Closed-Form Solution for the Nontrivial Zeros of the Riemann Zeta Function

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Recently it was conjectured that if the Bender-Brody-Müller (BBM) Hamiltonian can be shown to be self-adjoint, then the Riemann hypothesis holds true. Herein we discuss the domain and eigenvalues of the Bender-Brody-Müller conjecture.

I. INTRODUCTION

It was recently shown in [1] that the eigenvalues of a Bender-Brody-Müller (BBM) Hamiltonian operator correspond to the nontrivial zeroes of the Riemann zeta function [2]. Although the BBM Hamiltonian is pseudo-Hermitian, it is consistent with the Berry-Keating conjecture [3, 4]. The eigenvalues of the BBM Hamiltonian are conjectured to be the imaginary parts of the nontrivial zeroes of the zeta function

$$\begin{aligned}\zeta(z) &= \sum_{k=1}^{\infty} \frac{1}{k^z} \\ &= \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{\exp(t)-1} dt.\end{aligned} \tag{1}$$

The idea that the imaginary parts of the zeroes of Eq. (1) are given by a self-adjoint operator was conjectured by Hilbert and Pólya [5]. Formally, Hilbert and Pólya conjectured that if the eigenfunctions of a self-adjoint operator satisfy the boundary conditions $\psi_n(0) = 0 \ \forall \ n$, then the eigenvalues are the nontrivial zeroes of Eq. (1). The BBM Hamiltonian also satisfies the Berry-Keating conjecture, which states that when $\hat{x}$ and $\hat{p}$ commute, the Hamiltonian reduces to the classical $H = 2xp$.

Remark. *If there are nontrivial roots of Eq. (1) for which $\Re(z) \neq 1/2$, the corresponding eigenvalues and eigenstates are degenerate [1].*

II. STATEMENT OF PROBLEM

A. Bender-Brody-Müller Hamiltonian

Theorem 1. *The eigenvalues of the Hamiltonian*

$$\hat{H} = \frac{1}{1-e^{-i\hat{p}}}(\hat{x}\hat{p} + \hat{p}\hat{x})(1-e^{-i\hat{p}}) \tag{2}$$

are real, where $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, and $\hat{x} = x$.

Corollary 1.1. *[1] Solutions to the equation $\hat{H}\psi = E\psi$ are given by the Hurwitz zeta function*

$$\begin{aligned}\psi_z(x) &= -\zeta(z, x+1) \\ &= -\sum_{n=0}^{\infty} \frac{1}{(x+1+n)^z}\end{aligned} \tag{3}$$

on the positive half line $x \in \mathbb{R}^+$ with eigenvalues $i(2z-1)$, and $z \in \mathbb{C}$, for the boundary condition $\psi_z(0) = 0$. Moreover, $\Re(z) > 1$, and $\Re(x+1) > 0$. As $-\psi_z(0)$ is the Riemann zeta function, i.e., Eq. (1), this implies that z belongs to the discrete set of zeros of the Riemann zeta function.

Proof. Let $\psi_z(x)$ be an eigenfunction of Eq. (2) with an eigenvalue $\lambda = i(2z-1)$:

$$\hat{H}\psi_z(x) = \lambda\psi_z(x). \tag{4}$$

Then we have the relation

$$\frac{1}{1-e^{-i\hat{p}}}(\hat{x}\hat{p} + \hat{p}\hat{x})(1-e^{-i\hat{p}})\psi_z(x) = \lambda\psi_z(x). \tag{5}$$

