

Closed-Form Solution for the Nontrivial Zeros of the Riemann Zeta Function

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In the year 2017 it was formally conjectured that if the Bender-Brody-Müller (BBM) Hamiltonian can be shown to be self-adjoint, then the Riemann hypothesis holds true. Herein we discuss the domain and eigenvalues of the Bender-Brody-Müller conjecture. Moreover, a second quantization of the BBM Schrödinger equation is performed, and a closed-form solution for the nontrivial zeros of the Riemann zeta function is obtained.

I. INTRODUCTION

It was recently shown in [1] that the eigenvalues of a Bender-Brody-Müller (BBM) Hamiltonian operator correspond to the nontrivial zeroes of the Riemann zeta function [2]. Although the BBM Hamiltonian is pseudo-Hermitian, it is consistent with the Berry-Keating conjecture [3, 4]. The eigenvalues of the BBM Hamiltonian are taken to be the imaginary parts of the nontrivial zeroes of the zeta function

$$\begin{aligned}\zeta(z) &= \sum_{k=1}^{\infty} \frac{1}{k^z} \\ &= \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{\exp(t) - 1} dt.\end{aligned}\tag{1}$$

The idea that the imaginary parts of the zeroes of Eq. (1) are given by a self-adjoint operator was conjectured by Hilbert and Pólya [5]. Formally, Hilbert and Pólya determined that if the eigenfunctions of a self-adjoint operator satisfy the boundary conditions $\psi_n(0) = 0 \forall n$, then the eigenvalues are the nontrivial zeroes of Eq. (1). The BBM Hamiltonian also satisfies the Berry-Keating conjecture, which states that when \hat{x} and \hat{p} commute, the Hamiltonian reduces to the classical $H = 2xp$.

Remark. *If there are nontrivial roots of Eq. (1) for which $\Re(z) \neq 1/2$, the corresponding eigenvalues and eigenstates are degenerate [1].*

II. STATEMENT OF PROBLEM

A. Bender-Brody-Müller Hamiltonian

Theorem 1. *The eigenvalues of the Hamiltonian*

$$\hat{H} = \frac{1}{1 - e^{-i\hat{p}}} (\hat{x}\hat{p} + \hat{p}\hat{x})(1 - e^{-i\hat{p}})\tag{2}$$

are real, where $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, and $\hat{x} = x$.

Corollary 1.1. *[1] Solutions to the equation $\hat{H}\psi = E\psi$ are given by the Hurwitz zeta function*

$$\begin{aligned}\psi_z(x) &= -\zeta(z, x+1) \\ &= -\sum_{n=0}^{\infty} \frac{1}{(x+1+n)^z}\end{aligned}\tag{3}$$

on the positive half line $x \in \mathbb{R}^+$ with eigenvalues $i(2z-1)$, and $z \in \mathbb{C}$, for the boundary condition $\psi_z(0) = 0$. Moreover, $\Re(z) > 1$, and $\Re(x+1) > 0$. As $-\psi_z(0)$ is the Riemann zeta function, i.e., Eq. (1), this implies that z belongs to the discrete set of zeros of the Riemann zeta function.

Proof. Let $\psi_z(x)$ be an eigenfunction of Eq. (2) with an eigenvalue $\lambda = i(2z-1)$:

$$\hat{H}\psi_z(x) = \lambda\psi_z(x).\tag{4}$$

Then we have the relation

$$\frac{1}{1 - e^{-i\hat{p}}}(\hat{x}\hat{p} + \hat{p}\hat{x})(1 - e^{-i\hat{p}})\psi_z(x) = \lambda\psi_z(x). \quad (5)$$

Letting

$$\begin{aligned} \varphi_z(x) &= [1 - \exp(-\partial_x)]\psi_z(x), \\ &= \hat{\Delta}\psi_z(x), \end{aligned} \quad (6)$$

where $\hat{\Delta}\psi_z(x) = \psi_z(x) - \psi_z(x-1)$, and

$$\hat{\Delta} \equiv 1 - \exp(-\partial_x), \quad (7)$$

is a shift operator. Upon inserting Eq. (6) into Eq. (5) with $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, and $\hat{x} = x$, we obtain

$$[-ix\partial_x - i\partial_x x]\varphi_z(x) = \lambda\varphi_z(x). \quad (8)$$

Then we have

$$\int_{\mathbb{R}^+} (x\partial_x\varphi_z(x))^*\varphi_z(x)dx + \int_{\mathbb{R}^+} (\partial_x x\varphi_z(x))^*\varphi_z(x)dx = -i\lambda^* \int_{\mathbb{R}^+} \varphi_z^*(x)\varphi_z(x)dx. \quad (9)$$

As $\varphi_z(x \rightarrow \infty) \rightarrow 0$, next we integrate the first term on the LHS of Eq. (9) by parts to obtain

$$\int_{\mathbb{R}^+} x\varphi_z(x)\partial_x\varphi_z^*(x)dx = - \int_{\mathbb{R}^+} \varphi_z^*(x)\varphi_z(x)dx - \int_{\mathbb{R}^+} \varphi_z^*(x)x\frac{d}{dx}(\varphi_z(x))dx, \quad (10)$$

and the second term on the LHS of Eq. (9) by parts to obtain

$$\int_{\mathbb{R}^+} x\varphi_z(x)^*\partial_x\varphi_z(x)dx = - \int_{\mathbb{R}^+} \varphi_z(x)\varphi_z^*(x)dx - \int_{\mathbb{R}^+} \varphi_z(x)x\frac{d}{dx}(\varphi_z^*(x))dx. \quad (11)$$

