

# The Homology Groups of $G_{n,m}(\mathbb{R})$

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## Abstract

In this article, we computed the homology groups of real Grassmann manifold  $G_{n,m}(\mathbb{R})$  by Witten complex.

## 1 Introduction

For a Morse function  $f$  on a compact manifold  $M$ , and choosing a Riemannian metric  $g$  on  $M$ . In [1], Witten introduces a chain complex and the homology groups of  $M$  by studying the negative gradient vector field associated to  $(f, g)$ , and he claims that these groups are the ordinary homology groups. Salamon gives a more detailed and precise interpretation to Witten's method in [2], we can also see the books [3] and [4]. Here below we give a brief description to Witten's method.

Let  $M$  be a  $n$ -dimensional compact smooth manifold and  $f$  be a Morse function on  $M$ . We can choose a suitable Riemannian metric on  $M$  such that the negative gradient vector field  $-\nabla f$  to be Morse-Smale type (i.e.  $f$  is a Morse-Smale function), then the connecting orbits determine the following chain complex.

First choose an orientation of the vector space  $E^u(p) = T_p W^u(p)$  for every critical point  $p$  of  $f$  and denote by  $\langle p \rangle$  the pair consisting of the critical point  $p$  and the chosen orientation. For  $r = 1, 2, \dots, \dim M$ , denote by  $C_r$  the free group  $C_r = \bigoplus_p \mathbb{Z} \langle p \rangle$  where  $p$  runs over all critical points of index  $r$ . The function  $f$  being of Morse-Smale function implies that  $W^u(p) \cap W^s(q)$  consists of finitely many trajectories if  $\text{ind}(p) - \text{ind}(q) = 1$ . In this case one can define an integer  $n(p, q)$  by assigning a number  $+1$  or  $-1$  to every connecting orbit and taking the sum. Let  $\varphi(t)$  be such a connecting orbit meaning a solution of  $\dot{x} = -\nabla f$  with  $\lim_{t \rightarrow -\infty} \varphi(t) = p$  and  $\lim_{t \rightarrow +\infty} \varphi(t) = q$ . Then  $\langle p \rangle$  induces an orientation on the orthogonal complement  $E_\varphi^u(p)$  of  $v = \lim_{t \rightarrow -\infty} |\dot{\varphi}(t)|^{-1} \dot{\varphi}(t)$  in  $E^u(p)$ . Then the negative gradient flow induces an isomorphism between  $E_\varphi^u(p)$  onto  $E^u(p)$  and we define  $n_\varphi$  to be  $+1$  and  $-1$  according to whether this map preserved or reverse the orientation.

Define

$$n(p, q) = \sum_{\varphi} n_\varphi$$

where the sum runs over all orbits of  $\dot{x} = -\nabla f$  connecting  $p$  and  $q$ . Then Witten's boundary operator  $\partial : C_{r+1} \rightarrow C_r$  of the chain complex is defined by

$$\partial \langle p \rangle = \sum_q n(p, q) \langle q \rangle$$

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where  $q$  runs over all critical points of index  $r$ .

Witten claims that  $\partial$  is a boundary operator, i.e.  $\partial^2 = 0$ , therefore

$$0 \rightarrow C_n \rightarrow C_{n-1} \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow 0$$

is a chain complex. Define for  $r = 0, 1, 2, \dots, n$ ,

$$H_r^W(M, \mathbb{Z}) = \frac{\ker\{\partial : C_r \rightarrow C_{r-1}\}}{\partial(C_{r+1})}$$

1.  $H_*^W(M, \mathbb{Z})$  are independent of the choice of the Riemannian metric on  $M$ ;
2.  $H_*^W(M, \mathbb{Z}) = H_*(M, \mathbb{Z})$ , the usual homology groups of  $M$ .

To compute the homology groups of real Grassmann manifold by Witten complex comes from the work of Feng Hui-tao on  $G_{5,2}(\mathbb{R})$ (see [5]) for provide a nontrivial example of Witten's method. In [6], Qiao Pei-zhi gives the result on  $\mathbb{R}P^n$ , i.e.  $G_{n+1,1}(\mathbb{R})$ . In [7], Yang Ying gives the result on  $G_{n,2}(\mathbb{R})$ . In this article we will give the result on  $G_{n,m}(\mathbb{R})$ . Computation of the homology groups of  $G_{n,m}(\mathbb{C})$  by Witten complex see [4]. We must point out that the computation of the homology groups of Grassmann manifold has been known to topologist by use Schubert calculus(see [10] and [11]).

## 2 A Morse funtion on the Grassmann manifold $G_{n,m}(\mathbb{R})$

Let the Grassmann manifold  $G_{n,m}(\mathbb{R})$  is the set of all  $m$ -dimensional linear subspaces of  $n$ -dimensional real vector space  $\mathbb{R}^n$ . Set

$$M = \left\{ (x_{\alpha k}) = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \dots & \dots & \dots & \dots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix} \middle| x_{\alpha k} \in \mathbb{R}, \text{rank}(x_{\alpha k}) = m \right\},$$

then the Grassmann manifold  $G_{n,m}(\mathbb{R})$  can be defined by

$$G_{n,m}(\mathbb{R}) = M/GL(m, \mathbb{R}).$$

Let  $\pi : M \rightarrow G_{n,m}(\mathbb{R})$  is the standard projection and define

$$U_{i_1 i_2 \cdots i_m} = \left\{ (x_{\alpha k}) \middle| x_{\alpha k} \in M, \begin{pmatrix} x_{1i_1} & x_{1i_2} & \cdots & x_{1i_m} \\ x_{2i_1} & x_{2i_2} & \cdots & x_{2i_m} \\ \dots & \dots & \dots & \dots \\ x_{mi_1} & x_{mi_2} & \cdots & x_{mi_m} \end{pmatrix} = I_m \right\}, 1 \leq i_1 < i_2 < \cdots < i_m \leq n.$$

where  $I_m$  is the identity matrix of rank  $m$ .

$$\phi_{i_1 i_2 \cdots i_m} : U_{i_1 i_2 \cdots i_m} \rightarrow \mathbb{R}^{m(n-m)},$$

$$\phi_{i_1 i_2 \cdots i_m}(x_{\alpha k}) = (x_{1k_1}, x_{1k_2}, \cdots, x_{1k_{n-m}}, x_{2k_1}, x_{2k_2}, \cdots, x_{2k_{n-m}}, \cdots, x_{mk_1}, x_{mk_2}, \cdots, x_{mk_{n-m}}).$$

where  $k_s \neq i_1, i_2, \dots, i_m$ ;  $s = 1, 2, \dots, n-m$ ;  $1 \leq k_1 < \cdots < k_{n-m} \leq n$ .

Set

$$\tilde{U}_{i_1 i_2 \dots i_m} = \pi(U_{i_1 i_2 \dots i_m}), \quad \tilde{\phi}_{i_1 i_2 \dots i_m} = \phi_{i_1 i_2 \dots i_m} \circ \pi^{-1} : \tilde{U}_{i_1 i_2 \dots i_m} \rightarrow \mathbb{R}^{m(n-m)}.$$

Then  $\{(\tilde{U}_{i_1 i_2 \dots i_m}, \tilde{\phi}_{i_1 i_2 \dots i_m}) | 1 \leq i_1 < i_2 < \dots < i_m \leq n\}$  are the local coordinate covering of  $G_{n,m}(\mathbb{R})$ .

Here we will construct the Morse function on  $G_{n,m}(\mathbb{R})$  by the same way as in [5],[6],[7].

Let  $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n$  be fixed numbers, for any  $(x_{\alpha k}) \in M$ , set

$$\xi_\alpha = (x_{\alpha 1}, x_{\alpha 2}, \dots, x_{\alpha n}),$$

$$\bar{\xi}_\alpha = \left( \frac{x_{\alpha 1}}{\lambda_1}, \frac{x_{\alpha 2}}{\lambda_2}, \dots, \frac{x_{\alpha n}}{\lambda_n} \right), \quad \alpha = 1, 2, \dots, m.$$

$$\Delta = \det \begin{pmatrix} \langle \xi_1, \xi_1 \rangle & \langle \xi_1, \xi_2 \rangle & \cdots & \langle \xi_1, \xi_m \rangle \\ \langle \xi_2, \xi_1 \rangle & \langle \xi_2, \xi_2 \rangle & \cdots & \langle \xi_2, \xi_m \rangle \\ \dots & \dots & \dots & \dots \\ \langle \xi_m, \xi_1 \rangle & \langle \xi_m, \xi_2 \rangle & \cdots & \langle \xi_m, \xi_m \rangle \end{pmatrix}$$

$$\bar{\Delta} = \det \begin{pmatrix} \langle \bar{\xi}_1, \bar{\xi}_1 \rangle & \langle \bar{\xi}_1, \bar{\xi}_2 \rangle & \cdots & \langle \bar{\xi}_1, \bar{\xi}_m \rangle \\ \langle \bar{\xi}_2, \bar{\xi}_1 \rangle & \langle \bar{\xi}_2, \bar{\xi}_2 \rangle & \cdots & \langle \bar{\xi}_2, \bar{\xi}_m \rangle \\ \dots & \dots & \dots & \dots \\ \langle \bar{\xi}_m, \bar{\xi}_1 \rangle & \langle \bar{\xi}_m, \bar{\xi}_2 \rangle & \cdots & \langle \bar{\xi}_m, \bar{\xi}_m \rangle \end{pmatrix}$$

where

$$\langle \xi_i, \xi_j \rangle = x_{i1}x_{j1} + x_{i2}x_{j2} + \dots + x_{im}x_{jm}, \quad i, j = 1, 2, \dots, m.$$

we define a function  $f$  on  $M$  by

$$f \doteq \frac{\Delta}{\bar{\Delta}}$$

**Lemma 1.** *The function  $f$  is defined on the Grassmann manifold  $G_{n,m}(\mathbb{R})$ .*

*Proof.* Let  $A \in GL(m, \mathbb{R})$ , because

$$\begin{pmatrix} \langle \xi_1, \xi_1 \rangle & \langle \xi_1, \xi_2 \rangle & \cdots & \langle \xi_1, \xi_m \rangle \\ \langle \xi_2, \xi_1 \rangle & \langle \xi_2, \xi_2 \rangle & \cdots & \langle \xi_2, \xi_m \rangle \\ \dots & \dots & \dots & \dots \\ \langle \xi_m, \xi_1 \rangle & \langle \xi_m, \xi_2 \rangle & \cdots & \langle \xi_m, \xi_m \rangle \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \dots & \dots & \dots & \dots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \dots & \dots & \dots & \dots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}^T$$

and

$$\begin{pmatrix} \langle \bar{\xi}_1, \bar{\xi}_1 \rangle & \langle \bar{\xi}_1, \bar{\xi}_2 \rangle & \cdots & \langle \bar{\xi}_1, \bar{\xi}_m \rangle \\ \langle \bar{\xi}_2, \bar{\xi}_1 \rangle & \langle \bar{\xi}_2, \bar{\xi}_2 \rangle & \cdots & \langle \bar{\xi}_2, \bar{\xi}_m \rangle \\ \dots & \dots & \dots & \dots \\ \langle \bar{\xi}_m, \bar{\xi}_1 \rangle & \langle \bar{\xi}_m, \bar{\xi}_2 \rangle & \cdots & \langle \bar{\xi}_m, \bar{\xi}_m \rangle \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \dots & \dots & \dots & \dots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda_1^2} & & & \\ & \frac{1}{\lambda_2^2} & & \\ & & \ddots & \\ & & & \frac{1}{\lambda_{m-1}^2} \\ & & & & \frac{1}{\lambda_m^2} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \dots & \dots & \dots & \dots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}^T$$

where  $\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \dots & \dots & \dots & \dots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}^T$  is the transposed matrix of  $\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \dots & \dots & \dots & \dots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}$ .