Letting

$$\varphi_z(x) = [1 - \exp(-\partial_x)]\psi_z(x),$$
$$= \hat{\Delta}\psi_z(x), \tag{6}$$

where $\hat{\Delta}\psi_z(x) = \psi_z(x) - \psi_z(x-1)$, and

$$\hat{\Delta} \equiv 1 - \exp(-\partial_x), \tag{7}$$

is a shift operator. Upon inserting Eq. (6) into Eq. (5) with $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, and $\hat{x} = x$, we obtain

$$[-ix\partial_x - i\partial_x x]\varphi_z(x) = \lambda\varphi_z(x). \tag{8}$$

Then we have

$$\int_{\mathbb{R}^+} (x\partial_x\varphi_z(x))^*\varphi_z(x)dx + \int_{\mathbb{R}^+} (\partial_x x\varphi_z(x))^*\varphi_z(x)dx = -i\lambda^* \int_{\mathbb{R}^+} \varphi_z^*(x)\varphi_z(x)dx. \tag{9}$$

Now we integrate the first term on the LHS of Eq. (9) by parts to obtain

$$\int_{\mathbb{R}^+} x\varphi_z(x)\partial_x\varphi_z^*(x)dx = -\int_{\mathbb{R}^+} \varphi_z^*(x)\varphi_z(x)dx - \int_{\mathbb{R}^+} \varphi_z^*(x)x\frac{d}{dx}(\varphi_z(x))dx, \tag{10}$$

and the second term on the LHS of Eq. (9) by parts to obtain

$$\int_{\mathbb{R}^+} x\varphi_z(x)^*\partial_x\varphi_z(x)dx = -\int_{\mathbb{R}^+} \varphi_z(x)\varphi_z^*(x)dx - \int_{\mathbb{R}^+} \varphi_z(x)x\frac{d}{dx}(\varphi_z^*(x))dx. \tag{11}$$

Upon substituting Eqs. (10) and (11) into Eq. (9), we obtain

$$\int_{\mathbb{R}^+} \varphi_z^*(x)x\frac{d}{dx}(\varphi_z(x))dx + \int_{\mathbb{R}^+} \varphi_z(x)x\frac{d}{dx}(\varphi_z^*(x))dx = (i\lambda^* - 2)N, \tag{12}$$

where

$$N = \int_{\mathbb{R}^+} \varphi_z^*(x)\varphi_z(x)dx. \tag{13}$$

Next, we split $\varphi_z(x)$ into real and imaginary components, such that

$$\varphi = \Re(\varphi_z(x)) + i\Im(\varphi_z(x)), \tag{14}$$

and substitute Eq. (14) into Eq. (12) such that

$$\int_{\mathbb{R}^+} \Re(\varphi_z(x))x\frac{d}{dx}\Re(\varphi_z(x))dx + \int_{\mathbb{R}^+} \Im(\varphi_z(x))x\frac{d}{dx}\Im(\varphi_z(x))dx + N = \frac{i\lambda^*}{2}N. \tag{15}$$

Upon setting $\lambda = i(2z-1)$, Eq. (15) can be written

$$\int_{\mathbb{R}^+} \Re(\varphi_z(x))x\frac{d}{dx}\Re(\varphi_z(x))dx + \int_{\mathbb{R}^+} \Im(\varphi_z(x))x\frac{d}{dx}\Im(\varphi_z(x))dx + N = \frac{1-2z}{2}N. \tag{16}$$

It can be seen that all terms on the LHS of Eq. (15) are real, thereby verifying Theorem 1.

Q.E.D.

□

Remark. *If the Riemann hypothesis is correct [2], the the eigenvalues of Eq. (2) are degenerate [1].*

Lemma 1.1. *Under the boundary condition $\psi(0) = 0$, the n^{th} eigenstate of Eq. (2) is Eq. (3), and the nontrivial zeros of the Riemann zeta function are given by*

$$z_n = \frac{1}{N}\int_{\mathbb{R}^+} \Re(\varphi_n(x))x\frac{d}{dx}\Re(\varphi_n(x))dx + \frac{1}{N}\int_{\mathbb{R}^+} \Im(\varphi_n(x))x\frac{d}{dx}\Im(\varphi_n(x))dx + \frac{3}{2}. \tag{17}$$