Upon substituting Eqs. (10) and (11) into Eq. (9), we obtain

$$\int_{\mathbb{R}^+} \varphi_z^*(x)x\frac{d}{dx}(\varphi_z(x))dx + \int_{\mathbb{R}^+} \varphi_z(x)x\frac{d}{dx}(\varphi_z^*(x))dx = (i\lambda^* - 2)N, \quad (12)$$

where

$$N = \int_{\mathbb{R}^+} \varphi_z^*(x)\varphi_z(x)dx. \quad (13)$$

Next, we split $\varphi_z(x)$ into real and imaginary components, such that

$$\varphi_z(x) = \Re(\varphi_z(x)) + i\Im(\varphi_z(x)), \quad (14)$$

and substitute Eq. (14) into Eq. (12) such that

$$\int_{\mathbb{R}^+} \Re(\varphi_z(x))x\frac{d}{dx}\Re(\varphi_z(x))dx + \int_{\mathbb{R}^+} \Im(\varphi_z(x))x\frac{d}{dx}\Im(\varphi_z(x))dx + N = \frac{i\lambda^*}{2}N. \quad (15)$$

Upon setting $\lambda = i(2z - 1)$, Eq. (15) can be written

$$\int_{\mathbb{R}^+} \Re(\varphi_z(x))x\frac{d}{dx}\Re(\varphi_z(x))dx + \int_{\mathbb{R}^+} \Im(\varphi_z(x))x\frac{d}{dx}\Im(\varphi_z(x))dx + N = \frac{1 - 2z}{2}N. \quad (16)$$

It can be seen that all terms on the LHS of Eq. (15) are real, thereby verifying Theorem 1.

Q.E.D. □

Remark. If the Riemann hypothesis is correct [2], the the eigenvalues of Eq. (2) are degenerate [1].

Given that

$$\begin{aligned}
\psi_z(x) &= \hat{\Delta}\psi_z(x) \\
&= \psi_z(x) - \psi_n(x-1) \\
&= -\sum_{n=0}^{\infty} \frac{1}{(x+1+n)^z} + \sum_{n=0}^{\infty} \frac{1}{(x+n)^z},
\end{aligned} \tag{17}$$

the second term on the LHS of Eq. (16) goes to zero, as $\Im(\varphi_z(x)) = 0$. Hence, we are left with

$$z = \frac{1}{N} \int_{\mathbb{R}^+} \varphi_z(x) x \frac{d}{dx} \varphi_z(x) dx + \frac{3}{2}. \tag{18}$$

Moreover, it can be seen that

$$\begin{aligned}
x \frac{d}{dx} (\varphi_z(x)) &= x \frac{d}{dx} \psi_z(x) - x \frac{d}{dx} \psi_z(x-1) \\
&= -x \frac{d}{dx} \sum_{n=0}^{\infty} \frac{1}{(x+1+n)^z} + x \frac{d}{dx} \sum_{n=0}^{\infty} \frac{1}{(x+n)^z} \\
&= xz\zeta(z+1, x+1) - xz\zeta(z+1, x).
\end{aligned} \tag{19}$$

Multiplying Eq. (19) by $\varphi_n(x)$, we obtain

$$\begin{aligned}
\varphi_z(x)xz\zeta(z+1, x+1) - \varphi_z(x)xz\zeta(z+1, x) &= \varphi_z(x)[xz\zeta(z+1, x+1) - xz\zeta(z+1, x)] \\
&= -\zeta(z, x+1)xz\zeta(z+1, x+1) \\
&\quad + \zeta(z, x+1)xz\zeta(z+1, x) \\
&\quad + \zeta(z, x)xz\zeta(z+1, x+1) \\
&\quad - \zeta(z, x)xz\zeta(z+1, x).
\end{aligned} \tag{20}$$

From the RHS of Eq. (20), it can be seen that

$$-\int_{\mathbb{R}^+} \zeta(z, x+1)xz\zeta(z+1, x+1)dx = \sum_{n=0}^{\infty} \frac{(n+x+1)^{-2z}(n+2xz+1)}{2(2z-1)} + \text{const}, \tag{21}$$

$$-\int_{\mathbb{R}^+} \zeta(z, x)xz\zeta(z+1, x)dx = \sum_{n=0}^{\infty} \frac{(n+x)^{-2z}(n+2xz)}{2z(2z-1)} + \text{const}, \tag{22}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^+} \zeta(z, x+1)xz\zeta(z+1, x) + \zeta(z, x)xz\zeta(z+1, x+1)dx \\
&= \sum_{n=0}^{\infty} \frac{((n+x)^{-z}(n+x+1)^{-z}((n+x)_2F_1(1, 1-2z, 1-z, n+x+1) - n-2xz))}{(2z-1)} + \text{const},
\end{aligned} \tag{23}$$

where the hypergeometric function is

$${}_2F_1(1, 1-2z, 1-z, n+x+1) = \sum_{n=0}^{\infty} \frac{(1)_n(1-2z)_n}{(1-z)_n} \frac{(n+x+1)^n}{n!}. \tag{24}$$