By computation we have

$$\begin{pmatrix} \langle A\xi_1, A\xi_1 \rangle & \langle A\xi_1, A\xi_2 \rangle & \cdots & \langle A\xi_1, A\xi_m \rangle \\ \langle A\xi_2, A\xi_1 \rangle & \langle A\xi_2, A\xi_2 \rangle & \cdots & \langle A\xi_2, A\xi_m \rangle \\ \dots & \dots & \dots & \dots \\ \langle A\xi_m, A\xi_1 \rangle & \langle A\xi_m, A\xi_2 \rangle & \cdots & \langle A\xi_m, A\xi_m \rangle \end{pmatrix} = A \begin{pmatrix} \langle \xi_1, \xi_1 \rangle & \langle \xi_1, \xi_2 \rangle & \cdots & \langle \xi_1, \xi_m \rangle \\ \langle \xi_2, \xi_1 \rangle & \langle \xi_2, \xi_2 \rangle & \cdots & \langle \xi_2, \xi_m \rangle \\ \dots & \dots & \dots & \dots \\ \langle \xi_m, \xi_1 \rangle & \langle \xi_m, \xi_2 \rangle & \cdots & \langle \xi_m, \xi_m \rangle \end{pmatrix} A^T$$

and

$$\begin{pmatrix} \langle A\bar{\xi}_1, A\bar{\xi}_1 \rangle & \langle A\bar{\xi}_1, A\bar{\xi}_2 \rangle & \cdots & \langle A\bar{\xi}_1, A\bar{\xi}_m \rangle \\ \langle A\bar{\xi}_2, A\bar{\xi}_1 \rangle & \langle A\bar{\xi}_2, A\bar{\xi}_2 \rangle & \cdots & \langle A\bar{\xi}_2, A\bar{\xi}_m \rangle \\ \dots & \dots & \dots & \dots \\ \langle A\bar{\xi}_m, A\bar{\xi}_1 \rangle & \langle A\bar{\xi}_m, A\bar{\xi}_2 \rangle & \cdots & \langle A\bar{\xi}_m, A\bar{\xi}_m \rangle \end{pmatrix} = A \begin{pmatrix} \langle \bar{\xi}_1, \bar{\xi}_1 \rangle & \langle \bar{\xi}_1, \bar{\xi}_2 \rangle & \cdots & \langle \bar{\xi}_1, \bar{\xi}_m \rangle \\ \langle \bar{\xi}_2, \bar{\xi}_1 \rangle & \langle \bar{\xi}_2, \bar{\xi}_2 \rangle & \cdots & \langle \bar{\xi}_2, \bar{\xi}_m \rangle \\ \dots & \dots & \dots & \dots \\ \langle \bar{\xi}_m, \bar{\xi}_1 \rangle & \langle \bar{\xi}_m, \bar{\xi}_2 \rangle & \cdots & \langle \bar{\xi}_m, \bar{\xi}_m \rangle \end{pmatrix} A^T$$

where  $A^T$  is the transposed matrix of  $A$ . So we have the function  $\frac{\Delta}{\bar{\Delta}}$  is invariant under the natural left action of the group  $GL(m, \mathbb{R})$ , then we get  $f$  is defined on the Grassmann manifold  $G_{n,m}(\mathbb{R})$ .  $\square$

**Lemma 2.** *The function  $f$  is the Morse function on the Grassmann manifold  $G_{n,m}(\mathbb{R})$ , has and only has one critical point  $O_{i_1 i_2 \dots i_m} = \tilde{\phi}_{i_1 i_2 \dots i_m}^{-1}(\underbrace{0, 0, \dots, 0}_{m(n-m)})$  on each  $\tilde{U}_{i_1 i_2 \dots i_m}$ ,  $1 \leq i_1 < i_2 < \dots < i_m \leq n$ .*

*Proof.* By the definition

$$f = \frac{\Delta}{\bar{\Delta}}$$

so on the local coordinate systems  $\tilde{U}_{i_1 i_2 \dots i_m}$  we have

$$\frac{\partial f}{\partial x_{ij}} = \frac{\frac{\partial \Delta}{\partial x_{ij}} \cdot \bar{\Delta} - \Delta \cdot \frac{\partial \bar{\Delta}}{\partial x_{ij}}}{\bar{\Delta} \cdot \bar{\Delta}}. \quad (1)$$

where  $i = 1, 2, \dots, m$ ;  $j = k_s \neq i_1, i_2, \dots, i_m$ . By computation we have

$$\frac{\partial \Delta}{\partial x_{ij}} = 2 \det \begin{pmatrix} \langle \xi_1, \xi_1 \rangle & \langle \xi_1, \xi_2 \rangle & \cdots & \langle \xi_1, \xi_{i-1} \rangle & x_{1j} & \langle \xi_1, \xi_{i+1} \rangle & \cdots & \langle \xi_1, \xi_m \rangle \\ \langle \xi_2, \xi_1 \rangle & \langle \xi_2, \xi_2 \rangle & \cdots & \langle \xi_2, \xi_{i-1} \rangle & x_{2j} & \langle \xi_2, \xi_{i+1} \rangle & \cdots & \langle \xi_2, \xi_m \rangle \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \langle \xi_m, \xi_1 \rangle & \langle \xi_m, \xi_2 \rangle & \cdots & \langle \xi_m, \xi_{i-1} \rangle & x_{mj} & \langle \xi_m, \xi_{i+1} \rangle & \cdots & \langle \xi_m, \xi_m \rangle \end{pmatrix}$$

$$\frac{\partial \bar{\Delta}}{\partial x_{ij}} = 2 \det \begin{pmatrix} \langle \bar{\xi}_1, \bar{\xi}_1 \rangle & \langle \bar{\xi}_1, \bar{\xi}_2 \rangle & \cdots & \langle \bar{\xi}_1, \bar{\xi}_{i-1} \rangle & \frac{x_{1j}}{\lambda_j^2} & \langle \bar{\xi}_1, \bar{\xi}_{i+1} \rangle & \cdots & \langle \bar{\xi}_1, \bar{\xi}_m \rangle \\ \langle \bar{\xi}_2, \bar{\xi}_1 \rangle & \langle \bar{\xi}_2, \bar{\xi}_2 \rangle & \cdots & \langle \bar{\xi}_2, \bar{\xi}_{i-1} \rangle & \frac{x_{2j}}{\lambda_j^2} & \langle \bar{\xi}_2, \bar{\xi}_{i+1} \rangle & \cdots & \langle \bar{\xi}_2, \bar{\xi}_m \rangle \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \langle \bar{\xi}_m, \bar{\xi}_1 \rangle & \langle \bar{\xi}_m, \bar{\xi}_2 \rangle & \cdots & \langle \bar{\xi}_m, \bar{\xi}_{i-1} \rangle & \frac{x_{mj}}{\lambda_j^2} & \langle \bar{\xi}_m, \bar{\xi}_{i+1} \rangle & \cdots & \langle \bar{\xi}_m, \bar{\xi}_m \rangle \end{pmatrix}$$

then

$$\frac{\partial \Delta}{\partial x_{ij}} \cdot \bar{\Delta} - \Delta \cdot \frac{\partial \bar{\Delta}}{\partial x_{ij}} =$$

$$2(x_{1j}A_{1i} + x_{2j}A_{2i} + \cdots + x_{mj}A_{mi}) \cdot \bar{\Delta} - 2\left(\frac{x_{1j}\bar{A}_{1i}}{\lambda_j^2} + \frac{x_{2j}\bar{A}_{2i}}{\lambda_j^2} + \cdots + \frac{x_{mj}\bar{A}_{mi}}{\lambda_j^2}\right) \cdot \Delta \quad (2)$$

where  $A_{ij}$  is  $(i, j)$  cofactor of

$$\begin{pmatrix} \langle \xi_1, \xi_1 \rangle & \langle \xi_1, \xi_2 \rangle & \cdots & \langle \xi_1, \xi_m \rangle \\ \langle \xi_2, \xi_1 \rangle & \langle \xi_2, \xi_2 \rangle & \cdots & \langle \xi_2, \xi_m \rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle \xi_m, \xi_1 \rangle & \langle \xi_m, \xi_2 \rangle & \cdots & \langle \xi_m, \xi_m \rangle \end{pmatrix},$$

$\bar{A}_{ij}$  is  $(i, j)$  cofactor of

$$\begin{pmatrix} \langle \bar{\xi}_1, \bar{\xi}_1 \rangle & \langle \bar{\xi}_1, \bar{\xi}_2 \rangle & \cdots & \langle \bar{\xi}_1, \bar{\xi}_m \rangle \\ \langle \bar{\xi}_2, \bar{\xi}_1 \rangle & \langle \bar{\xi}_2, \bar{\xi}_2 \rangle & \cdots & \langle \bar{\xi}_2, \bar{\xi}_m \rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle \bar{\xi}_m, \bar{\xi}_1 \rangle & \langle \bar{\xi}_m, \bar{\xi}_2 \rangle & \cdots & \langle \bar{\xi}_m, \bar{\xi}_m \rangle \end{pmatrix}$$

so we have  $O_{i_1 i_2 \dots i_m} = \tilde{\phi}_{i_1 i_2 \dots i_m}^{-1}(\underbrace{0, 0, \dots, 0}_{m(n-m)})$  is a critical point of  $f$ . In the following we can

proof that  $O_{i_1 i_2 \dots i_m}$  is the only one critical point on each  $\tilde{U}_{i_1 i_2 \dots i_m}$ .

If there exists another critical point  $p \neq O_{i_1 i_2 \dots i_m}$ , then for some  $i, j$  we have  $x_{ij} \neq 0$ , and according to the definition of critical point  $\frac{\partial f}{\partial x_{ij}}(p) = 0$ , then  $\frac{\partial \Delta}{\partial x_{ij}} \cdot \bar{\Delta} - \Delta \cdot \frac{\partial \bar{\Delta}}{\partial x_{ij}} = 0$ , so

$$(x_{1j}A_{1i} + x_{2j}A_{2i} + \cdots + x_{mj}A_{mi}) \cdot \bar{\Delta} - \left(\frac{x_{1j}\bar{A}_{1i}}{\lambda_j^2} + \frac{x_{2j}\bar{A}_{2i}}{\lambda_j^2} + \cdots + \frac{x_{mj}\bar{A}_{mi}}{\lambda_j^2}\right) \cdot \Delta = 0$$

If we get  $i$  from 1 to  $m$ , by them we can get a linear system of equations, because  $x_{ij} \neq 0$  so we get

$$\det \begin{pmatrix} A_{11} \cdot \bar{\Delta} - \frac{1}{\lambda_j^2} \bar{A}_{11} \cdot \Delta & A_{21} \cdot \bar{\Delta} - \frac{1}{\lambda_j^2} \bar{A}_{21} \cdot \Delta & \cdots & A_{m1} \cdot \bar{\Delta} - \frac{1}{\lambda_j^2} \bar{A}_{m1} \cdot \Delta \\ A_{12} \cdot \bar{\Delta} - \frac{1}{\lambda_j^2} \bar{A}_{12} \cdot \Delta & A_{22} \cdot \bar{\Delta} - \frac{1}{\lambda_j^2} \bar{A}_{22} \cdot \Delta & \cdots & A_{m2} \cdot \bar{\Delta} - \frac{1}{\lambda_j^2} \bar{A}_{m2} \cdot \Delta \\ \cdots & \cdots & \cdots & \cdots \\ A_{1m} \cdot \bar{\Delta} - \frac{1}{\lambda_j^2} \bar{A}_{1m} \cdot \Delta & A_{2m} \cdot \bar{\Delta} - \frac{1}{\lambda_j^2} \bar{A}_{2m} \cdot \Delta & \cdots & A_{mm} \cdot \bar{\Delta} - \frac{1}{\lambda_j^2} \bar{A}_{mm} \cdot \Delta \end{pmatrix} = 0$$

we denote the above matrix by  $B$ , because

$$\begin{pmatrix} \langle \xi_1, \xi_1 \rangle & \langle \xi_1, \xi_2 \rangle & \cdots & \langle \xi_1, \xi_m \rangle \\ \langle \xi_2, \xi_1 \rangle & \langle \xi_2, \xi_2 \rangle & \cdots & \langle \xi_2, \xi_m \rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle \xi_m, \xi_1 \rangle & \langle \xi_m, \xi_2 \rangle & \cdots & \langle \xi_m, \xi_m \rangle \end{pmatrix} B \\ = \begin{pmatrix} \langle \bar{\xi}_1, \bar{\xi}_1 \rangle - \frac{\langle \xi_1, \xi_1 \rangle}{\lambda_j^2} & \langle \bar{\xi}_1, \bar{\xi}_2 \rangle - \frac{\langle \xi_1, \xi_2 \rangle}{\lambda_j^2} & \cdots & \langle \bar{\xi}_1, \bar{\xi}_m \rangle - \frac{\langle \xi_1, \xi_m \rangle}{\lambda_j^2} \\ \langle \bar{\xi}_2, \bar{\xi}_1 \rangle - \frac{\langle \xi_2, \xi_1 \rangle}{\lambda_j^2} & \langle \bar{\xi}_2, \bar{\xi}_2 \rangle - \frac{\langle \xi_2, \xi_2 \rangle}{\lambda_j^2} & \cdots & \langle \bar{\xi}_2, \bar{\xi}_m \rangle - \frac{\langle \xi_2, \xi_m \rangle}{\lambda_j^2} \\ \cdots & \cdots & \cdots & \cdots \\ \langle \bar{\xi}_m, \bar{\xi}_1 \rangle - \frac{\langle \xi_m, \xi_1 \rangle}{\lambda_j^2} & \langle \bar{\xi}_m, \bar{\xi}_2 \rangle - \frac{\langle \xi_m, \xi_2 \rangle}{\lambda_j^2} & \cdots & \langle \bar{\xi}_m, \bar{\xi}_m \rangle - \frac{\langle \xi_m, \xi_m \rangle}{\lambda_j^2} \end{pmatrix} C$$