Proof. Given that

$$\begin{aligned}\psi_n(x) &= \hat{\Delta}\psi_n(x)\\ &= \psi_n(x) - \psi_n(x-1)\\ &= -\sum_{n=0}^{\infty}\frac{1}{(x+1+n)^z} + \sum_{n=0}^{\infty}\frac{1}{(x+n)^z},\end{aligned} \tag{18}$$

the second term on the RHS of Eq. (17) goes to zero, as $\Im(\varphi_n(x)) = 0$. Hence, we are left with

$$z_n = \frac{1}{N}\int_{\mathbb{R}^+}\varphi_n(x)x\frac{d}{dx}\varphi_n(x)dx + \frac{3}{2}. \tag{19}$$

Moreover, it can be seen that

$$\begin{aligned}x\frac{d}{dx}(\varphi_n(x)) &= x\frac{d}{dx}\psi_n(x) - x\frac{d}{dx}\psi_n(x-1)\\ &= -x\frac{d}{dx}\sum_{n=0}^{\infty}\frac{1}{(x+1+n)^z} + x\frac{d}{dx}\sum_{n=0}^{\infty}\frac{1}{(x+n)^z}\\ &= xz\zeta(z+1,x+1) - xz\zeta(z+1,x).\end{aligned} \tag{20}$$

Multiplying Eq. (20) by $\varphi_n(x)$, we obtain

$$\begin{aligned}\varphi_n(x)xz\zeta(z+1,x+1) - \varphi_n(x)xz\zeta(z+1,x) &= \varphi_n(x)[xz\zeta(z+1,x+1) - xz\zeta(z+1,x)]\\ &= -\zeta(z,x+1)xz\zeta(z+1,x+1)\\ &+ \zeta(z,x+1)xz\zeta(z+1,x)\\ &+ \zeta(z,x)xz\zeta(z+1,x+1)\\ &- \zeta(z,x)xz\zeta(z+1,x).\end{aligned} \tag{21}$$

From the RHS of Eq. (21), it can be seen that

$$-\int_{\mathbb{R}^+}\zeta(z,x+1)xz\zeta(z+1,x+1)dx = \sum_{n=0}^{\infty}\frac{(n+x+1)^{-2z}(n+2xz+1)}{2(2z-1)} + \text{const} \tag{22}$$

$$-\int_{\mathbb{R}^+}\zeta(z,x)xz\zeta(z+1,x)dx = \sum_{n=0}^{\infty}\frac{(n+x)^{-2z}(n+2xz)}{2z(2z-1)} + \text{const} \tag{23}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^+}\zeta(z,x+1)xz\zeta(z+1,x) + \zeta(z,x)xz\zeta(z+1,x+1)dx\\ &= \sum_{n=0}^{\infty}\frac{((n+x)^{-z}(n+x+1)^{-z}((n+x){}_2F_1(1,1-2z,1-z,n+x+1) - n - 2xz))}{(2z-1)} + \text{const},\end{aligned} \tag{24}$$

where the hypergeometric function is

$${}_2F_1(1,1-2z,1-z,n+x+1) = \sum_{n=0}^{\infty}\frac{(1)_n(1-2z)_n}{(1-z)_n}\frac{(n+x+1)^n}{n!}. \tag{25}$$

Since

$$N = \int_{\mathbb{R}^+} \varphi_n^*(x)\varphi_n(x)dx$$
$$= \int_{\mathbb{R}^+} [\psi_n(x) - \psi_n(x-1)]^2 dx$$
$$= \int_{\mathbb{R}^+} [\psi_n^2(x) - 2\psi_n(x-1)\psi_n(x) + \psi_n^2(x-1)]dx$$
$$= \sum_{n=0}^{\infty} \int_{\mathbb{R}^+} [(n+x+1)^{-2z} - 2(n+x)^{-z}(n+x+1)^{-z} + (n+x)^{-2z}]dx$$
$$= \frac{\zeta(2z-1,x)}{(1-2z)} + \sum_{n=0}^{\infty} \frac{2(-n-x)^z(n+x)^{-z}(n+x+1)^{1-z}\ {}_2F_1(1-z,z,2-z,n+x+1)}{z-1}$$
$$+ \sum_{n=0}^{\infty} \frac{(n+x+1)^{1-2z}}{(1-2z)},\ \Re(z) > 1, \tag{26}$$

with the hypergeometric function

$$ {}_2F_1(1-z,z,2-z,n+x+1) = \sum_{n=0}^{\infty} \frac{(1-z)_n(z)_n}{(2-z)_n} \frac{(n+x+1)^n}{n!}, \tag{27}$$