Since

$$\begin{aligned}
N &= \int_{\mathbb{R}^+} \varphi_z^*(x) \varphi_z(x) dx \\
&= \int_{\mathbb{R}^+} [\psi_z(x) - \psi_z(x-1)]^2 dx \\
&= \int_{\mathbb{R}^+} [\psi_z^2(x) - 2\psi_z(x-1)\psi_z(x) + \psi_z^2(x-1)] dx \\
&= \int_{\mathbb{R}^+} [(n+x+1)^{-2z} - 2(n+x)^{-z}(n+x+1)^{-z} + (n+x)^{-2z}] dx \\
&= \frac{(n+x)^{1-2z}}{(1-2z)} + \frac{2(-n-x)^z(n+x)^{-z}(n+x+1)^{1-z} {}_2F_1(1-z, z, 2-z, n+x+1)}{z-1} \\
&\quad + \frac{(n+x+1)^{1-2z}}{(1-2z)} + \text{const}, \tag{25}
\end{aligned}$$

with the hypergeometric function

$${}_2F_1(1-z, z, 2-z, n+x+1) = \sum_{n=0}^{\infty} \frac{(1-z)_n (z)_n (n+x+1)^n}{(2-z)_n n!}, \tag{26}$$

Eq. (18) can be rewritten

$$\begin{aligned}
z_n &= \sum_{n=0}^{\infty} \left[\frac{(1-2z)}{(n+x)^{1-2z}} + \frac{z-1}{2(-n-x)^z(n+x)^{-z}(n+x+1)^{1-z} {}_2F_1(1-z, z, 2-z, n+x+1)} \right. \\
&\quad + \frac{(1-2z)}{(n+x+1)^{1-2z}} \left[\frac{(n+x+1)^{-2z}(n+2xz+1)}{2(2z-1)} + \frac{(n+x)^{-2z}(n+2xz)}{2z(2z-1)} \right. \\
&\quad \left. \left. + \frac{((n+x)^{-z}(n+x+1)^{-z}((n+x) {}_2F_1(1, 1-2z, 1-z, n+x+1) - n-2xz))}{(2z-1)} \right] \right] + \frac{3}{2} \pm \text{const}, \tag{27}
\end{aligned}$$

for $\Re(z) > 1$, and $\Re(x+1) > 0$. Upon imposing the boundary condition

$$\begin{aligned}
\psi_n(0) &= - \sum_{n=1}^{\infty} \frac{1}{n^{z_n}} \\
&= - \frac{1}{\Gamma(z_n)} \int_0^{\infty} \frac{t^{z_n-1}}{\exp(t_n) - 1} dt \\
&= 0, \tag{28}
\end{aligned}$$

Eq. (27) are the nontrivial zeros of Eq. (1), and for $z \in \mathbb{C}$ where z must belong to the discrete set of zeros of Eq. (1). Consequently, for the boundary condition $\psi(0) = 0$, the n^{th} eigenstate of Eq. (2) is

$$\begin{aligned}
\psi_n(x) &= -\zeta(z_n, x+1) \\
&= - \sum_{n=0}^{\infty} \frac{1}{(x+1+n)^{z_n}}, \tag{29}
\end{aligned}$$

where z_n is given by Eq. (27). The Riemann hypothesis states [2] that the *nontrivial* zeros are located at $\Re(z) = 1/2$. As such, taking $x = 0$ according to Eq. (28), Eq. (27) can be written

$$\begin{aligned}
z_n &= \sum_{n=0}^{\infty} \left[\frac{(n)^{1-2z}}{(1-2z)} + \frac{2(-n)^z(n)^{-z}(n+1)^{1-z} {}_2F_1(1-z, z, 2-z, n+1)}{z-1} \right. \\
&\quad + \frac{(n+1)^{1-2z}}{(1-2z)} \left. \right]^{-1} \cdot \left[\frac{(n+1)^{-2z}(n+1)}{2(2z-1)} + \frac{(n)^{-2z}(n)}{2z(2z-1)} \right. \\
&\quad \left. + \frac{((n)^{-z}(n+1)^{-z}((n) {}_2F_1(1, 1-2z, 1-z, n+1) - n))}{(2z-1)} \right] + \frac{3}{2} \pm \text{const} \\
&= \frac{1}{2}(1 - i\lambda_n), \tag{30}
\end{aligned}$$

where the nontrivial zeros of Eq. (1) are located at $\Re(z) = 1/2$, and $z \in \mathbb{C}$.

B. Convergence

For brevity, let us examine the convergence of the integral representation of the discrete nontrivial zeros of the Riemann zeta function on the positive half line $x \in \mathbb{R}^+$, $z \in \mathbb{C}$, $\Re(z) > 1$, and $\Re(x+1) > 0$. From Eq. (18), the integral representation of the discrete nontrivial zeros of the Riemann zeta function are given by

$$\begin{aligned} z_n = & -\frac{1}{N} \int_{\mathbb{R}^+} \zeta(z, x+1)xz\zeta(z+1, x+1)dx \\ & -\frac{1}{N} \int_{\mathbb{R}^+} \zeta(z, x)xz\zeta(z+1, x)dx \\ & +\frac{1}{N} \int_{\mathbb{R}^+} \zeta(z, x+1)xz\zeta(z+1, x) + \zeta(z, x)xz\zeta(z+1, x+1)dx + \frac{3}{2}, \end{aligned} \quad (31)$$

where

$$N = \int_{\mathbb{R}^+} \left[(n+x+1)^{-2z} - 2(n+x)^{-z}(n+x+1)^{-z} + (n+x)^{-2z} \right] dx. \quad (32)$$

Lemma 1.1. *From the first term on the RHS of Eq. (31), if*

$$\int_0^t \zeta(z, x+1)xz\zeta(z+1, x+1)dx \quad (33)$$

exists for every number $t \geq 0$, then

$$\int_0^\infty \zeta(z, x+1)xz\zeta(z+1, x+1)dx = \lim_{t \rightarrow \infty} \int_0^t \zeta(z, x+1)xz\zeta(z+1, x+1)dx, \quad (34)$$

provided this limit exists as a finite number.