where

$$C = \begin{pmatrix} \bar{A}_{11} \cdot \Delta & \bar{A}_{21} \cdot \Delta & \cdots & \bar{A}_{m1} \cdot \Delta \\ \bar{A}_{12} \cdot \Delta & \bar{A}_{22} \cdot \Delta & \cdots & \bar{A}_{m2} \cdot \Delta \\ \cdots & \cdots & \cdots & \cdots \\ \bar{A}_{1m} \cdot \Delta & \bar{A}_{2m} \cdot \Delta & \cdots & \bar{A}_{mm} \cdot \Delta \end{pmatrix} = \Delta \cdot \begin{pmatrix} \bar{A}_{11} & \bar{A}_{21} & \cdots & \bar{A}_{m1} \\ \bar{A}_{12} & \bar{A}_{22} & \cdots & \bar{A}_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ \bar{A}_{1m} & \bar{A}_{2m} & \cdots & \bar{A}_{mm} \end{pmatrix}.$$

by computation we have

$$= X \begin{pmatrix} \langle \bar{\xi}_1, \bar{\xi}_1 \rangle - \frac{\langle \xi_1, \xi_1 \rangle}{\lambda_j^2} & \langle \bar{\xi}_1, \bar{\xi}_2 \rangle - \frac{\langle \xi_1, \xi_2 \rangle}{\lambda_j^2} & \dots & \langle \bar{\xi}_1, \bar{\xi}_m \rangle - \frac{\langle \xi_1, \xi_m \rangle}{\lambda_j^2} \\ \langle \bar{\xi}_2, \bar{\xi}_1 \rangle - \frac{\langle \xi_2, \xi_1 \rangle}{\lambda_j^2} & \langle \bar{\xi}_2, \bar{\xi}_2 \rangle - \frac{\langle \xi_2, \xi_2 \rangle}{\lambda_j^2} & \dots & \langle \bar{\xi}_2, \bar{\xi}_m \rangle - \frac{\langle \xi_2, \xi_m \rangle}{\lambda_j^2} \\ \dots & \dots & \dots & \dots \\ \langle \bar{\xi}_m, \bar{\xi}_1 \rangle - \frac{\langle \xi_m, \xi_1 \rangle}{\lambda_j^2} & \langle \bar{\xi}_m, \bar{\xi}_2 \rangle - \frac{\langle \xi_m, \xi_2 \rangle}{\lambda_j^2} & \dots & \langle \bar{\xi}_m, \bar{\xi}_m \rangle - \frac{\langle \xi_m, \xi_m \rangle}{\lambda_j^2} \end{pmatrix} \\ \\ \begin{pmatrix} \frac{1}{\lambda_1^2} - \frac{1}{\lambda_j^2} & & & & & & \\ & \frac{1}{\lambda_2^2} - \frac{1}{\lambda_j^2} & & & & & \\ & & \dots & & & & \\ & & & \frac{1}{\lambda_{j-1}^2} - \frac{1}{\lambda_j^2} & & & \\ & & & & \frac{1}{\lambda_{j+1}^2} - \frac{1}{\lambda_j^2} & & \\ & & & & & \dots & \\ & & & & & & \frac{1}{\lambda_{n-1}^2} - \frac{1}{\lambda_j^2} \\ & & & & & & & \frac{1}{\lambda_n^2} - \frac{1}{\lambda_j^2} \end{pmatrix} X^T$$

where

$$X = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1(j-1)} & x_{1(j+1)} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2(j-1)} & x_{2(j+1)} & \dots & x_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x_{m1} & x_{m2} & \dots & x_{m(j-1)} & x_{m(j+1)} & \dots & x_{mn} \end{pmatrix}$$

$X^T$  is the transposed matrix of  $X$ . Because  $\det B = 0$ , so we have

$$\det \begin{pmatrix} \langle \bar{\xi}_1, \bar{\xi}_1 \rangle - \frac{\langle \xi_1, \xi_1 \rangle}{\lambda_j^2} & \langle \bar{\xi}_1, \bar{\xi}_2 \rangle - \frac{\langle \xi_1, \xi_2 \rangle}{\lambda_j^2} & \dots & \langle \bar{\xi}_1, \bar{\xi}_m \rangle - \frac{\langle \xi_1, \xi_m \rangle}{\lambda_j^2} \\ \langle \bar{\xi}_2, \bar{\xi}_1 \rangle - \frac{\langle \xi_2, \xi_1 \rangle}{\lambda_j^2} & \langle \bar{\xi}_2, \bar{\xi}_2 \rangle - \frac{\langle \xi_2, \xi_2 \rangle}{\lambda_j^2} & \dots & \langle \bar{\xi}_2, \bar{\xi}_m \rangle - \frac{\langle \xi_2, \xi_m \rangle}{\lambda_j^2} \\ \dots & \dots & \dots & \dots \\ \langle \bar{\xi}_m, \bar{\xi}_1 \rangle - \frac{\langle \xi_m, \xi_1 \rangle}{\lambda_j^2} & \langle \bar{\xi}_m, \bar{\xi}_2 \rangle - \frac{\langle \xi_m, \xi_2 \rangle}{\lambda_j^2} & \dots & \langle \bar{\xi}_m, \bar{\xi}_m \rangle - \frac{\langle \xi_m, \xi_m \rangle}{\lambda_j^2} \end{pmatrix} = 0$$

because it is independent of the value of  $x_{kl}$  ( $k = 1, 2, \dots, m; l = 1, 2, \dots, j-1, j+1, \dots, n.$ ), so we must have  $\frac{1}{\lambda_l^2} - \frac{1}{\lambda_j^2} = 0$  for some  $l$ , then we get  $\lambda_l = \lambda_j$ , which contradicts to  $\lambda_l \neq \lambda_j$ , ( $l \neq j$ ). So we get  $x_{ij} = 0$ , there non-exists another critical point  $p \neq O_{i_1 i_2 \dots i_m}$  on  $\tilde{U}_{i_1 i_2 \dots i_m}$ .  $\square$

**Lemma 3.** *The critical point  $O_{i_1 i_2 \dots i_m}$  on each  $\tilde{U}_{i_1 i_2 \dots i_m}$  ( $1 \leq i_1 < i_2 < \dots < i_m \leq n$ ) is non-degenerate, and  $\text{ind}(O_{i_1 i_2 \dots i_m}) = i_1 + i_2 + \dots + i_m - \frac{1}{2}m(m+1)$ .*

*Proof.* On  $\tilde{U}_{i_1 i_2 \dots i_m}$  we have

$$\frac{\partial^2 f}{\partial x_{kl} \partial x_{ij}} = \frac{\partial}{\partial x_{kl}} \left( \frac{\frac{\partial \Delta}{\partial x_{ij}} \cdot \bar{\Delta} - \Delta \cdot \frac{\partial \bar{\Delta}}{\partial x_{ij}}}{\bar{\Delta} \cdot \bar{\Delta}} \right) \\ = \frac{1}{\Delta^4} \left( \frac{\partial}{\partial x_{kl}} \left( \frac{\partial \Delta}{\partial x_{ij}} \cdot \bar{\Delta} - \Delta \cdot \frac{\partial \bar{\Delta}}{\partial x_{ij}} \right) \cdot \bar{\Delta}^2 - \left( \frac{\partial \Delta}{\partial x_{ij}} \cdot \bar{\Delta} - \Delta \cdot \frac{\partial \bar{\Delta}}{\partial x_{ij}} \right) \frac{\partial}{\partial x_{kl}} \bar{\Delta}^2 \right)$$

$$= \frac{1}{\bar{\Delta}^4} \left( \frac{\partial^2 \Delta}{\partial x_{kl} \partial x_{ij}} \cdot \bar{\Delta}^3 - \frac{\partial \Delta}{\partial x_{ij}} \frac{\partial \bar{\Delta}}{\partial x_{kl}} \cdot \bar{\Delta}^2 - \frac{\partial \Delta}{\partial x_{kl}} \frac{\partial \bar{\Delta}}{\partial x_{ij}} \cdot \bar{\Delta}^2 - \Delta \frac{\partial^2 \bar{\Delta}}{\partial x_{kl} \partial x_{ij}} \bar{\Delta}^2 - 2\Delta \frac{\partial \bar{\Delta}}{\partial x_{ij}} \frac{\partial \bar{\Delta}}{\partial x_{kl}} \cdot \bar{\Delta} \right)$$

When  $k < i, l < j$ , then

$$\frac{\partial^2 \Delta}{\partial x_{kl} \partial x_{ij}} =$$

$$2 \det \begin{pmatrix} \langle \xi_1, \xi_1 \rangle & \cdots & \langle \xi_1, \xi_k \rangle & \cdots & \langle \xi_1, \xi_{i-1} \rangle & x_{1j} & \langle \xi_1, \xi_{i+1} \rangle & \cdots & \langle \xi_1, \xi_m \rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle \xi_{k-1}, \xi_1 \rangle & \cdots & \langle \xi_{k-1}, \xi_k \rangle & \cdots & \langle \xi_{k-1}, \xi_{i-1} \rangle & x_{(k-1)j} & \langle \xi_{k-1}, \xi_{i+1} \rangle & \cdots & \langle \xi_{k-1}, \xi_m \rangle \\ x_{1l} & \cdots & x_{kl} & \cdots & x_{(i-1)l} & 0 & x_{(i+1)l} & \cdots & x_{ml} \\ \langle \xi_{k+1}, \xi_1 \rangle & \cdots & \langle \xi_{k+1}, \xi_k \rangle & \cdots & \langle \xi_{k+1}, \xi_{i-1} \rangle & x_{(k+1)j} & \langle \xi_{k+1}, \xi_{i+1} \rangle & \cdots & \langle \xi_{k+1}, \xi_m \rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle \xi_m, \xi_1 \rangle & \cdots & \langle \xi_m, \xi_k \rangle & \cdots & \langle \xi_m, \xi_{i-1} \rangle & x_{mj} & \langle \xi_m, \xi_{i+1} \rangle & \cdots & \langle \xi_m, \xi_m \rangle \end{pmatrix}$$

$$+ 2 \det \begin{pmatrix} \langle \xi_1, \xi_1 \rangle & \cdots & \langle \xi_1, \xi_{k-1} \rangle & x_{1l} & \cdots & \langle \xi_1, \xi_{i-1} \rangle & x_{1j} & \cdots & \langle \xi_1, \xi_m \rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle \xi_{k-1}, \xi_1 \rangle & \cdots & \langle \xi_{k-1}, \xi_{k-1} \rangle & x_{(k-1)l} & \cdots & \langle \xi_{k-1}, \xi_{i-1} \rangle & x_{(k-1)j} & \cdots & \langle \xi_{k-1}, \xi_m \rangle \\ \langle \xi_k, \xi_1 \rangle & \cdots & \langle \xi_k, \xi_1 \rangle & x_{kl} & \cdots & \langle \xi_k, \xi_{i-1} \rangle & x_{kj} & \cdots & \langle \xi_k, \xi_m \rangle \\ \langle \xi_{k+1}, \xi_1 \rangle & \cdots & \langle \xi_{k+1}, \xi_{k-1} \rangle & x_{(k+1)l} & \cdots & \langle \xi_{k+1}, \xi_{i-1} \rangle & x_{(k+1)j} & \cdots & \langle \xi_{k+1}, \xi_m \rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle \xi_m, \xi_1 \rangle & \cdots & \langle \xi_m, \xi_{k_1} \rangle & x_{ml} & \cdots & \langle \xi_m, \xi_{i-1} \rangle & x_{mj} & \cdots & \langle \xi_m, \xi_m \rangle \end{pmatrix}.$$

so we have  $\frac{\partial^2 \Delta}{\partial x_{kl} \partial x_{ij}}(O_{i_1 i_2 \dots i_m}) = 0$ . When  $k > i, l < j$  and  $k \neq i, l > j$  the result is the same.