Eq. (19) can be rewritten

$$z_n = \Big[\frac{(1-2z)}{\zeta(2z-1,x)} + \sum_{n=0}^{\infty} \frac{z-1}{2(-n-x)^z(n+x)^{-z}(n+x+1)^{1-z}\ {}_2F_1(1-z,z,2-z,n+x+1)}$$
$$+ \sum_{n=0}^{\infty} \frac{(1-2z)}{(n+x+1)^{1-2z}}\Big]\Big[\sum_{n=0}^{\infty} \frac{(n+x+1)^{-2z}(n+2xz+1)}{2(2z-1)} + \sum_{n=0}^{\infty} \frac{(n+x)^{-2z}(n+2xz)}{2z(2z-1)}$$
$$+ \sum_{n=0}^{\infty} \frac{((n+x)^{-z}(n+x+1)^{-z}((n+x)_2F_1(1,1-2z,1-z,n+x+1)-n-2xz))}{(2z-1)}\Big] + \frac{3}{2}, \tag{28}$$

for $\Re(z) > 1$, and $\Re(x+1) > 0$. Upon imposing the boundary condition

$$\psi_n(0) = -\sum_{n=1}^{\infty} \frac{1}{n^z}$$
$$= -\frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{\exp(t)-1}, \tag{29}$$

Eq. (28) are the nontrivial zeros of Eq. (1).

□

B. Convergence

For brevity, let us examine the convergence of the integral representation of the discrete nontrivial zeros of the Riemann zeta function on the positive half line $x \in \mathbb{R}^+$, $z \in \mathbb{C}$, $\Re(z) > 1$, and $\Re(x+1) > 0$. From Eq. (19), the integral representation of the discrete nontrivial zeros of the Riemann zeta function are given by

$$z_n = -\frac{1}{N}\int_{\mathbb{R}^+} \zeta(z,x+1)xz\zeta(z+1,x+1)dx$$
$$-\frac{1}{N}\int_{\mathbb{R}^+} \zeta(z,x)xz\zeta(z+1,x)dx$$
$$+\frac{1}{N}\int_{\mathbb{R}^+} \zeta(z,x+1)xz\zeta(z+1,x) + \zeta(z,x)xz\zeta(z+1,x+1)dx + \frac{3}{2}, \tag{30}$$

where

$$N = \int_{\mathbb{R}^+} [(n+x+1)^{-2z} - 2(n+x)^{-z}(n+x+1)^{-z} + (n+x)^{-2z}]dx. \tag{31}$$

Lemma 1.2. *From the first term on the RHS of Eq. (30), if*

$$\int_0^t \zeta(z,x+1)xz\zeta(z+1,x+1)dx \tag{32}$$

exists for every number $t \geq 0$, then

$$\int_0^\infty \zeta(z,x+1)xz\zeta(z+1,x+1)dx = \lim_{t\to\infty}\int_0^t \zeta(z,x+1)xz\zeta(z+1,x+1)dx, \tag{33}$$

provided this limit exists as a finite number.