Proof.

$$\int_0^t \zeta(z, x+1)xz\zeta(z+1, x+1)dx = \frac{((n+t+1)^{-2z}((n+1)(\frac{t}{n+1})+1)^{2z} - n - 2tz - 1)}{(2z(2z-1))}. \quad (35)$$

From L'Hospital's Rule, we have

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{((n+t+1)^{-2z}((n+1)(\frac{t}{n+1})+1)^{2z} - n - 2tz - 1)}{(2z(2z-1))} \\ = & \lim_{t \rightarrow \infty} \frac{((n+t+1)^{-2z}((n+1)(\frac{t}{n+1})+1)^{2z} - n - 2tz - 1)}{(2z(2z-1))} \cdot \frac{(n+t+1)^{-2z}}{(n+t+1)^{-2z}} \\ = & \lim_{t \rightarrow \infty} \frac{-2z(n+t+1)^{-4z-1}(n(\frac{t}{n+1})+1)^{2z} + (\frac{t}{n+1})+1)^{2z} - n - 4tz + t - 1}{4(1-2z)z^2(n+t+1)^{-2z-1}}. \end{aligned} \quad (36)$$

Upon evaluating Eq. (36) with a series expansion at $t = \infty$, we obtain

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{(-1-n+t+(1+n)(1+\frac{t}{1+n})^{2z} - 4tz)}{(2(1+n+t)^{2z}z(-1+2z))} \\ = & \frac{((n+t+1)^{-2z}((n+1)(\frac{t}{n+1})+1)^{2z} - n + t(1-4z) - 1)}{(2z(2z-1))}. \end{aligned} \quad (37)$$

Hence, it can be seen that the first term on the RHS of Eq. (31) is convergent, given that the limit seen in Eq. (36) exists as a finite number as seen in Eq. (37). Here, it should be pointed out that as $t = -n$, and $\Re(z) = 1/2$, Eq. (37) is of indeterminate form. As such, we apply L'Hopital's rule to obtain

$$\begin{aligned} & \frac{(n+t+1)^{-2z}((n+1)(\frac{t}{n+1})+1)^{2z} - n + t(1-4z) - 1)}{(2z(2z-1))} \\ = & \frac{2(n+1)(1-\frac{n}{n+1})^{(2z)} \log(1-\frac{n}{n+1}) + 4n}{8z-2}. \end{aligned} \quad (38)$$

□

Lemma 1.2. *From the second term on the RHS of Eq. (31), if*

$$\int_0^t \zeta(z, x)xz\zeta(z+1, x)dx \quad (39)$$

exists for every number $t \geq 0$, then

$$\int_0^\infty \zeta(z, x)xz\zeta(z+1, x)dx = \lim_{t \rightarrow \infty} \int_0^t \zeta(z, x)xz\zeta(z+1, x)dx, \quad (40)$$

provided this limit exists as a finite number.

Proof.

$$\int_0^t \zeta(z, x)xz\zeta(z+1, x)dx = -\frac{((n+t)^{-2z}(-n(\frac{n+t}{n})^{2z} + n + 2tz))}{(2(2z-1))}. \quad (41)$$

From L'Hospital's Rule, we have

$$\begin{aligned} & -\lim_{t \rightarrow \infty} \frac{((n+t)^{-2z}(-n(\frac{n+t}{n})^{2z} + n + 2tz))}{(2(2z-1))} \\ &= -\lim_{t \rightarrow \infty} \frac{((n+t)^{-2z}(-n(\frac{n+t}{n})^{2z} + n + 2tz))}{(2(2z-1))} \cdot \frac{(n+t)^{-2z}}{(n+t)^{-2z}} \\ &= -\lim_{t \rightarrow \infty} \frac{(n+t)^{-4z}(-n(\frac{n+t}{n})^{2z} + n + 2tz)}{2(2z-1)(n+t)^{-2z}} \end{aligned} \quad (42)$$

Upon evaluating Eq. (42) with a series expansion at $t = \infty$, we obtain

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{((n+t)^{-2z}(-n(\frac{n+t}{n})^{2z} + n + 2tz))}{((n+t)^{-2z}(-n(\frac{n+t}{n})^{2z} + n + 2tz))} \\ &= \frac{(n+t)^{-2z}(-n(\frac{n+t}{n})^{2z} + n + 2tz)}{(2(2z-1))}. \end{aligned} \quad (43)$$

Hence, it can be seen that the second term on the RHS of Eq. (31) is convergent, given that the limit seen in Eq. (42) exists as a finite number as seen in Eq. (43). Here, it should be pointed out that as $t = -n$, and $\Re(z) = 1/2$, Eq. (43) is undefined. \square

Lemma 1.3. *From the third term on the RHS of Eq. (31), if*

$$\int_0^t \zeta(z, x+1)xz\zeta(z+1, x) + \zeta(z, x)xz\zeta(z+1, x+1)dx \quad (44)$$

exists for every number $t \geq 0$, then

$$\begin{aligned} & \int_0^\infty \zeta(z, x+1)xz\zeta(z+1, x) + \zeta(z, x)xz\zeta(z+1, x+1)dx \\ &= \lim_{t \rightarrow \infty} \int_0^t \zeta(z, x+1)xz\zeta(z+1, x) + \zeta(z, x)xz\zeta(z+1, x+1)dx, \end{aligned} \quad (45)$$

provided this limit exists as a finite number.