By the same way we can get  $\frac{\partial^2 \bar{\Delta}}{\partial x_{kl} \partial x_{ij}}(O_{i_1 i_2 \dots i_m}) = 0$ .

When  $k \neq i, l = j$ , then

$$\frac{\partial^2 \Delta}{\partial x_{kj} \partial x_{ij}} =$$

$$2 \det \begin{pmatrix} \langle \xi_1, \xi_1 \rangle & \cdots & \langle \xi_1, \xi_k \rangle & \cdots & \langle \xi_1, \xi_{i-1} \rangle & x_{1j} & \langle \xi_1, \xi_{i+1} \rangle & \cdots & \langle \xi_1, \xi_m \rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle \xi_{k-1}, \xi_1 \rangle & \cdots & \langle \xi_{k-1}, \xi_k \rangle & \cdots & \langle \xi_{k-1}, \xi_{i-1} \rangle & x_{(k-1)j} & \langle \xi_{k-1}, \xi_{i+1} \rangle & \cdots & \langle \xi_{k-1}, \xi_m \rangle \\ x_{1j} & \cdots & x_{kj} & \cdots & x_{(i-1)j} & 1 & x_{(i+1)j} & \cdots & x_{mj} \\ \langle \xi_{k+1}, \xi_1 \rangle & \cdots & \langle \xi_{k+1}, \xi_k \rangle & \cdots & \langle \xi_{k+1}, \xi_{i-1} \rangle & x_{(k+1)j} & \langle \xi_{k+1}, \xi_{i+1} \rangle & \cdots & \langle \xi_{k+1}, \xi_m \rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle \xi_m, \xi_1 \rangle & \cdots & \langle \xi_m, \xi_k \rangle & \cdots & \langle \xi_m, \xi_{i-1} \rangle & x_{mj} & \langle \xi_m, \xi_{i+1} \rangle & \cdots & \langle \xi_m, \xi_m \rangle \end{pmatrix}$$

$$= 2A_{ki}$$

because  $A_{ki}(O_{i_1 i_2 \dots i_m}) = 0$ , so we get  $\frac{\partial^2 \Delta}{\partial x_{kj} \partial x_{ij}}(O_{i_1 i_2 \dots i_m}) = 0$ , by the same way we can get

$\frac{\partial^2 \bar{\Delta}}{\partial x_{kj} \partial x_{ij}}(O_{i_1 i_2 \dots i_m}) = 0$ .

When  $k = i, l = j$ , then

$$\frac{\partial^2 \Delta}{\partial x_{ij} \partial x_{ij}} = 2 \det \begin{pmatrix} \langle \xi_1, \xi_1 \rangle & \cdots & \langle \xi_1, \xi_{i-1} \rangle & x_{1j} & \langle \xi_1, \xi_{i+1} \rangle & \cdots & \langle \xi_1, \xi_m \rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle \xi_{i-1}, \xi_1 \rangle & \cdots & \langle \xi_{i-1}, \xi_{i-1} \rangle & x_{(i-1)j} & \langle \xi_{i-1}, \xi_{i+1} \rangle & \cdots & \langle \xi_{i-1}, \xi_m \rangle \\ x_{1j} & \cdots & x_{(i-1)j} & 1 & x_{(i+1)j} & \cdots & x_{mj} \\ \langle \xi_{i+1}, \xi_1 \rangle & \cdots & \langle \xi_{i+1}, \xi_{i-1} \rangle & x_{(i+1)j} & \langle \xi_{i+1}, \xi_{i+1} \rangle & \cdots & \langle \xi_{i+1}, \xi_m \rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle \xi_m, \xi_1 \rangle & \cdots & \langle \xi_m, \xi_{i-1} \rangle & x_{mj} & \langle \xi_m, \xi_{i+1} \rangle & \cdots & \langle \xi_m, \xi_m \rangle \end{pmatrix}.$$

$$\frac{\partial^2 \bar{\Delta}}{\partial x_{ij} \partial x_{ij}} = 2 \det \begin{pmatrix} \langle \bar{\xi}_1, \bar{\xi}_1 \rangle & \cdots & \langle \bar{\xi}_1, \bar{\xi}_{i-1} \rangle & \frac{x_{1j}}{\lambda_j^2} & \langle \bar{\xi}_1, \bar{\xi}_{i+1} \rangle & \cdots & \langle \bar{\xi}_1, \bar{\xi}_m \rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle \bar{\xi}_{i-1}, \bar{\xi}_1 \rangle & \cdots & \langle \bar{\xi}_{i-1}, \bar{\xi}_{i-1} \rangle & \frac{x^{(i-1)j}}{\lambda_j^2} & \langle \bar{\xi}_{i-1}, \bar{\xi}_{i+1} \rangle & \cdots & \langle \bar{\xi}_{i-1}, \bar{\xi}_m \rangle \\ \frac{x_{1j}}{\lambda_j^2} & \cdots & \frac{x^{(i-1)j}}{\lambda_j^2} & \frac{1}{\lambda_j^2} & \frac{x^{(i+1)j}}{\lambda_j^2} & \cdots & \frac{x_{mj}}{\lambda_j^2} \\ \langle \bar{\xi}_{i+1}, \bar{\xi}_1 \rangle & \cdots & \langle \bar{\xi}_{i+1}, \bar{\xi}_{i-1} \rangle & \frac{x^{(i+1)j}}{\lambda_j^2} & \langle \bar{\xi}_{i+1}, \bar{\xi}_{i+1} \rangle & \cdots & \langle \bar{\xi}_{i+1}, \bar{\xi}_m \rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle \bar{\xi}_m, \bar{\xi}_1 \rangle & \cdots & \langle \bar{\xi}_m, \bar{\xi}_{i-1} \rangle & \frac{x_{mj}}{\lambda_j^2} & \langle \bar{\xi}_m, \bar{\xi}_{i+1} \rangle & \cdots & \langle \bar{\xi}_m, \bar{\xi}_m \rangle \end{pmatrix}.$$

so

$$\frac{\partial^2 \Delta}{\partial x_{ij} \partial x_{ij}}(O_{i_1 i_2 \cdots i_m}) = 2$$

$$\frac{\partial^2 \bar{\Delta}}{\partial x_{ij} \partial x_{ij}}(O_{i_1 i_2 \cdots i_m}) = 2 \frac{1}{\lambda_{i_1}^2 \lambda_{i_2}^2 \cdots \lambda_{i_{k-1}}^2 \lambda_j^2 \lambda_{i_{k+1}}^2 \cdots \lambda_{i_m}^2}$$

then by computation we have

$$\frac{\partial^2 f}{\partial x_{ij} \partial x_{ij}}(O_{i_1 i_2 \cdots i_m}) = 2 \lambda_{i_1}^2 \lambda_{i_2}^2 \cdots \lambda_{i_m}^2 \left( 1 - \frac{\lambda_{i_k}^2}{\lambda_j^2} \right),$$

here  $j = k_s$ .

So at the critical point  $O_{i_1 i_2 \cdots i_m}$  of  $f$  on each  $\tilde{U}_{i_1 i_2 \cdots i_m}$ , we have the Hessian matrix  $H_{O_{i_1 i_2 \cdots i_m}}(f)$

$$H_{O_{i_1 i_2 \cdots i_m}}(f) = 2 \lambda_{i_1}^2 \lambda_{i_2}^2 \cdots \lambda_{i_m}^2 \text{diag} \left( 1 - \frac{\lambda_{i_1}^2}{\lambda_{k_1}^2}, \cdots, 1 - \frac{\lambda_{i_1}^2}{\lambda_{k_{n-m}}^2}, \cdots, 1 - \frac{\lambda_{i_m}^2}{\lambda_{k_1}^2}, \cdots, 1 - \frac{\lambda_{i_m}^2}{\lambda_{k_{n-m}}^2} \right)$$

then the critical point  $O_{i_1 i_2 \cdots i_m}$  is non-degenerate. By the definition of Morse index of  $f$ , we have  $\text{ind}(O_{i_1 i_2 \cdots i_m}) = (i_1 - 1) + (i_2 - 2) + \cdots + (i_m - m) = i_1 + i_2 + \cdots + i_m - \frac{1}{2}m(m+1)$ .  $\square$

### 3 Riemannian metric on the Grassmann manifold $G_{n,m}(\mathbb{R})$

Lu Qi-keng introduce a Riemannian metric  $g$  on the Grassmann manifold  $G_{n,m}(\mathbb{R})$  in [8]. It has the following form

$$g = \text{tr}[(I + ZZ^T)^{-1} dZ(I + Z^T Z)^{-1} dZ^T]$$

where  $I$  is the identity matrix,

$$Z = \begin{pmatrix} x_{1k_1} & x_{1k_2} & \cdots & x_{1k_{n-m}} \\ x_{2k_1} & x_{2k_2} & \cdots & x_{2k_{n-m}} \\ \cdots & \cdots & \cdots & \cdots \\ x_{mk_1} & x_{mk_2} & \cdots & x_{mk_{n-m}} \end{pmatrix},$$

$Z^T$  is the transposed matrix of  $Z$ .



we have

$$I + ZZ^T = \begin{pmatrix} \langle \xi_1, \xi_1 \rangle & \langle \xi_1, \xi_2 \rangle & \cdots & \langle \xi_1, \xi_m \rangle \\ \langle \xi_2, \xi_1 \rangle & \langle \xi_2, \xi_2 \rangle & \cdots & \langle \xi_2, \xi_m \rangle \\ \dots & \dots & \dots & \dots \\ \langle \xi_m, \xi_1 \rangle & \langle \xi_m, \xi_2 \rangle & \cdots & \langle \xi_m, \xi_m \rangle \end{pmatrix}$$

$$(I + ZZ^T)^{-1} = \frac{1}{\Delta} \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{m1} \\ A_{12} & A_{22} & \cdots & A_{m2} \\ \dots & \dots & \dots & \dots \\ A_{1m} & A_{2m} & \cdots & A_{mm} \end{pmatrix}$$

Set

$$\begin{cases} \eta_{k_1} = (x_{1k_1}, x_{2k_1}, \dots, x_{mk_1}, 1, 0, \dots, 0) \\ \eta_{k_2} = (x_{1k_2}, x_{2k_2}, \dots, x_{mk_2}, 0, 1, \dots, 0) \\ \dots \\ \eta_{k_{n-m}} = (x_{1k_{n-m}}, x_{2k_{n-m}}, \dots, x_{mk_{n-m}}, 0, 0, \dots, 1) \end{cases}$$

so

$$I + Z^T Z = \begin{pmatrix} \langle \eta_{k_1}, \eta_{k_1} \rangle & \langle \eta_{k_1}, \eta_{k_2} \rangle & \cdots & \langle \eta_{k_1}, \eta_{k_{n-m}} \rangle \\ \langle \eta_{k_2}, \eta_{k_1} \rangle & \langle \eta_{k_2}, \eta_{k_2} \rangle & \cdots & \langle \eta_{k_2}, \eta_{k_{n-m}} \rangle \\ \dots & \dots & \dots & \dots \\ \langle \eta_{k_{n-m}}, \eta_{k_1} \rangle & \langle \eta_{k_{n-m}}, \eta_{k_2} \rangle & \cdots & \langle \eta_{k_{n-m}}, \eta_{k_{n-m}} \rangle \end{pmatrix}$$

$$R \doteq (I + Z^T Z)^{-1} = \frac{1}{\det(I + Z^T Z)} \begin{pmatrix} D_{11} & D_{21} & \cdots & D_{(n-m)1} \\ D_{12} & D_{22} & \cdots & D_{(n-m)2} \\ \dots & \dots & \dots & \dots \\ D_{1(n-m)} & D_{2(n-m)} & \cdots & D_{(n-m)(n-m)} \end{pmatrix}$$

where  $D_{ij}$  is  $(i, j)$  cofactor of  $I + Z^T Z$ .