Proof.

$$\int_0^t \zeta(z,x+1)xz\zeta(z+1,x+1)dx = \frac{((n+t+1)^{-2z}((n+1)(\frac{t}{(n+1)}+1)^{2z} - n - 2tz - 1))}{(2z(2z-1))} \tag{34}$$

From L'Hospital's Rule, we have

$$\begin{aligned}
&\lim_{t\to\infty} \frac{((n+t+1)^{-2z}((n+1)(\frac{t}{(n+1)}+1)^{2z} - n - 2tz - 1))}{(2z(2z-1))} \\
&= \lim_{t\to\infty} \frac{((n+t+1)^{-2z}((n+1)(\frac{t}{(n+1)}+1)^{2z} - n - 2tz - 1))}{(2z(2z-1))} \cdot \frac{(n+t+1)^{-2z}}{(n+t+1)^{-2z}} \\
&= \lim_{t\to\infty} \frac{-2z(n+t+1)^{-4z-1}(n(\frac{t}{(n+1)}+1)^{2z} + (\frac{t}{(n+1)}+1)^{2z} - n - 4tz + t - 1)}{4(1-2z)z^2(n+t+1)^{-2z-1}}.
\end{aligned} \tag{35}$$

Upon evaluating Eq. (35) with a series expansion at $t = \infty$, we obtain

$$\begin{aligned}
&\lim_{t\to\infty} \frac{(-1-n+t+(1+n)(1+t/(1+n))^{2z} - 4tz)}{(2(1+n+t)^{2z}z(-1+2z))} \\
&= \frac{((n+t+1)^{-2z}((n+1)(\frac{t}{(n+1)}+1)^{2z} - n + t(1-4z) - 1))}{(2z(2z-1))}.
\end{aligned} \tag{36}$$

Hence, it can be seen that the first term on the RHS of Eq. (30) is convergent, given that the limit seen in Eq. (35) exists as a finite number as seen in Eq. (36). □

Lemma 1.3. *From the second term on the RHS of Eq. (30), if*

$$\int_0^t \zeta(z,x)xz\zeta(z+1,x)dx \tag{37}$$

exists for every number $t \geq 0$, then

$$\int_0^\infty \zeta(z,x)xz\zeta(z+1,x)dx = \lim_{t\to\infty}\int_0^t \zeta(z,x)xz\zeta(z+1,x)dx, \tag{38}$$

provided this limit exists as a finite number.

Proof.

$$\int_0^t \zeta(z,x)xz\zeta(z+1,x)dx = -\frac{((n+t)^{-2z}(-n(\frac{(n+t)}{n})^{2z} + n + 2tz))}{(2(2z-1))} \tag{39}$$

From L'Hospital's Rule, we have

$$
\begin{aligned}
&-\lim_{t\to\infty}\frac{((n+t)^{-2z}(-n(\frac{(n+t)}{n}^{2z}+n+2tz))}{(2(2z-1))}\\
&=-\lim_{t\to\infty}\frac{((n+t)^{-2z}(-n(\frac{(n+t)}{n}^{2z}+n+2tz))}{(2(2z-1))}\cdot\frac{(n+t)^{-2z}}{(n+t)^{-2z}}\\
&=-\lim_{t\to\infty}\frac{(n+t)^{-4z}(-n(\frac{(n+t)}{n})^{2z}+n+2tz)}{2(2z-1)(n+t)^{-2z}}
\end{aligned}
\tag{40}
$$

Upon evaluating Eq. (40) with a series expansion at $t=\infty$, we obtain

$$
\begin{aligned}
&\lim_{t\to\infty}\frac{((n+t)^{-2z}(-n((n+t)/n)^{2z}+n+2tz))}{((n+t)^{-2z}(-n((n+t)/n)^{2z}+n+2tz))}\\
&=\frac{((n+t)^{-2z}(-n(\frac{(n+t)}{n})^{2z}+n+2tz))}{(2(2z-1))}.
\end{aligned}
\tag{41}
$$

Hence, it can be seen that the second term on the RHS of Eq. (30) is convergent, given that the limit seen in Eq. (40) exists as a finite number as seen in Eq. (41). □

Lemma 1.4. *From the third term on the RHS of Eq. (30), if*

$$
\int_0^t \zeta(z,x+1)xz\zeta(z+1,x)+\zeta(z,x)xz\zeta(z+1,x+1)dx \tag{42}
$$

exists for every number $t\geq 0$, then

$$
\begin{aligned}
&\int_0^\infty \zeta(z,x+1)xz\zeta(z+1,x)+\zeta(z,x)xz\zeta(z+1,x+1)dx\\
&=\lim_{t\to\infty}\int_0^t \zeta(z,x+1)xz\zeta(z+1,x)+\zeta(z,x)xz\zeta(z+1,x+1)dx,
\end{aligned}
\tag{43}
$$

provided this limit exists as a finite number.