Proof. From the RHS of Eq. (23) it can be seen that

$$\begin{aligned} & \int_0^t \zeta(z, x+1)xz\zeta(z+1, x) + \zeta(z, x)xz\zeta(z+1, x+1)dx \\ &= \frac{((n+t)^{-z}(n+t+1)^{-z}((n+t)_2F_1(1, 1-2z, 1-z, n+t+1) - n - 2tz))}{(2z-1)} \\ &= \frac{((n)^{-z}(n+1)^{-z}((n)_2F_1(1, 1-2z, 1-z, n+1) - n))}{(2z-1)}. \end{aligned} \quad (46)$$

Since the second term on the RHS of Eq. (46) is independent of t , we are only concerned with the limit of the first term on the RHS of Eq. (46). As such, we consider the limit

$$\lim_{t \rightarrow \infty} \frac{((n+t)_2F_1(1, 1-2z, 1-z, n+t+1) - n - 2tz)}{(n+t)^z(n+t+1)^z(2z-1)}. \quad (47)$$

Here, it is useful to employ Gauss' theorem, i.e.,

$${}_2F_1(1, 1-2z, 1-z, n+t+1) = \frac{\Gamma(1-z)\Gamma(z-1)}{\Gamma(-z)\Gamma(z)} \quad (48)$$

where $\Re(z) > 1$, $n = -t$, and

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx \quad (49)$$

is the gamma function. Therefore, Eq. (47) can be written

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{((n+t) \frac{\Gamma(1-z)\Gamma(z-1)}{\Gamma(-z)\Gamma(z)} - n - 2tz)}{(n+t)^z(n+t+1)^z(2z-1)} \\ &= - \lim_{t \rightarrow \infty} \frac{(n+t)^{-z}(n+t+1)^{-z}(n+t)}{(z-1)}. \end{aligned} \quad (50)$$

Upon evaluating Eq. (50) with a series expansion at $t = \infty$, we obtain

$$\lim_{t \rightarrow \infty} \frac{((n+t)_2F_1(1, 1-2z, 1-z, n+t+1) - n - 2tz)}{(n+t)^z(n+t+1)^z(2z-1)} = (n+t)^{-z}(n+t+1)^{-z} \left[-\frac{n}{(z-1)} - \frac{(tz)}{(z-1)} \right]. \quad (51)$$

Hence, it can be seen that the third term on the RHS of Eq. (31) is convergent, given that the limit seen in Eq. (47) exists as a finite number as seen in Eq. (51). Here, it should be pointed out that as $t = -n$, and $\Re(z) = 1/2$, Eq. (51) is undefined. Moreover, the second term on the RHS of Eq. (46) is indeterminate at $\Re(z)$. \square

Finally, we must consider the convergence of the normalization factor N .

Lemma 1.4. *From the first three terms on the RHS of Eq. (31), if*

$$\int_0^t \left[(n+x+1)^{-2z} - 2(n+x)^{-z}(n+x+1)^{-z} + (n+x)^{-2z} \right] dx \quad (52)$$

exists for every number $t \geq 0$, then

$$\begin{aligned} & \int_0^\infty \left[(n+x+1)^{-2z} - 2(n+x)^{-z}(n+x+1)^{-z} + (n+x)^{-2z} \right] dx \\ &= \lim_{t \rightarrow \infty} \int_0^t \left[(n+x+1)^{-2z} - 2(n+x)^{-z}(n+x+1)^{-z} + (n+x)^{-2z} \right] dx \end{aligned} \quad (53)$$

provided this limit exists as a finite number.

Proof.

$$\begin{aligned} & \int_0^\infty \left[(n+x+1)^{-2z} - 2(n+x)^{-z}(n+x+1)^{-z} + (n+x)^{-2z} \right] dx \\ &= \lim_{t \rightarrow \infty} \frac{((n+t+1)^{-2z}((n+1)(\frac{t}{(n+1)} + 1)^{2z} - n - t - 1))}{(2z-1)} \\ &+ \lim_{t \rightarrow \infty} \frac{((n+t)^{-2z}(n((\frac{n+t}{n})^{2z} - 1) - t))}{(2z-1)} \\ &+ \lim_{t \rightarrow \infty} \frac{(2(-n-t)^z(n+t)^{-z}(n+t+1)^{1-z} {}_2F_1(1-z, z, 2-z, n+t+1))}{(z-1)} \\ &- \frac{(2(-n)^z(n)^{-z}(n+1)^{1-z} {}_2F_1(1-z, z, 2-z, n+1))}{(z-1)}, \end{aligned} \quad (54)$$

where the last term on the RHS of Eq. (54) omits the limit, as it is independent of t . The limits seen on the RHS of Eq. (54) can be evaluated in a similar manner to those seen in Eqs. (37), (43), and (47), respectively. \square

C. Domain of the Bender-Brody-Müller Hamiltonian

For the BBM Hamiltonian operator as given by Eq. (2), the Hilbert space is $\mathcal{H} = L^2(\mathbb{R}^+, dx)$. Moreover, \hat{p} and \hat{x} are self-adjoint operators that act in \mathcal{H} . In order to study the domain of the BBM Hamiltonian operator, we first introduce an auxiliary operator \hat{O} , such that

$$\hat{O} = \hat{p}\hat{p} + \hat{x}\hat{x}, \quad (55)$$

where $\hat{p}\hat{p} = -\nabla^2$, and $\hat{x}\hat{x} = x^2$. The set of finite linear combinations of Hermite functions is a core of \hat{O} , and therefore the Schwartz space \mathcal{S} is also a core of \hat{O} .