**Lemma 4.** On  $\{(\tilde{U}_{i_1 i_2 \dots i_m}, \tilde{\phi}_{i_1 i_2 \dots i_m}) | 1 \leq i_1 < i_2 < \dots < i_m \leq n\}$ , the local matrix expression of the Riemannian metric  $g$  of  $G_{n,m}(\mathbb{R})$  is given by

$$G = \frac{1}{\Delta} \begin{pmatrix} A_{11}(I + Z^T Z)^{-1} & A_{21}(I + Z^T Z)^{-1} & \cdots & A_{m1}(I + Z^T Z)^{-1} \\ A_{12}(I + Z^T Z)^{-1} & A_{22}(I + Z^T Z)^{-1} & \cdots & A_{m2}(I + Z^T Z)^{-1} \\ \dots & \dots & \dots & \dots \\ A_{1m}(I + Z^T Z)^{-1} & A_{2m}(I + Z^T Z)^{-1} & \cdots & A_{mm}(I + Z^T Z)^{-1} \end{pmatrix}$$

*Proof.* By definition

$$g = \text{tr}[(I + ZZ^T)^{-1} dZ(I + Z^T Z)^{-1} dZ^T]$$

$$= \text{tr} \left[ \frac{1}{\Delta} \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{m1} \\ A_{12} & A_{22} & \cdots & A_{m2} \\ \dots & \dots & \dots & \dots \\ A_{1m} & A_{2m} & \cdots & A_{mm} \end{pmatrix} \begin{pmatrix} dX_1 \\ dX_2 \\ \vdots \\ dX_m \end{pmatrix} (I + Z^T Z)^{-1} \begin{pmatrix} dX_1^T & dX_2^T & \cdots & dX_m^T \end{pmatrix} \right]$$

$$= \frac{1}{\Delta} \begin{pmatrix} dX_1 & dX_2 & \cdots & dX_m \end{pmatrix} \begin{pmatrix} A_{11}R & A_{21}R & \cdots & A_{m1}R \\ A_{12}R & A_{22}R & \cdots & A_{m2}R \\ \dots & \dots & \dots & \dots \\ A_{1m}R & A_{2m}R & \cdots & A_{mm}R \end{pmatrix} \begin{pmatrix} dX_1^T \\ dX_2^T \\ \vdots \\ dX_m^T \end{pmatrix}$$

where  $X_\alpha = \text{row}_\alpha Z$ ,  $\alpha = 1, 2, \dots, m$ . So we get the result

$$G = \frac{1}{\Delta} \begin{pmatrix} A_{11}(I + Z^T Z)^{-1} & A_{21}(I + Z^T Z)^{-1} & \cdots & A_{m1}(I + Z^T Z)^{-1} \\ A_{12}(I + Z^T Z)^{-1} & A_{22}(I + Z^T Z)^{-1} & \cdots & A_{m2}(I + Z^T Z)^{-1} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1m}(I + Z^T Z)^{-1} & A_{2m}(I + Z^T Z)^{-1} & \cdots & A_{mm}(I + Z^T Z)^{-1} \end{pmatrix}$$

we can see that the matrix  $G$  is rank of  $m(n - m)$ .  $\square$

By computation we have

$$G^{-1} = \begin{pmatrix} \langle \xi_1, \xi_1 \rangle (I + Z^T Z) & \langle \xi_1, \xi_2 \rangle (I + Z^T Z) & \cdots & \langle \xi_1, \xi_m \rangle (I + Z^T Z) \\ \langle \xi_2, \xi_1 \rangle (I + Z^T Z) & \langle \xi_2, \xi_2 \rangle (I + Z^T Z) & \cdots & \langle \xi_2, \xi_m \rangle (I + Z^T Z) \\ \cdots & \cdots & \cdots & \cdots \\ \langle \xi_m, \xi_1 \rangle (I + Z^T Z) & \langle \xi_m, \xi_2 \rangle (I + Z^T Z) & \cdots & \langle \xi_m, \xi_m \rangle (I + Z^T Z) \end{pmatrix} \quad (3)$$

By simple computation, it is obviously that  $G$  is the unit matrix of  $m(n - m) \times m(n - m)$  at critical point  $O_{i_1 i_2 \dots i_m}$ .

## 4 The study of the dynamical system $\dot{x} = -\nabla f$

We can use the above-mentioned Riemannian metric to define the negative gradient vector field  $-\nabla f$  on  $G_{n,m}(\mathbb{R})$  for the Morse function  $f \doteq \frac{\Delta}{\Delta} : G_{n,m}(\mathbb{R}) \rightarrow \mathbb{R}$ , which has the following local expression on  $\{(\tilde{U}_{i_1 i_2 \dots i_m}, \tilde{\phi}_{i_1 i_2 \dots i_m}) | 1 \leq i_1 < i_2 < \dots < i_m \leq n\}$

$$-\nabla f = -((\nabla f)_{1k_1}, (\nabla f)_{1k_2}, \dots, (\nabla f)_{1k_{n-m}}, \dots, (\nabla f)_{mk_1}, (\nabla f)_{mk_2}, \dots, (\nabla f)_{mk_{n-m}}).$$

we can get the computation formula of  $-\nabla f$  by  $g(\nabla f, V) = Vf$ ,

$$-\nabla f = -g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i}$$

where  $g^{ij}$  is the inverse of the matrix expression of the Riemannian metric  $g$ .

By computation we have

$$\begin{aligned} (\nabla f)_{ik_s} &= \langle \xi_i, \xi_1 \rangle \langle \eta_{k_s}, \eta_{k_1} \rangle \frac{\partial f}{\partial x_{1k_1}} + \cdots + \langle \xi_i, \xi_1 \rangle \langle \eta_{k_s}, \eta_{k_{n-m}} \rangle \frac{\partial f}{\partial x_{1k_{n-m}}} \\ &+ \langle \xi_i, \xi_2 \rangle \langle \eta_{k_s}, \eta_{k_1} \rangle \frac{\partial f}{\partial x_{2k_1}} + \cdots + \langle \xi_i, \xi_2 \rangle \langle \eta_{k_s}, \eta_{k_{n-m}} \rangle \frac{\partial f}{\partial x_{2k_{n-m}}} \\ &+ \cdots \\ &+ \langle \xi_i, \xi_m \rangle \langle \eta_{k_s}, \eta_{k_1} \rangle \frac{\partial f}{\partial x_{mk_1}} + \cdots + \langle \xi_i, \xi_m \rangle \langle \eta_{k_s}, \eta_{k_{n-m}} \rangle \frac{\partial f}{\partial x_{mk_{n-m}}}. \end{aligned}$$

Because (1) and (2) we have

$$\frac{\partial f}{\partial x_{ij}} = \frac{1}{\Delta^2} \left( 2(x_{1j} A_{1i} + x_{2j} A_{2i} + \cdots + x_{mj} A_{mi}) \cdot \bar{\Delta} - 2 \left( \frac{x_{1j}}{\lambda_j^2} \bar{A}_{1i} + \frac{x_{2j}}{\lambda_j^2} \bar{A}_{2i} + \cdots + \frac{x_{mj}}{\lambda_j^2} \bar{A}_{mi} \right) \cdot \Delta \right)$$

so we get

$$\begin{aligned}
(\nabla f)_{ik_s} &= \frac{2}{\bar{\Delta}^2} \sum_{t=1}^{n-m} [\langle \xi_i, \xi_1 \rangle \langle \eta_{k_s}, \eta_{k_t} \rangle (x_{1k_t} A_{11} + x_{2k_t} A_{21} + \cdots + x_{mk_t} A_{m1}) \cdot \bar{\Delta} \\
&+ \langle \xi_i, \xi_2 \rangle \langle \eta_{k_s}, \eta_{k_t} \rangle (x_{1k_t} A_{12} + x_{2k_t} A_{22} + \cdots + x_{mk_t} A_{m2}) \cdot \bar{\Delta} + \cdots \\
&+ \langle \xi_i, \xi_m \rangle \langle \eta_{k_s}, \eta_{k_t} \rangle (x_{1k_t} A_{1m} + x_{2k_t} A_{2m} + \cdots + x_{mk_t} A_{mm}) \cdot \bar{\Delta}] \\
&- \frac{2}{\bar{\Delta}^2} \sum_{t=1}^{n-m} [\langle \xi_i, \xi_1 \rangle \langle \eta_{k_s}, \eta_{k_t} \rangle (\frac{x_{1k_t}}{\lambda_{k_t}^2} \bar{A}_{11} + \frac{x_{2k_t}}{\lambda_{k_t}^2} \bar{A}_{21} + \cdots + \frac{x_{mk_t}}{\lambda_{k_t}^2} \bar{A}_{m1}) \cdot \Delta \\
&+ \langle \xi_i, \xi_2 \rangle \langle \eta_{k_s}, \eta_{k_t} \rangle (\frac{x_{1k_t}}{\lambda_{k_t}^2} \bar{A}_{12} + \frac{x_{2k_t}}{\lambda_{k_t}^2} \bar{A}_{22} + \cdots + \frac{x_{mk_t}}{\lambda_{k_t}^2} \bar{A}_{m2}) \cdot \Delta + \cdots \\
&+ \langle \xi_i, \xi_m \rangle \langle \eta_{k_s}, \eta_{k_t} \rangle (\frac{x_{1k_t}}{\lambda_{k_t}^2} \bar{A}_{1m} + \frac{x_{2k_t}}{\lambda_{k_t}^2} \bar{A}_{2m} + \cdots + \frac{x_{mk_t}}{\lambda_{k_t}^2} \bar{A}_{mm}) \cdot \Delta] \\
&= \frac{2}{\bar{\Delta}^2} [\langle \xi_i, \xi_1 \rangle \cdot \Delta \cdot \bar{\Delta} \cdot x_{1k_s} + \langle \xi_i, \xi_2 \rangle \cdot \Delta \cdot \bar{\Delta} \cdot x_{2k_s} + \cdots + \langle \xi_i, \xi_m \rangle \cdot \Delta \cdot \bar{\Delta} \cdot x_{mk_s}] \\
&- \frac{2}{\bar{\Delta}^2} [\langle \xi_i, \xi_1 \rangle \cdot \Delta \cdot \bar{\Delta} \cdot x_{1k_s} + \langle \xi_i, \xi_1 \rangle \cdot \Delta \left( \bar{A}_{11} (\frac{1}{\lambda_{k_s}^2} - \frac{1}{\lambda_{i_1}^2}) x_{1k_s} + \cdots + \bar{A}_{m1} (\frac{1}{\lambda_{k_s}^2} - \frac{1}{\lambda_{i_m}^2}) x_{mk_s} \right) \\
&+ \langle \xi_i, \xi_2 \rangle \cdot \Delta \cdot \bar{\Delta} \cdot x_{2k_s} + \langle \xi_i, \xi_2 \rangle \cdot \Delta \left( \bar{A}_{12} (\frac{1}{\lambda_{k_s}^2} - \frac{1}{\lambda_{i_1}^2}) x_{1k_s} + \cdots + \bar{A}_{m2} (\frac{1}{\lambda_{k_s}^2} - \frac{1}{\lambda_{i_m}^2}) x_{mk_s} \right) + \cdots \\
&+ \langle \xi_i, \xi_m \rangle \cdot \Delta \cdot \bar{\Delta} \cdot x_{mk_s} + \langle \xi_i, \xi_m \rangle \cdot \Delta \left( \bar{A}_{1m} (\frac{1}{\lambda_{k_s}^2} - \frac{1}{\lambda_{i_1}^2}) x_{1k_s} + \cdots + \bar{A}_{mm} (\frac{1}{\lambda_{k_s}^2} - \frac{1}{\lambda_{i_m}^2}) x_{mk_s} \right)] \\
&= -\frac{2\Delta}{\bar{\Delta}^2} [(\frac{1}{\lambda_{k_s}^2} - \frac{1}{\lambda_{i_1}^2}) \cdot (\langle \xi_i, \xi_1 \rangle \bar{A}_{11} + \langle \xi_i, \xi_2 \rangle \bar{A}_{12} + \cdots + \langle \xi_i, \xi_m \rangle \bar{A}_{1m}) \cdot x_{1k_s} \\
&+ (\frac{1}{\lambda_{k_s}^2} - \frac{1}{\lambda_{i_2}^2}) \cdot (\langle \xi_i, \xi_1 \rangle \bar{A}_{21} + \langle \xi_i, \xi_2 \rangle \bar{A}_{22} + \cdots + \langle \xi_i, \xi_m \rangle \bar{A}_{2m}) \cdot x_{2k_s} + \cdots \\
&+ (\frac{1}{\lambda_{k_s}^2} - \frac{1}{\lambda_{i_m}^2}) \cdot (\langle \xi_i, \xi_1 \rangle \bar{A}_{m1} + \langle \xi_i, \xi_2 \rangle \bar{A}_{m2} + \cdots + \langle \xi_i, \xi_m \rangle \bar{A}_{mm}) \cdot x_{mk_s}]
\end{aligned}$$