Proof. From the RHS of Eq. (24) it can be seen that

$$
\begin{aligned}
&\int_0^t \zeta(z,x+1)xz\zeta(z+1,x)+\zeta(z,x)xz\zeta(z+1,x+1)dx\\
&=\frac{((n+t)^{-z}(n+t+1)^{-z}((n+t)_2F_1(1,1-2z,1-z,n+t+1)-n-2tz))}{(2z-1)}\\
&-\frac{((n)^{-z}(n+1)^{-z}((n)_2F_1(1,1-2z,1-z,n+1)-n))}{(2z-1)}.
\end{aligned}
\tag{44}
$$

Since the second term on the RHS of Eq. (44) is independent of t, we are only concerned with the limit of the first term on the RHS of Eq. (44). As such,

$$
\begin{aligned}
&\lim_{t\to\infty}\frac{((n+t)_2F_1(1,1-2z,1-z,n+t+1)-n-2tz)}{(n+t)^z(n+t+1)^z(2z-1)}\\
&=\lim_{t\to\infty}\frac{((n+t)_2F_1(1,1-2z,1-z,n+t+1)-n-2tz)}{(n+t)^z(n+t+1)^z(2z-1)}\cdot\frac{{}_2F_1(1,1-2z,1-z,n+t+1)}{{}_2F_1(1,1-2z,1-z,n+t+1)}\\
&=\lim_{t\to\infty}\frac{{}_2F_1(1,1-2z,1-z,n+t+1)(n\,{}_2F_1(1,1-2z,1-z,n+t+1)+t\,{}_2F_1(1,1-2z,1-z,n+t+1)-n-2tz)}{2z(n+t)^z(n+t+1)^z\,{}_2F_1(1,1-2z,1-z,n+t+1)-(n+t)^z(n+t+1)^z\,{}_2F_1(1,1-2z,1-z,n+t+1)}
\end{aligned}
\tag{45}
$$

□

C. Domain of the Bender-Brody-Müller Hamiltonian

For the BBM Hamiltonian operator as given by Eq. (2), the Hilbert space is $\mathscr{H} = L^2(\mathbb{R}^+, dx)$. Moreover, $\hat{p}$ and $\hat{x}$ are self-adjoint operators that act in $\mathscr{H}$. In order to study the domain of the BBM Hamiltonian operator, we first introduce an auxiliary operator $\hat{O}$, such that

$$\hat{O} = \hat{p}\hat{p} + \hat{x}\hat{x}, \tag{46}$$

where $\hat{p}\hat{p} = -\nabla^2$, and $\hat{x}\hat{x} = x^2$. The set of finite linear combinations of Hermite functions is a core of $\hat{O}$, and therefore the Schwartz space $\mathscr{S}$ is also a core of $\hat{O}$.

Lemma 1.5. *[6] If φ is in $\mathscr{D}(\hat{O})$, then*

$$\|\hat{p}\hat{p}\varphi\|^2 + \|\hat{x}\hat{x}\varphi\|^2 \leq \|\hat{O}\varphi\|^2 + c\|\varphi\|^2. \tag{47}$$

Proof. [6] We estimate φ for a core of $\hat{O}$ via a double commutator to make a double commutator estimate [7],

$$\begin{aligned}\hat{O}\hat{O} &= \hat{p}\hat{p}\hat{p}\hat{p} + \hat{x}\hat{x}\hat{x}\hat{x} + \hat{p}\hat{p}\hat{x}\hat{x} + \hat{x}\hat{x}\hat{p}\hat{p} \\ &= \hat{p}\hat{p}\hat{p}\hat{p} + \hat{x}\hat{x}\hat{x}\hat{x} + 2\sum_{i=1}^{n}\Big[\hat{x}_i\hat{p}\hat{p}\hat{x}_i + [\hat{x}_i, [\hat{x}_i, \hat{p}\hat{p}]]\Big] \\ &\geq \hat{p}\hat{p}\hat{p}\hat{p} + \hat{x}\hat{x}\hat{x}\hat{x} - 2n,\end{aligned} \tag{48}$$