Lemma 1.5. [6] *If φ is in $\mathcal{D}(\hat{O})$, then*

$$\|\hat{p}\hat{p}\varphi\|^2 + \|\hat{x}\hat{x}\varphi\|^2 \leq \|\hat{O}\varphi\|^2 + c\|\varphi\|^2. \quad (56)$$

Proof. [6] We estimate φ for a core of \hat{O} via a double commutator to make the estimate [7],

$$\begin{aligned} \hat{O}\hat{O} &= \hat{p}\hat{p}\hat{p}\hat{p} + \hat{x}\hat{x}\hat{x}\hat{x} + \hat{p}\hat{p}\hat{x}\hat{x} + \hat{x}\hat{x}\hat{p}\hat{p} \\ &= \hat{p}\hat{p}\hat{p}\hat{p} + \hat{x}\hat{x}\hat{x}\hat{x} + 2 \sum_{i=1}^n \left[\hat{x}_i \hat{p}\hat{p}\hat{x}_i + [\hat{x}_i, [\hat{x}_i, \hat{p}\hat{p}]] \right] \\ &\geq \hat{p}\hat{p}\hat{p}\hat{p} + \hat{x}\hat{x}\hat{x}\hat{x} - 2n, \end{aligned} \quad (57)$$

Therefore, in Eq. (56) $c = 2n$. □

After rewriting Eq. (8) as

$$[x\partial_x + \partial_x x]\varphi = (1 - 2z)\varphi, \quad (58)$$

then $\hat{p}\hat{p} = x\partial_x$ and $f(\hat{x}) = \partial_x x$ are self-adjoint operators acting in $\mathcal{H} = L^2(\mathbb{R}^+, dx)$. Setting

$$\hat{H} = \hat{p}\hat{p} + f(\hat{x}), \quad (59)$$

defined on

$$\mathcal{D}(\hat{p}\hat{p}) \cap \mathcal{D}(f(\hat{x})). \quad (60)$$

If $f(\hat{x})$ is local in \mathcal{H} , then Eq. (59) is dense and Hermitian.

Theorem 2. *The BBM Hamiltonian operator in Eq. (2) is essentially self-adjoint, given that $|\nabla f(\hat{x})| \leq a|\hat{x}| + b$.*

The BBM Hamiltonian operator in Eq. (2) is real-valued on the positive half line \mathbb{R}^+ , after being reduced to Eq. (58). From $|\nabla f(\hat{x})| \leq a|\hat{x}| + b$ we have

$$\begin{aligned} |f(\hat{x})| &\leq \frac{1}{2}\hat{x}\hat{x} + b|\hat{x}| \\ &\leq c\hat{x}\hat{x} + d. \end{aligned} \quad (61)$$

Let us examine the uniqueness.

Proof. As shown in [6], if \hat{H} is Hermitian, and \hat{O} is a positive self-adjoint operator, then \mathcal{C} is a core of \hat{O} such that $\mathcal{C} \subset \mathcal{D}(\hat{H})$. As such,

$$\|(\hat{p}\hat{p} + f(\hat{x}))\varphi\|^2 \leq a\|(\hat{p}\hat{p} + \hat{x}\hat{x})\varphi\|^2 + b\|\varphi\|^2, \quad (62)$$

where $\varphi \in \mathcal{S}$. Since $(1 + \hat{x}\hat{x})\varphi \in L^2$, $f(\hat{x})\varphi \in L^2$. Therefore, $\mathcal{S} \subset \mathcal{D}(\hat{H})$. Moreover, since $f(\hat{x})^2 \leq r\hat{x}\hat{x}\hat{x}\hat{x} + s$,

$$\|f(\hat{x})\varphi\|^2 \leq r\|\hat{x}\hat{x}\varphi\|^2 + s\|\varphi\|^2. \quad (63)$$

As such, from Eq. (56), Eq. (62) is satisfied. If $\varphi \in \mathcal{S}$, then $\nabla(f(\hat{x})\varphi) \in L^2$. Since,

$$\pm i[\hat{H}, \hat{O}] \leq c\hat{O} \quad (64)$$

as quadratic forms on \mathcal{C} , we thus have

$$\begin{aligned}
\pm i[\hat{H}, \hat{O}] &= \pm i\{[\hat{p}\hat{p}, \hat{x}\hat{x}] + [f(\hat{x}), \hat{p}\hat{p}]\} \\
&= \pm\{2(\hat{p} \cdot \hat{x} + \hat{x} \cdot \hat{p}) - (\hat{p} \cdot \nabla f(\hat{x}) + \nabla f(\hat{x}) \cdot \hat{p})\} \\
&\leq 2(\hat{p}\hat{p} + \hat{x}\hat{x}) + \hat{p}\hat{p} + (\nabla f(\hat{x}))^2 \\
&\leq 2(\hat{p}\hat{p} + \hat{x}\hat{x}) + \hat{p}\hat{p} + 2(a^2\hat{x}\hat{x} + b^2) \\
&\leq c\hat{O},
\end{aligned} \tag{65}$$

for constant c .