by the computation we have

$$\begin{pmatrix} (\nabla f)_{1k_1} \\ (\nabla f)_{1k_2} \\ \vdots \\ (\nabla f)_{1k_{n-m}} \\ \vdots \\ (\nabla f)_{mk_1} \\ (\nabla f)_{mk_2} \\ \vdots \\ (\nabla f)_{mk_{n-m}} \end{pmatrix} = -\frac{2\Delta}{\bar{\Delta}^2} \begin{pmatrix} \langle \xi_1, \xi_1 \rangle I_{n-m} & \cdots & \langle \xi_1, \xi_m \rangle I_{n-m} \\ \langle \xi_2, \xi_1 \rangle I_{n-m} & \cdots & \langle \xi_2, \xi_m \rangle I_{n-m} \\ \dots & \dots & \dots \\ \langle \xi_m, \xi_1 \rangle I_{n-m} & \cdots & \langle \xi_m, \xi_m \rangle I_{n-m} \end{pmatrix} \cdot F \cdot H \cdot X \quad (4)$$

where

$$X = (x_{1k_1}, x_{1k_2}, \cdots, x_{1k_{n-m}}, x_{2k_1}, x_{2k_2}, \cdots, x_{2k_{n-m}}, \cdots, x_{mk_1}, x_{mk_2}, \cdots, x_{mk_{n-m}})^T.$$

$$F = \begin{pmatrix} \bar{A}_{11} I_{n-m} & \bar{A}_{21} I_{n-m} & \cdots & \bar{A}_{m1} I_{n-m} \\ \bar{A}_{12} I_{n-m} & \bar{A}_{22} I_{n-m} & \cdots & \bar{A}_{m1} I_{n-m} \\ \dots & \dots & \dots & \dots \\ \bar{A}_{1m} I_{n-m} & \bar{A}_{2m} I_{n-m} & \cdots & \bar{A}_{mm} I_{n-m} \end{pmatrix}$$

$$H = \begin{pmatrix} H_{11} & & & 0 \\ & H_{22} & & \\ & & \ddots & \\ 0 & & & H_{(n-m)(n-m)} \end{pmatrix}$$

where

$$H_{ss} = \begin{pmatrix} (\frac{1}{\lambda_{k_s}^2} - \frac{1}{\lambda_{i_1}^2}) & & & 0 \\ & (\frac{1}{\lambda_{k_s}^2} - \frac{1}{\lambda_{i_2}^2}) & & \\ & & \ddots & \\ 0 & & & (\frac{1}{\lambda_{k_s}^2} - \frac{1}{\lambda_{i_m}^2}) \end{pmatrix}$$

$s = 1, 2, \dots, n - m$ .

**Lemma 5.**  $O_{i_1 i_2 \dots i_m}$  is a hyperbolic singular point of  $\dot{x} = -\nabla f$ , and the linear part of  $\dot{x} = -\nabla f$  at  $O_{i_1 i_2 \dots i_m}$  has the expression  $AX$  on  $(\tilde{U}_{i_1 i_2 \dots i_m}, \tilde{\phi}_{i_1 i_2 \dots i_m})$  where

$$A = -2\lambda_{i_1}^2 \lambda_{i_2}^2 \cdots \lambda_{i_m}^2 \text{diag} \left( 1 - \frac{\lambda_{i_1}^2}{\lambda_{k_1}^2}, \dots, 1 - \frac{\lambda_{i_1}^2}{\lambda_{k_{n-m}}^2}, \dots, 1 - \frac{\lambda_{i_m}^2}{\lambda_{k_1}^2}, \dots, 1 - \frac{\lambda_{i_m}^2}{\lambda_{k_{n-m}}^2} \right)$$

and  $X = (x_{1k_1}, x_{1k_2}, \dots, x_{1k_{n-m}}, x_{2k_1}, x_{2k_2}, \dots, x_{2k_{n-m}}, \dots, x_{mk_1}, x_{mk_2}, \dots, x_{mk_{n-m}})^T$ .

*Proof.* Because at  $O_{i_1 i_2 \dots i_m}$  by computation  $\frac{2\Delta}{\Delta^2}(O_{i_1 i_2 \dots i_m}) = 2\lambda_{i_1}^4 \lambda_{i_2}^4 \cdots \lambda_{i_m}^4$ ,

$$F(O_{i_1 i_2 \dots i_m}) = \begin{pmatrix} (\frac{1}{\lambda_{i_2}^2} \frac{1}{\lambda_{i_3}^2} \cdots \frac{1}{\lambda_{i_m}^2}) I_{n-m} & & & 0 \\ & (\frac{1}{\lambda_{i_1}^2} \frac{1}{\lambda_{i_3}^2} \cdots \frac{1}{\lambda_{i_m}^2}) I_{n-m} & & \\ & & \ddots & \\ 0 & & & (\frac{1}{\lambda_{i_1}^2} \frac{1}{\lambda_{i_2}^2} \cdots \frac{1}{\lambda_{i_{m-1}}^2}) I_{n-m} \end{pmatrix}$$

then by (4) we can get

$$A = -2\lambda_{i_1}^2 \lambda_{i_2}^2 \cdots \lambda_{i_m}^2 \text{diag} \left( 1 - \frac{\lambda_{i_1}^2}{\lambda_{k_1}^2}, \dots, 1 - \frac{\lambda_{i_1}^2}{\lambda_{k_{n-m}}^2}, \dots, 1 - \frac{\lambda_{i_m}^2}{\lambda_{k_1}^2}, \dots, 1 - \frac{\lambda_{i_m}^2}{\lambda_{k_{n-m}}^2} \right)$$

□

Let  $M$  be a compact smooth Riemannian manifold with metric  $g$ , and  $f$  be a Morse function on  $M$ , then the negative gradient vector field  $-\nabla f$  determines a smooth flow  $\varphi : \mathbb{R} \times M \rightarrow M$ , and  $\varphi_t$  is a diffeomorphism of  $M$  for all  $t \in \mathbb{R}$  (see [4]).

**Definition 1** (see [4]). Let  $p \in M$  be a non-degenerate critical point of Morse function  $f$ , the stable manifold of  $p$  is defined to be

$$W^s(p) = \{x \in M \mid \lim_{t \rightarrow +\infty} \varphi_t(x) = p\}$$

the unstable manifold of  $p$  is defined to be

$$W^u(p) = \{x \in M \mid \lim_{t \rightarrow -\infty} \varphi_t(x) = p\}$$

Obviously

$$\partial_{1k_1}, \partial_{1k_2}, \dots, \partial_{1k_{n-m}}, \partial_{2k_1}, \dots, \partial_{2k_{n-m}}, \partial_{mk_1}, \dots, \partial_{mk_{n-m}}$$

is the orthonormal basis of the tangent space on  $\tilde{U}_{i_1 i_2 \dots i_m}$ , where  $\partial_{ij} = \frac{\partial}{\partial x_{ij}}$ .

By the computation in lemma 3., we have linear part  $A = -H_{O_{i_1 i_2 \dots i_m}}(f)$ , so use the stabel and unstable manifold theorem for a Morse function(see [4]), we identified the positive definite eigenvalue subspace

$$\begin{aligned} E^s(O_{i_1 i_2 \dots i_m}) = \\ \text{span}_{\mathbb{R}}\{ & \partial_{1(i_1+1)}, \partial_{1(i_1+2)}, \dots, \widehat{\partial_{1i_2}}, \dots, \widehat{\partial_{1i_3}}, \dots, \widehat{\partial_{1i_m}}, \dots, \partial_{1n}, \\ & \partial_{2(i_2+1)}, \partial_{2(i_2+2)}, \dots, \widehat{\partial_{2i_3}}, \dots, \widehat{\partial_{2i_m}}, \dots, \partial_{2n}, \\ & \partial_{3(i_3+1)}, \partial_{3(i_3+2)}, \dots, \widehat{\partial_{3i_4}}, \dots, \widehat{\partial_{3i_m}}, \dots, \partial_{3n}, \\ & \dots, \\ & \partial_{m(i_m+1)}, \partial_{m(i_m+2)}, \dots, \partial_{mn}\} \end{aligned}$$

and negative definite eigenvalue subspace

$$\begin{aligned} E^u(O_{i_1 i_2 \dots i_m}) = \\ \text{span}_{\mathbb{R}}\{ & \partial_{11}, \partial_{12}, \dots, \partial_{1(i_1-1)}, \\ & \partial_{21}, \partial_{22}, \dots, \widehat{\partial_{2i_1}}, \dots, \partial_{2(i_2-1)}, \\ & \partial_{31}, \partial_{32}, \dots, \widehat{\partial_{3i_1}}, \dots, \widehat{\partial_{3i_2}}, \dots, \partial_{3(i_3-1)}, \\ & \dots, \\ & \partial_{m1}, \partial_{m2}, \dots, \widehat{\partial_{mi_1}}, \dots, \widehat{\partial_{mi_2}}, \dots, \widehat{\partial_{mi_3}}, \dots, \widehat{\partial_{mi_{m-1}}}, \dots, \partial_{m(i_m-1)}\} \end{aligned}$$

and

$$E^s(O_{i_1 i_2 \dots i_m}) = T_{O_{i_1 i_2 \dots i_m}} W^s(O_{i_1 i_2 \dots i_m}), \quad E^u(O_{i_1 i_2 \dots i_m}) = T_{O_{i_1 i_2 \dots i_m}} W^u(O_{i_1 i_2 \dots i_m}).$$

where  $\widehat{\partial_{ij}}$  means this one is empty.

**Definition 2** (see [6]). *An invariant manifold  $N$  of a vector field  $V$  on a manifold  $M$  and of the corresponding differential equation  $\dot{x} = V(x)$  is defined to be a submanifold of  $M$  which is tangent to the vector field  $V$  at each of its points.*

*An invariant manifold  $N$  is global if the initial value problem*

$$\dot{x} = V(x), \quad x(0) = p$$

*has a global solution  $x = x(t)$ ,  $(-\infty < t < +\infty)$  for any  $p \in N$ .*

**Lemma 6.** *The following sets as global invariant manifold of  $\dot{x} = -\nabla f$  on  $G_{n,m}(\mathbb{R})$ ,*

$$\begin{aligned} & \tilde{U}_{i_1 i_2 \dots i_m}(x_{\beta k_{s_1}}, x_{\beta k_{s_2}}, \dots, x_{\beta k_{s_t}}) \\ & = \pi(\{(x_{\alpha k}) \in U_{i_1 i_2 \dots i_m} \mid x_{\alpha k} = 0, (\alpha, k) \neq (\beta, k_{s_p}), p = 1, \dots, t\}), \\ & \tilde{U}_{i_1 i_2 \dots i_m}(x_{\beta_1 k_{s_1}^1}, x_{\beta_1 k_{s_2}^1}, \dots, x_{\beta_1 k_{s_t_1}^1}, x_{\beta_2 k_{s_1}^2}, x_{\beta_2 k_{s_2}^2}, \dots, x_{\beta_2 k_{s_t_2}^2}) \\ & = \pi(\{(x_{\alpha k}) \in U_{i_1 i_2 \dots i_m} \mid x_{\alpha k} = 0, (\alpha, k) \neq (\beta_j, k_{s_p}^j), j = 1, 2; p = 1, \dots, t_j\}), \end{aligned}$$

$$\begin{aligned}
& \tilde{U}_{i_1 i_2 \dots i_m}(x_{\beta_1 k_{s_1^1}}, x_{\beta_1 k_{s_1^2}}, \dots, x_{\beta_1 k_{s_1^{t_1}}}, x_{\beta_2 k_{s_2^1}}, x_{\beta_2 k_{s_2^2}}, \dots, x_{\beta_2 k_{s_2^{t_2}}}, x_{\beta_3 k_{s_3^1}}, x_{\beta_3 k_{s_3^2}}, \dots, x_{\beta_3 k_{s_3^{t_3}}}) \\
& = \pi(\{(x_{\alpha k}) \in U_{i_1 i_2 \dots i_m} \mid x_{\alpha k} = 0, (\alpha, k) \neq (\beta_j, k_{s_p^j}), j = 1, 2, 3; p = 1, \dots, t_j\}), \\
& \quad \dots \dots \dots \\
& \tilde{U}_{i_1 i_2 \dots i_m}(x_{\beta_j k_{s_1^j}}, x_{\beta_j k_{s_2^j}}, \dots, x_{\beta_j k_{s_{t_j}^j}}) \\
& = \pi(\{(x_{\alpha k}) \in U_{i_1 i_2 \dots i_m} \mid x_{\alpha k} = 0, (\alpha, k) \neq (\beta_j, k_{s_p^j}), j = 1, 2, 3, \dots, m; p = 1, \dots, t_j\}).
\end{aligned}$$

*Proof.* For any  $\tilde{U}_{i_1 i_2 \dots i_m}(\dots)$ , when  $x_{\beta k} \notin (\dots)$ , then  $x_{\beta k} = 0$ , so  $\dot{x}_{\beta k}(t) = 0$  on  $\tilde{U}_{i_1 i_2 \dots i_m}(\dots)$  and  $-(\nabla f)_{\beta k} |_{\tilde{U}_{i_1 i_2 \dots i_m}(\dots)} = \dot{x}_{\beta k}(t) = 0$ , then  $-(\nabla f) |_{\tilde{U}_{i_1 i_2 \dots i_m}(\dots)}$  is tangent vector field on  $\tilde{U}_{i_1 i_2 \dots i_m}(\dots)$ . So  $\tilde{U}_{i_1 i_2 \dots i_m}(\dots)$  is invariant manifold of  $\dot{x} = -\nabla f$  on  $G_{n,m}(\mathbb{R})$ .

