the results of the set theory. In the contemporary set theory, $1/c = 0$ identically. This is also true with $1/\aleph_0 = 0$. So the question that arises is – where is the error?

The error lies with the excommunication of infinitesimals. At some point in the history of mathematics infinitesimals were replaced by limits. For instance, $\lim_{x\to 0} x = 0$. But what exactly is the meaning of this limit?

Any limit denotes some limiting process. In our example, $\lim_{x\to 0} x = 0$, variable x is allowed to take values ever closer to 0. What values exactly? Well... this is not specified! This very important detail is left vague. The use of limits came about with the study of sequences. So, for instance, if variable x takes values $1/n$ as naturals n grow ever larger and larger, then obviously x tends to zero.

But does such x ever reach zero? No, it does not. It can not. If it did reach zero, then in the definition of a derivative

$$f'(x) = \frac{\Delta f(x)}{\Delta x} = \lim_{\epsilon\to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}$$

one would end up with dividing two zeros, that are both zero identically. That's exactly why our ancestors who invented higher mathematics used infinitesimals. Infinitesimals are not identically zero. They are exactly what $\lim_{x\to 0} x$ should mean. It should mean $\lim_{x\to 0} x = \epsilon$. Not $\lim_{x\to 0} x = 0$.

So what happens when one introduces the faulty definition $\lim_{x\to 0} x = 0$ and then treats it as if it were true? Let us return back to our example with variable x assuming values $1/n$ as naturals n grow ever larger. The quantity $1/n$ never reaches zero, because there is no fixed point at infinity. We can always add 1 to anything we might call infinity, and thus n would grow still. The result would change. But the result would change beyond any measuring. Thus, the result would be an infinitesimal. This is because there is no fixed infinity. One can always change any fixed infinity by adding 1 to it, or whatever other quantity different than zero.

So, if one introduces the definition $\lim_{x\to 0} x = 0$, one has, without even knowing, fixed the value of infinity! Fixed value of infinity will produce a fixed unchanging result in $1/n$ as n reaches infinity that is fixed. Instead of ever changing $\lim_{x\to 0} x = \epsilon$, one has $\lim_{x\to 0} x = 0$ now with a fixed infinity.

This is exactly what Cantor had to establish. There was no other way. Because Cantor believed in sequences and he believed in $\lim_{x\to 0} x = 0$. Cantor simply had no choice but to discover fixed infinities. Infinities such as $\aleph_0 + 1 = \aleph_0$, that don't change no matter the quantity one enlarges them by.

So what other errors stem from this error then?

One way to look at this is to start from the logical beginning of the set theory. The very fundamental notion is the bijection. If one can map elements from set S_1 to elements of set S_2 in a 1-on-1 fashion, that mapping is a bijection then. There's nothing wrong with this definition of bijection, of course. The error comes from the way Cantor generalized bijection to infinitely large sets.

What did Cantor do? For instance, Cantor said there's a bijection between the set of all natural numbers $\mathbb{N}$ and the set of all even numbers $\mathbb{E}$. From this, he concluded that there are as many even numbers as there are natural numbers.

But this conclusion is against the very obvious and simple every day experience. There are twice as many natural numbers than there are even numbers. Just consider the set $\{1, 2, ..., 2n\}$ of the first $2n$ natural numbers. How many even numbers are there? Obviously only n even numbers in this set, whilst there are altogether $2n$ elements. The ratio of the number of naturals and the number of evens is $2n/n = 2$.

Now let n grow without bounds. What result do we get for the ratio of naturals and evens? The result remains the same regardless the value of n. The result remain 2. There are twice as many naturals than evens. No matter how large n may be.

But Cantor concluded otherwise. This is probably logically the most fundamental error is Cantor's set theory. Cantor added the extra axiom, saying – there is a bijection between infinitely large sets, even though the bijection need not exist in the finite case. When does this miracle of bijection occur? Whenever one needs to demonstrate that $\aleph_0$ is unique and fixed. So, circularly, Cantor then from the faulty bijection axiom, the axiom he never explicitly admitted to, deduces the uniqueness of the fixed $\aleph_0$. Not realising he was actually led subconsciously to this result because he firmly, firmly believed in $\lim_{x\to 0} x = 0$.

5 Lovely but faulty proofs

We shall demonstrate here now that Cantor's common sense proofs about his theories are not common sense, unfortunately.

To do so, we begin with the diagonal argument. We analyse the finite case here, very simple one too. Consider two digit numbers in binary base. So, we only consider numbers that have two digits only, and digits may assume values from $\{0, 1\}$ only. It's easy to list these numbers:

$$\begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array}$$

Do notice that this list is not in the form of a square. There are four rows, but only two columns.

Now we pay attention to the diagonal running from the left digit of the first row to the right digit of the second row. We pick the first digit to be different than the first digit of the number 00, thus 1, and we pick the second digit to be different from the second digit of the number 01, thus 0. So, the two digits we just picked are 10. The diagonal argument states that the number 10 isn't in the list of two digit binary numbers. But clearly, clearly, our list contains the number 10 – it's the third row of the list.

So, the diagonal argument doesn't make common sense.

We now pay attention to the book argument. Imagine we have a book that contains all the subsets of the set $\{0, 1\}$ written in it. This book has only four pages, obviously. They read:

$$\{\}$$
$$\{0\}$$
$$\{1\}$$
$$\{0, 1\}$$

The book arguement tells us to pick number 0 if it's not listed on page 1, and to pick number 1 if it's not listed on page 2. We do so, and we form the set $\{0, 1\}$. The book argument says the set $\{0, 1\}$ is not listed in our book. But it clearly is listed – on the forth page.

So, the book argument doesn't make common sense either.

One may object that our lists are not square list, because they are incomplete. Our lists do not contain all the possible numbers. If we included all the possible numbers, the lists may become square lists. Then, our argument would fail, and Cantor's arguments would stand true.

But this is refuted easily. Consider the book example. If we allow for n elements, the number of subsets is 2^n. The departure of the list from the square shape can be measured by the ratio $r_n = p_n/n_n$ of the number of pages p_n and the number of elements n_n. For instance, with two elements only in our initial example, the ratio is $r_2 = p_2/n_2 = 4/2 = 2$. There are twice as many pages than elements. For n elements, the ratio is $r_n = p_n/n_n = 2^n/n$. This obviously grows without bounds as n grows large. This ratio is definitely not closing on to number 1 required for the list to be a square one.

6 Conclusion

Unfortunately, Cantor was wrong. His notion of transfinite bijection is flawed. Cantor introduced this notion of transfinite bijection as the additional axiom, even though without even realising this. From this error, other errors sprung into the existence. He did all this in the heroic effort to justify the death of infinitesimals, even though he wasn't aware of this either. Cantor went bravely on to defend the established error in higher mathematics before his mentors and peers who banished infinitesimals. Instead, he demonstrated the error of it. He never realised this as well. This paper elucidates this link between Cantor's errors and infinitesimals.