Therefore, in Eq. (47) $c = 2n$. □

After rewriting Eq. (8) as

$$[x\partial_x + \partial_x x]\varphi = (1 - 2z)\varphi, \tag{49}$$

then $\hat{p}\hat{p} = x\partial_x$ and $f(\hat{x}) = \partial_x x$ are self-adjoint operators acting in $\mathscr{H} = L^2(\mathbb{R}^+, dx)$. Setting

$$\hat{H} = \hat{p}\hat{p} + f(\hat{x}), \tag{50}$$

defined on

$$\hat{p}\hat{p} : X \bigcap f(\hat{x}) : Y. \tag{51}$$

If $f(\hat{x})$ is local in $\mathscr{H}$, then Eq. (50) is dense and Hermitian.

Theorem 2. *The BBM Hamiltonian operator in Eq. (2) is essentially self-adjoint, given that $|\nabla f(\hat{x})| \leq a|\hat{x}| + b$.*

The BBM Hamiltonian operator in Eq. (2) is real-valued on the positive half line $\mathbb{R}^+$, after being reduced to Eq. (49). From $|\nabla f(\hat{x})| \leq a|\hat{x}| + b$ we have

$$\begin{aligned}|f(\hat{x})| &\leq \frac{1}{2}\hat{x}\hat{x} + b|\hat{x}| \\ &\leq c\hat{x}\hat{x} + d.\end{aligned} \tag{52}$$

Let us examine the uniqueness.

Proof. As shown in [6], if $\hat{H}$ is Hermitian, and $\hat{O}$ is a positive self-adjoint operator, then $\mathscr{C}$ is a core of $\hat{O}$ such that $\mathscr{C} \subset \mathscr{D}(\hat{H})$. As such,

$$\|(\hat{p}\hat{p} + f(\hat{x}))\varphi\|^2 \leq a\|(\hat{p}\hat{p} + \hat{x}\hat{x})\varphi\|^2 + b\|\varphi\|^2, \tag{53}$$

where $\varphi \in \mathscr{S}$. Since $(1 + \hat{x}\hat{x})\varphi \in L^2$, $f(\hat{x})\varphi \in L^2$. Therefore, $\mathscr{S} \subset \mathscr{D}(\hat{H})$. Moreover, since $f(\hat{x})^2 \leq r\hat{x}\hat{x}\hat{x}\hat{x} + s$,

$$\|f(\hat{x})\varphi\|^2 \leq r\|\hat{x}\hat{x}\varphi\|^2 + s\|\varphi\|^2. \tag{54}$$

As such, from Eq. (47), Eq. (53) is satisfied. If $\varphi \in \mathscr{S}$, then $\nabla(f(\hat{x})\varphi) \in L^2$. Since,

$$\pm i[\hat{H}, \hat{O}] \leq c\hat{O} \tag{55}$$

as quadratic forms on $\mathscr{C}$, we thus have

$$\begin{aligned}\pm i[\hat{H},\hat{O}] &= \pm i\{[\hat{p}\hat{p},\hat{x}\hat{x}] + [f(\hat{x}),\hat{p}\hat{p}]\} \\ &= \pm\{2(\hat{p}\cdot\hat{x}+\hat{x}\cdot\hat{p}) - (\hat{p}\cdot\nabla f(\hat{x}) + \nabla f(\hat{x})\cdot\hat{p})\} \\ &\leq 2(\hat{p}\hat{p}+\hat{x}\hat{x}) + \hat{p}\hat{p} + (\nabla f(\hat{x}))^2 \\ &\leq 2(\hat{p}\hat{p}+\hat{x}\hat{x}) + \hat{p}\hat{p} + 2(a^2\hat{x}\hat{x}+b^2) \\ &\leq c\hat{O}, \end{aligned} \tag{56}$$

for constant c.