Because  $G_{n,m}(\mathbb{R})$  is compact smooth manifold, so the initial value problem

$$\dot{x} = -\nabla f, \quad x(0) = p$$

on  $G_{n,m}(\mathbb{R})$  has global solution.

$$\tilde{U}_{i_1 i_2 \dots i_m}(\dots) \in G_{n,m}(\mathbb{R})$$

the initial value problem

$$\dot{x} = -(\nabla f) |_{\tilde{U}_{i_1 i_2 \dots i_m}(\dots)}, \quad x(0) = p, \quad p \in \tilde{U}_{i_1 i_2 \dots i_m}(\dots)$$

on  $\tilde{U}_{i_1 i_2 \dots i_m}(\dots)$  also has global solution. Then  $\tilde{U}_{i_1 i_2 \dots i_m}(\dots)$  is the global invariant manifold.  $\square$

**Lemma 7** (see [6]). *Let  $V$  be a smooth vector field on a manifold  $M$  and  $p \in M$  be a hyperbolic singular point of  $V$ . Let  $N$  be a global invariant manifold of  $V$  in  $M$  and  $p \in N$ . Then we have the following sets equalities*

$$W_N^s(p) = W^s(p) \cap N, \quad W_N^u(p) = W^u(p) \cap N,$$

where  $W_N^s(p), W_N^u(p)$  are the stable and unstable manifold of  $V |_N$ , the restriction of  $V$  on  $N$  at  $p$ , particularly we have

$$W_N^s(p) = W^s(p) \quad \text{if} \quad \dim W_N^s(p) = \dim W^s(p)$$

$$W_N^u(p) = W^u(p) \quad \text{if} \quad \dim W_N^u(p) = \dim W^u(p).$$

**Lemma 8.** *The stable and unstable manifold of  $O_{i_1 i_2 \dots i_m}$ , have the following results*

a)

$$W^s(O_{i_1 i_2 \dots i_m}) \subset \tilde{U}_{i_1 i_2 \dots i_m}(x_{\beta_j k_{s_1^j}}, x_{\beta_j k_{s_2^j}}, \dots, x_{\beta_j k_{s_{t_j}^j}})$$

where  $j = 1, 2, \dots, m$ ;

when  $j = 1$ , let  $\beta_1 = 1$ ,  $k_{s_1^1} = i_1 + 1, i_1 + 2, \dots, \widehat{i_2}, \dots, \widehat{i_3}, \dots, \widehat{i_m}, \dots, n$ ;

when  $j = 2$ , let  $\beta_2 = 2$ ,  $k_{s_2^2} = i_2 + 1, i_2 + 2, \dots, \widehat{i_3}, \dots, \widehat{i_4}, \dots, \widehat{i_m}, \dots, n$ ;

.....

when  $j = m$ , let  $\beta_m = m$ ,  $k_{s_m^m} = i_m + 1, i_m + 2, \dots, n$ .

b)

$$W^u(O_{i_1 i_2 \dots i_m}) \subset \tilde{U}_{i_1 i_2 \dots i_m}(x_{\beta_j k_{s_1^j}}, x_{\beta_j k_{s_2^j}}, \dots, x_{\beta_j k_{s_{t_j}^j}})$$

where  $j = 1, 2, \dots, m$ ;

when  $j = 1$ , let  $\beta_1 = 1$ ,  $k_{s_1^1} = 1, 2, \dots, i_1 - 1$ ;

when  $j = 2$ , let  $\beta_2 = 2$ ,  $k_{s_2^2} = 1, 2, \dots, \widehat{i_1}, \dots, i_2 - 1$ ;

.....

when  $j = m$ , let  $\beta_m = m$ ,  $k_{s_p^m} = 1, 2, \dots, \widehat{i_1}, \dots, \widehat{i_2}, \dots, \widehat{i_{m-1}}, \dots, i_m - 1$ .

*Proof.* Because  $O_{i_1 i_2 \dots i_m} \in \tilde{U}_{i_1 i_2 \dots i_m}(x_{\beta_j k_{s_1^j}}, x_{\beta_j k_{s_2^j}}, \dots, x_{\beta_j k_{s_{t_j}^j}})$ , where  $j = 1, 2, \dots, m$ ; so  $O_{i_1 i_2 \dots i_m}$  is a singular point of  $-\nabla f|_{\tilde{U}_{i_1 i_2 \dots i_m}(x_{\beta_j k_{s_1^j}}, x_{\beta_j k_{s_2^j}}, \dots, x_{\beta_j k_{s_{t_j}^j})}$ .

By lemma 6.,  $\tilde{U}_{i_1 i_2 \dots i_m}(x_{\beta_j k_{s_1^j}}, x_{\beta_j k_{s_2^j}}, \dots, x_{\beta_j k_{s_{t_j}^j}})$  is a global invariant manifold, so  $O_{i_1 i_2 \dots i_m}$  is the hyperbolic singular point of  $-\nabla f|_{\tilde{U}_{i_1 i_2 \dots i_m}(x_{\beta_j k_{s_1^j}}, x_{\beta_j k_{s_2^j}}, \dots, x_{\beta_j k_{s_{t_j}^j})}$  and the linear part at  $O_{i_1 i_2 \dots i_m}$  is

$$A = -2\lambda_{i_1}^2 \lambda_{i_2}^2 \dots \lambda_{i_m}^2 \text{diag} \left( 1 - \frac{\lambda_{i_1}^2}{\lambda_{k_1^1}^2}, \dots, 1 - \frac{\lambda_{i_1}^2}{\lambda_{k_{t_1}^1}^2}, \dots, 1 - \frac{\lambda_{i_m}^2}{\lambda_{k_1^m}^2}, \dots, 1 - \frac{\lambda_{i_m}^2}{\lambda_{k_{t_m}^m}^2} \right)$$

where  $k_{t_p}^1 = i_1 + 1, i_1 + 2, \dots, \widehat{i_2}, \dots, \widehat{i_3}, \dots, \widehat{i_m}, \dots, n$ ;

$k_{t_p}^2 = i_2 + 1, i_2 + 2, \dots, \widehat{i_3}, \dots, \widehat{i_4}, \dots, \widehat{i_m}, \dots, n$ ; .....;  $k_{t_p}^m = i_m + 1, i_m + 2, \dots, n$ .

Then the positive definite eigenvalue subspace is

$$\begin{aligned} E_{\tilde{U}_{i_1 i_2 \dots i_m}(x_{\beta_j k_{s_1^j}}, x_{\beta_j k_{s_2^j}}, \dots, x_{\beta_j k_{s_{t_j}^j})}^s(O_{i_1 i_2 \dots i_m}) = \\ \text{span}_{\mathbb{R}} \{ \partial_{1(i_1+1)}, \partial_{1(i_1+2)}, \dots, \widehat{\partial_{1i_2}}, \dots, \widehat{\partial_{1i_3}}, \dots, \widehat{\partial_{1i_m}}, \dots, \partial_{1n}, \\ \partial_{2(i_2+1)}, \partial_{2(i_2+2)}, \dots, \widehat{\partial_{2i_3}}, \dots, \widehat{\partial_{2i_m}}, \dots, \partial_{2n}, \\ \partial_{3(i_3+1)}, \partial_{3(i_3+2)}, \dots, \widehat{\partial_{3i_4}}, \dots, \widehat{\partial_{3i_m}}, \dots, \partial_{3n}, \\ \dots, \\ \partial_{m(i_m+1)}, \partial_{m(i_m+2)}, \dots, \partial_{mn} \}. \end{aligned}$$

So we have  $E_{\tilde{U}_{i_1 i_2 \dots i_m}(x_{\beta_j k_{s_1^j}}, x_{\beta_j k_{s_2^j}}, \dots, x_{\beta_j k_{s_{t_j}^j})}^s(O_{i_1 i_2 \dots i_m}) = E^s(O_{i_1 i_2 \dots i_m})$ , and we get

$$\begin{aligned} \dim W_{\tilde{U}_{i_1 i_2 \dots i_m}(x_{\beta_j k_{s_1^j}}, x_{\beta_j k_{s_2^j}}, \dots, x_{\beta_j k_{s_{t_j}^j})}^s(O_{i_1 i_2 \dots i_m}) &= \dim E_{\tilde{U}_{i_1 i_2 \dots i_m}(x_{\beta_j k_{s_1^j}}, x_{\beta_j k_{s_2^j}}, \dots, x_{\beta_j k_{s_{t_j}^j})}^s(O_{i_1 i_2 \dots i_m}) \\ &= \dim E^s(O_{i_1 i_2 \dots i_m}) = \dim W^s(O_{i_1 i_2 \dots i_m}) \end{aligned}$$

by Lemma 7., we have

$$W^s(O_{i_1 i_2 \dots i_m}) \subset \tilde{U}_{i_1 i_2 \dots i_m}(x_{\beta_j k_{s_1^j}}, x_{\beta_j k_{s_2^j}}, \dots, x_{\beta_j k_{s_{t_j}^j}});$$

where  $j = 1, 2, \dots, m$ ;

when  $j = 1$ , let  $\beta_1 = 1, k_{s_p^1} = i_1 + 1, i_1 + 2, \dots, \widehat{i_2}, \dots, \widehat{i_3}, \dots, \widehat{i_m}, \dots, n$ ;

when  $j = 2$ , let  $\beta_2 = 2, k_{s_p^2} = i_2 + 1, i_2 + 2, \dots, \widehat{i_3}, \dots, \widehat{i_4}, \dots, \widehat{i_m}, \dots, n$ ;

.....

when  $j = m$ , let  $\beta_m = m, k_{s_p^m} = i_m + 1, i_m + 2, \dots, n$ .

By the same way, we can get

$$W^u(O_{i_1 i_2 \dots i_m}) \subset \widetilde{U}_{i_1 i_2 \dots i_m}(x_{\beta_j k_{s_1^j}}, x_{\beta_j k_{s_2^j}}, \dots, x_{\beta_j k_{s_{t_j}^j}});$$

where  $j = 1, 2, \dots, m$ ;

when  $j = 1$ , let  $\beta_1 = 1, k_{s_p^1} = 1, 2, \dots, i_1 - 1$ ;

when  $j = 2$ , let  $\beta_2 = 2, k_{s_p^2} = 1, 2, \dots, \widehat{i_1}, \dots, i_2 - 1$ ;

.....

when  $j = m$ , let  $\beta_m = m, k_{s_p^m} = 1, 2, \dots, \widehat{i_1}, \dots, \widehat{i_2}, \dots, \widehat{i_{m-1}}, \dots, i_m - 1$ . □

## 5 The Morse-Smale transversality condition

In this section we will to proof  $f = \frac{\Delta}{\Delta}$  is a Morse-Smale funtion. For all critical points  $O_{i_1 i_2 \dots i_m}, O_{l_1 l_2 \dots l_m}$  of  $f = \frac{\Delta}{\Delta}$  we need to proof the stable and unstable manifolds of  $f$  intersect transversally(see [4] and [9]).