□

D. Second Quantization

We begin with the Bender-Brody-Müller (BBM) Schrödinger equation

$$-\frac{\hbar}{i} \frac{d}{dz} \psi(x, z) = \left[\hat{\Delta}^{-1} \hat{x} \hat{p} \hat{\Delta} + \hat{\Delta}^{-1} \hat{p} \hat{x} \hat{\Delta} \right] \psi(x, z), \tag{66}$$

where $\hat{\Delta}$ is given by Eq. (7), $\hat{x} = x$, $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, $x \in \mathbb{R}^+$, and $z \in \mathbb{C}$. Furthermore, let

$$\begin{aligned}
\psi_n(x) &= -\zeta(z_n, x+1) \\
&= -\sum_{n=0}^{\infty} \frac{1}{(x+1+n)^z}
\end{aligned} \tag{67}$$

be the solution of

$$\left(\hat{\Delta}^{-1} \hat{x} \hat{p} \hat{\Delta} + \hat{\Delta}^{-1} \hat{p} \hat{x} \hat{\Delta} \right) \psi_n(x) = \lambda_n \psi_n(x), \tag{68}$$

where z_n are the nontrivial zeros of the Riemann zeta function given by Eq. (27), λ_n are the eigenvalues, $\Re(z) > 1$, and $\Re(x+1) > 0$. Letting

$$\begin{aligned}
\varphi(x, z) &= [1 - \exp(-\partial_x)] \psi(x, z), \\
&= \hat{\Delta} \psi(x, z),
\end{aligned} \tag{69}$$

where $\hat{\Delta} \psi(x, z) = \psi(x, z) - \psi(x-1, z)$, and

$$\hat{\Delta} \equiv 1 - \exp(-\partial_x), \tag{70}$$

is a shift operator. Upon inserting Eq. (69) into Eq. (66) with $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, and $\hat{x} = x$, we obtain

$$-\hbar \frac{d}{dz} \varphi(x, z) = [x\partial_x + \partial_x x] \varphi(x, z). \tag{71}$$

Next, we write

$$\varphi(x, z) = \sum_n b_n(z) \varphi_n(x). \tag{72}$$

From Eq. (71) we find

$$-\hbar \frac{d}{dz} b_n(z) = \lambda_n b_n(z). \tag{73}$$

We now find a Hamiltonian that yields Eq. (73) as the equation of motion. Hence, we take

$$\hat{H} = \int_{\mathbb{R}^+} \varphi^*(x, z) [x\partial_x + \partial_x x] \varphi(x, z) dx \tag{74}$$

as the expectation value. Upon substituting Eq. (72) into Eq. (74) and using Eq. (68) we obtain the harmonic oscillator

$$\hat{H} = \sum_n \lambda_n b_n^*(z) b_n(z). \quad (75)$$

Taking $b_n(z)$ as an operator, and $b_n^*(z)$ as the adjoint, we obtain the usual properties:

$$\begin{aligned} [\hat{b}_n, \hat{b}_m] &= [\hat{b}_n^\dagger, \hat{b}_m^\dagger] = 0, \\ [\hat{b}_n, \hat{b}_m^\dagger] &= \delta_{nm}. \end{aligned} \quad (76)$$

From the analogous Heisenberg equations of motion,

$$\begin{aligned} -\hbar \frac{d}{dz} \hat{b}_n &= [\hat{b}_n, \hat{H}]_- \\ &= \sum_m E_m (\hat{b}_n \hat{b}_m^\dagger \hat{b}_m - \hat{b}_m^\dagger \hat{b}_m \hat{b}_n) \\ &= \sum_m E_m (\delta_{nm} \hat{b}_m - \hat{b}_m^\dagger \hat{b}_n \hat{b}_m - \hat{b}_m^\dagger \hat{b}_m \hat{b}_n) \\ &= \sum_m E_m (\delta_{nm} \hat{b}_m + \hat{b}_m^\dagger \hat{b}_m \hat{b}_n - \hat{b}_m^\dagger \hat{b}_m \hat{b}_n) \\ &= \lambda_n \hat{b}_n. \end{aligned} \quad (77)$$

The eigenvalues of \hat{H} are

$$\hat{H} = \sum_n \lambda_n N_n, \quad (78)$$

where $N_n = 0, 1, 2, 3, \dots, \infty$. Since, $\lambda_n = i(2z_n - 1)$, we can rewrite Eq. (78) as

$$\hat{H} = i \sum_n (2z_n - 1) N_n. \quad (79)$$

However, from Eq. (77) it can be seen that

$$-\hbar \frac{d}{dz} \hat{b}_n = i(2z_n - 1) \hat{b}_n. \quad (80)$$

As such,

$$\boxed{\frac{d}{dz} \hat{b}_n = -\frac{i}{\hbar} (2z_n - 1) \hat{b}_n.} \quad (81)$$

E. \mathcal{PT} -symmetric Bender-Brody-Müller Hamiltonian

Theorem 3. *The eigenvalues of the Hamiltonian*

$$i\hat{H} = \frac{i}{1 - e^{-i\hat{p}}} (\hat{x}\hat{p} + \hat{p}\hat{x})(1 - e^{-i\hat{p}}) \quad (82)$$

are imaginary, where $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, and $\hat{x} = x$.