□

D. $\mathcal{PT}$-symmetric Bender-Brody-Müller Hamiltonian

Theorem 3. *The eigenvalues of the Hamiltonian*

$$i\hat{H} = \frac{i}{1-e^{-i\hat{p}}}(\hat{x}\hat{p}+\hat{p}\hat{x})(1-e^{-i\hat{p}}) \tag{57}$$

are imaginary, where $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, and $\hat{x} = x$.

Corollary 3.1. *[1] Solutions to the equation $i\hat{H}\psi = E\psi$ are given by the Hurwitz zeta function*

$$\begin{aligned}\psi_z(x) &= -\zeta(z,x+1) \\ &= -\sum_{n=0}^{\infty}\frac{1}{(x+1+n)^z}\end{aligned} \tag{58}$$

on the positive half line $\mathbb{R}^+$ with eigenvalues $i(2z-1)$.

Proof. Let ψ be an eigenfunction of Eq. (57) with an eigenvalue $\lambda = i(2z-1)$:

$$i\hat{H}\psi = \lambda\psi. \tag{59}$$

Then we have the relation

$$\frac{i}{1-e^{-i\hat{p}}}(\hat{x}\hat{p}+\hat{p}\hat{x})(1-e^{-i\hat{p}})\psi = \lambda\psi. \tag{60}$$

Letting

$$\varphi = [1-\exp(-\partial_x)]\psi, \tag{61}$$

and inserting Eq. (61) into Eq. (60) with $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, and $\hat{x} = x$, we obtain

$$[x\partial_x + \partial_x x]\varphi = \lambda\varphi. \tag{62}$$

Then we have

$$\int (x\partial_x\varphi)^*\varphi dx + \int(\partial_x x\varphi)^*\varphi dx = \lambda^*\int\varphi^*\varphi dx. \tag{63}$$

Now we integrate the first term on the LHS of Eq. (63) by parts to obtain

$$\int x\varphi\partial_x\varphi^* dx = -\int\varphi^*\varphi dx - \int\varphi^* x\frac{d}{dx}(\varphi)dx, \tag{64}$$

and the second term on the LHS of Eq. (63) by parts to obtain

$$\int x\varphi^*\partial_x\varphi dx = -\int\varphi\varphi^* dx - \int\varphi x\frac{d}{dx}(\varphi^*)dx. \tag{65}$$

Upon substituting Eqs. (64) and (65) into Eq. (63), we obtain

$$\int \varphi^* x \frac{d}{dx}(\varphi)dx + \int \varphi x \frac{d}{dx}(\varphi^*)dx = -(\lambda^* + 2)N, \tag{66}$$

where

$$N = \int \varphi^* \varphi dx. \tag{67}$$

Next, we split φ into real and imaginary components, such that

$$\varphi = \varphi_{\Re} + i\varphi_{\Im}, \tag{68}$$

and substitute Eq. (68) into Eq. (66) such that

$$\int \varphi_{\Re} x \frac{d}{dx} \varphi_{\Re} dx + \int \varphi_{\Im} x \frac{d}{dx} \varphi_{\Im} dx + N = -\frac{\lambda^*}{2}. \tag{69}$$

Upon setting $\lambda = i(2z-1)$, Eq. (69) can be written

$$\int \varphi_{\Re} x \frac{d}{dx} \varphi_{\Re} dx + \int \varphi_{\Im} x \frac{d}{dx} \varphi_{\Im} dx + N = \frac{i(1-2z)}{2}. \tag{70}$$

It can be seen that all terms on the LHS of Eq. (69) are real, thereby verifying Theorem 3.

Q.E.D.

□

III. CONCLUSION

In this note, we have discussed the domain and eigenvalues of the BBM Hamiltonian.

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