Corollary 3.1. [1] *Solutions to the equation $i\hat{H}\psi = E\psi$ are given by the Hurwitz zeta function*

$$\begin{aligned} \psi_z(x) &= -\zeta(z, x+1) \\ &= -\sum_{n=0}^{\infty} \frac{1}{(x+1+n)^z} \end{aligned} \quad (83)$$

on the positive half line $x \in \mathbb{R}^+$ with eigenvalues $i(2z-1)$, and $z \in \mathbb{C}$, for the boundary condition $\psi_z(0) = 0$. Moreover, $\Re(z) > 1$, and $\Re(x+1) > 0$. As $-\psi_z(0)$ is the Riemann zeta function, i.e., Eq. (1), this implies that z belongs to the discrete set of zeros of the Riemann zeta function.

Proof. Let ψ be an eigenfunction of Eq. (82) with an eigenvalue $\lambda = i(2z - 1)$:

$$i\hat{H}\psi = \lambda\psi. \quad (84)$$

Then we have the relation

$$\frac{i}{1 - e^{-i\hat{p}}}(\hat{x}\hat{p} + \hat{p}\hat{x})(1 - e^{-i\hat{p}})\psi = \lambda\psi. \quad (85)$$

Letting

$$\begin{aligned} \varphi_z(x) &= [1 - \exp(-\partial_x)]\psi_z(x), \\ &= \hat{\Delta}\psi_z(x), \end{aligned} \quad (86)$$

where $\hat{\Delta}\psi_z(x) = \psi_z(x) - \psi_z(x-1)$, and inserting Eq. (86) into Eq. (85) with $\hat{p} = -i\hbar\partial_x$, $\hbar = 1$, and $\hat{x} = x$, we obtain

$$[x\partial_x + \partial_x x]\varphi_z(x) = \lambda\varphi_z(x). \quad (87)$$

Then we have

$$\int_{\mathbb{R}^+} (x\partial_x\varphi_z(x))^*\varphi_z(x)dx + \int_{\mathbb{R}^+} (\partial_x x\varphi_z(x))^*\varphi_z(x)dx = \lambda^* \int_{\mathbb{R}^+} \varphi_z^*(x)\varphi_z(x)dx. \quad (88)$$

As $\varphi_z(x \rightarrow \infty) \rightarrow 0$, next we integrate the first term on the LHS of Eq. (88) by parts to obtain

$$\int_{\mathbb{R}^+} x\varphi_z(x)\partial_x\varphi_z^*(x)dx = - \int_{\mathbb{R}^+} \varphi_z^*(x)\varphi_z(x)dx - \int_{\mathbb{R}^+} \varphi_z^*(x)x\frac{d}{dx}(\varphi_z(x))dx, \quad (89)$$

and the second term on the LHS of Eq. (88) by parts to obtain

$$\int_{\mathbb{R}^+} x\varphi_z^*(x)\partial_x\varphi_z(x)dx = - \int_{\mathbb{R}^+} \varphi_z(x)\varphi_z^*(x)dx - \int_{\mathbb{R}^+} \varphi_z(x)x\frac{d}{dx}(\varphi_z^*(x))dx. \quad (90)$$

Upon substituting Eqs. (89) and (90) into Eq. (88), we obtain

$$\int_{\mathbb{R}^+} \varphi_z^*(x)x\frac{d}{dx}(\varphi_z(x))dx + \int_{\mathbb{R}^+} \varphi_z(x)x\frac{d}{dx}(\varphi_z^*(x))dx = -(\lambda^* + 2)N, \quad (91)$$

where

$$N = \int_{\mathbb{R}^+} \varphi_z^*(x)\varphi_z(x)dx. \quad (92)$$

Next, we split $\varphi_z(x)$ into real and imaginary components, such that

$$\varphi_z(x) = \Re(\varphi_z(x)) + i\Im(\varphi_z(x)), \quad (93)$$

and substitute Eq. (93) into Eq. (91) such that

$$\int_{\mathbb{R}^+} \Re(\varphi_z(x))x\frac{d}{dx}\Re(\varphi_z(x))dx + \int_{\mathbb{R}^+} \Im(\varphi_z(x))x\frac{d}{dx}\Im(\varphi_z(x))dx + N = -\frac{\lambda^*}{2}N. \quad (94)$$

Upon setting $\lambda = i(2z - 1)$, Eq. (94) can be written

$$\int_{\mathbb{R}^+} \Re(\varphi_z(x))x\frac{d}{dx}\Re(\varphi_z(x))dx + \int_{\mathbb{R}^+} \Im(\varphi_z(x))x\frac{d}{dx}\Im(\varphi_z(x))dx + N = \frac{i(1 - 2z)}{2}N. \quad (95)$$

It can be seen that all terms on the LHS of Eq. (94) are real, thereby verifying Theorem 3.

Q.E.D.

□

III. CONCLUSION

In this study, we have discussed the domain and eigenvalues of the BBM Hamiltonian. Moreover, a second quantization procedure was performed for the BBM Schrödinger analogue equation. Finally, a closed-form expression for the nontrivial zeros of the Riemann zeta function was obtained, and a convergence test for the closed-form expression was performed.

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