**Lemma 9.** *Let  $O_{i_1 i_2 \dots i_m}, O_{l_1 l_2 \dots l_m}$  be the different critical points of  $f$ , and let  $l_k \geq i_k$  for some  $k \in 1, 2, \dots, m$ , then have the following result*

$$W^u(O_{i_1 i_2 \dots i_m}) \cap W^s(O_{l_1 l_2 \dots l_m}) = \emptyset$$

*Proof.* If  $W^u(O_{i_1 i_2 \dots i_m}) \cap W^s(O_{l_1 l_2 \dots l_m}) \neq \emptyset$ , there is  $p \in W^u(O_{i_1 i_2 \dots i_m}) \cap W^s(O_{l_1 l_2 \dots l_m})$  and  $\exists \varphi(t) (-\infty < t < +\infty)$  is the solution of  $\dot{x} = -\nabla f$ , with  $\varphi(0) = p$  and

$$\lim_{t \rightarrow +\infty} \varphi(t) = O_{l_1 l_2 \dots l_m}, \quad \lim_{t \rightarrow -\infty} \varphi(t) = O_{i_1 i_2 \dots i_m}$$

so  $\varphi(t) \in W^u(O_{i_1 i_2 \dots i_m}) \cap W^s(O_{l_1 l_2 \dots l_m})$ .

Because  $O_{l_1 l_2 \dots l_m} \in \widetilde{U}_{l_1 l_2 \dots l_m}$ , so  $\exists t_0 > 0$  with  $t > t_0, \varphi(t) \in \widetilde{U}_{l_1 l_2 \dots l_m}$ . By Lemma 8., we have

$$\varphi(t) \in \widetilde{U}_{l_1 l_2 \dots l_m} \cap \widetilde{U}_{i_1 i_2 \dots i_m}(x_{\beta_j k_{s_1^j}}, x_{\beta_j k_{s_2^j}}, \dots, x_{\beta_j k_{s_{t_j}^j}}), \quad t > t_0,$$

where  $j = 1, 2, \dots, m$ ;

when  $j = 1$ , let  $\beta_1 = 1, k_{s_p^1} = 1, 2, \dots, i_1 - 1$ ;

when  $j = 2$ , let  $\beta_2 = 2, k_{s_p^2} = 1, 2, \dots, \widehat{i_1}, \dots, i_2 - 1$ ;

.....

when  $j = m$ , let  $\beta_m = m, k_{s_p^m} = 1, 2, \dots, \widehat{i_1}, \dots, \widehat{i_2}, \dots, \widehat{i_{m-1}}, \dots, i_m - 1$ .

But because  $l_k \geq i_k$  for some  $k \in 1, 2, \dots, m$ , so we have

$$\widetilde{U}_{l_1 l_2 \dots l_m} \cap \widetilde{U}_{i_1 i_2 \dots i_m}(x_{\beta_j k_{s_1^j}}, x_{\beta_j k_{s_2^j}}, \dots, x_{\beta_j k_{s_{t_j}^j}}) = \emptyset,$$



where  $j = 1, 2, \dots, m$ ;

when  $j = 1$ , let  $\beta_1 = 1, k_{s_p^1} = 1, 2, \dots, i_1 - 1$ ;

when  $j = 2$ , let  $\beta_2 = 2, k_{s_p^2} = 1, 2, \dots, \widehat{i_1}, \dots, i_2 - 1$ ;

.....

when  $j = m$ , let  $\beta_m = m, k_{s_p^m} = 1, 2, \dots, \widehat{i_1}, \dots, \widehat{i_2}, \dots, \widehat{i_{m-1}}, \dots, i_m - 1$ .

Then we get

$$W^u(O_{i_1 i_2 \dots i_m}) \cap W^s(O_{l_1 l_2 \dots l_m}) = \emptyset$$

.

□

**Lemma 10.** *The function  $f = \frac{\Delta}{\Delta}$  is Morse-Smale function.*

*Proof.* Let  $O_{i_1 i_2 \dots i_m}, O_{l_1 l_2 \dots l_m}$  be any two different critical points of  $f$ , and with

$$W^u(O_{i_1 i_2 \dots i_m}) \cap W^s(O_{l_1 l_2 \dots l_m}) \neq \emptyset.$$

By the lemma 9., we know that  $l_1 < i_1, l_2 < i_2, \dots, l_m < i_m$ . Let  $p \in W^u(O_{i_1 i_2 \dots i_m}) \cap W^s(O_{l_1 l_2 \dots l_m})$ , the tangent space of  $W^u(O_{i_1 i_2 \dots i_m})$  at  $p$  is

$$\begin{aligned} T_p W^u(O_{i_1 i_2 \dots i_m}) = \text{span}_{\mathbb{R}} \{ & \partial_{11}, \partial_{12}, \dots, \partial_{1(i_1-1)}, \\ & \partial_{21}, \partial_{22}, \dots, \widehat{\partial_{2i_1}}, \dots, \partial_{2(i_2-1)}, \\ & \partial_{31}, \partial_{32}, \dots, \widehat{\partial_{3i_1}}, \dots, \widehat{\partial_{3i_2}}, \dots, \partial_{3(i_3-1)}, \\ & \dots, \\ & \partial_{m1}, \partial_{m2}, \dots, \widehat{\partial_{mi_1}}, \dots, \widehat{\partial_{mi_2}}, \dots, \widehat{\partial_{mi_3}}, \dots, \widehat{\partial_{mi_{m-1}}}, \dots, \partial_{m(i_m-1)} \}; \end{aligned}$$

The tangent space of  $W^s(O_{l_1 l_2 \dots l_m})$  at  $p$  is

$$\begin{aligned} T_p W^s(O_{l_1 l_2 \dots l_m}) = \text{span}_{\mathbb{R}} \{ & \partial_{1(l_1+1)}, \partial_{1(l_1+2)}, \dots, \widehat{\partial_{1l_2}}, \dots, \widehat{\partial_{1l_3}}, \dots, \widehat{\partial_{1l_m}}, \dots, \partial_{1n}, \\ & \partial_{2(l_2+1)}, \partial_{2(l_2+2)}, \dots, \widehat{\partial_{2l_3}}, \dots, \widehat{\partial_{2l_m}}, \dots, \partial_{2n}, \\ & \partial_{3(l_3+1)}, \partial_{3(l_3+2)}, \dots, \widehat{\partial_{3l_4}}, \dots, \widehat{\partial_{3l_m}}, \dots, \partial_{3n}, \\ & \dots, \\ & \partial_{m(l_m+1)}, \partial_{m(l_m+2)}, \dots, \partial_{mn} \}. \end{aligned}$$

So we have

$$T_p G_{n,m}(\mathbb{R}) \subset T_p W^u(O_{i_1 i_2 \dots i_m}) + T_p W^s(O_{l_1 l_2 \dots l_m})$$

and because

$$W^u(O_{i_1 i_2 \dots i_m}) \subset G_{n,m}(\mathbb{R}), \quad W^s(O_{l_1 l_2 \dots l_m}) \subset G_{n,m}(\mathbb{R})$$

so

$$T_p G_{n,m}(\mathbb{R}) \supset T_p W^u(O_{i_1 i_2 \dots i_m}) + T_p W^s(O_{l_1 l_2 \dots l_m})$$

Then we get

$$T_p G_{n,m}(\mathbb{R}) = T_p W^u(O_{i_1 i_2 \dots i_m}) + T_p W^s(O_{l_1 l_2 \dots l_m}).$$

It means that the stable and unstable manifolds of  $f$  intersect transversally,  $f = \frac{\Delta}{\Delta}$  is a Morse-Smale function.

□

## 6 The trajectories connect the critical points

**Theorem 1.** Let  $O_{i_1 i_2 \dots i_m}, O_{l_1 l_2 \dots l_m}$  be any two different critical points of  $f$ , with

$$\text{ind}(O_{i_1 i_2 \dots i_m}) - \text{ind}(O_{l_1 l_2 \dots l_m}) = 1,$$

and  $i_k - l_k > 1$  for some  $k \in 1, 2, \dots, m$ ; then

$$W^u(O_{i_1 i_2 \dots i_m}) \cap W^s(O_{l_1 l_2 \dots l_m}) = \emptyset$$

*Proof.* If  $W^u(O_{i_1 i_2 \dots i_m}) \cap W^s(O_{l_1 l_2 \dots l_m}) \neq \emptyset$ , there is  $p \in W^u(O_{i_1 i_2 \dots i_m}) \cap W^s(O_{l_1 l_2 \dots l_m})$  and  $\exists \varphi(t) (-\infty < t < +\infty)$  is the solution of  $\dot{x} = -\nabla f$ , with  $\varphi(0) = p$  and

$$\lim_{t \rightarrow +\infty} \varphi(t) = O_{l_1 l_2 \dots l_m}, \quad \lim_{t \rightarrow -\infty} \varphi(t) = O_{i_1 i_2 \dots i_m}$$

so  $\varphi(t) \in W^u(O_{i_1 i_2 \dots i_m}) \cap W^s(O_{l_1 l_2 \dots l_m})$ .

Because  $O_{l_1 l_2 \dots l_m} \in \tilde{U}_{l_1 l_2 \dots l_m}$ , so  $\exists t_0 > 0$  with  $t > t_0, \varphi(t) \in \tilde{U}_{l_1 l_2 \dots l_m}$ . By Lemma 8., we have

$$\varphi(t) \in \tilde{U}_{l_1 l_2 \dots l_m} \cap \tilde{U}_{i_1 i_2 \dots i_m}(x_{\beta_j k_{s_1^j}}, x_{\beta_j k_{s_2^j}}, \dots, x_{\beta_j k_{s_{t_j}^j}}), \quad t > t_0,$$

where  $j = 1, 2, \dots, m$ ;

when  $j = 1$ , let  $\beta_1 = 1, k_{s_1^1} = 1, 2, \dots, i_1 - 1$ ;

when  $j = 2$ , let  $\beta_2 = 2, k_{s_2^2} = 1, 2, \dots, \widehat{i_1}, \dots, i_2 - 1$ ;

.....

when  $j = m$ , let  $\beta_m = m, k_{s_m^m} = 1, 2, \dots, \widehat{i_1}, \dots, \widehat{i_2}, \dots, \widehat{i_{m-1}}, \dots, i_m - 1$ .

Because  $\text{ind}(O_{i_1 i_2 \dots i_m}) - \text{ind}(O_{l_1 l_2 \dots l_m}) = 1$ , so we have  $i_1 + i_2 + \dots + i_m - (l_1 + l_2 + \dots + l_m) = 1$ , and by  $i_k - l_k > 1$  for some  $k \in 1, 2, \dots, m$ , we get there is a  $j \neq k, j \in 1, 2, \dots, m$  with  $l_j > i_j$ . By it, we have

$$\tilde{U}_{l_1 l_2 \dots l_m} \cap \tilde{U}_{i_1 i_2 \dots i_m}(x_{\beta_j k_{s_1^j}}, x_{\beta_j k_{s_2^j}}, \dots, x_{\beta_j k_{s_{t_j}^j}}) = \emptyset,$$

where  $j = 1, 2, \dots, m$ ;

when  $j = 1$ , let  $\beta_1 = 1, k_{s_1^1} = 1, 2, \dots, i_1 - 1$ ;

when  $j = 2$ , let  $\beta_2 = 2, k_{s_2^2} = 1, 2, \dots, \widehat{i_1}, \dots, i_2 - 1$ ;

.....

when  $j = m$ , let  $\beta_m = m, k_{s_m^m} = 1, 2, \dots, \widehat{i_1}, \dots, \widehat{i_2}, \dots, \widehat{i_{m-1}}, \dots, i_m - 1$ . which contradicts to

$$\varphi(t) \in \tilde{U}_{l_1 l_2 \dots l_m} \cap \tilde{U}_{i_1 i_2 \dots i_m}(x_{\beta_j k_{s_1^j}}, x_{\beta_j k_{s_2^j}}, \dots, x_{\beta_j k_{s_{t_j}^j}}), \quad t > t_0.$$

So we get

$$W^u(O_{i_1 i_2 \dots i_m}) \cap W^s(O_{l_1 l_2 \dots l_m}) = \emptyset.$$

□

By Theorem 1., we know that if  $\text{ind}(O_{i_1 i_2 \dots i_m}) - \text{ind}(O_{l_1 l_2 \dots l_m}) = 1$  and  $W^u(O_{i_1 i_2 \dots i_m}) \cap W^s(O_{l_1 l_2 \dots l_m}) \neq \emptyset$ , so  $\exists k$  with  $i_k - l_k = 1$ , where  $k \in 1, 2, \dots, m$ .

**Theorem 2.** Let  $O_{i_1 i_2 \dots i_m}, O_{l_1 l_2 \dots l_m}$  be any two different critical points of  $f$ , with

$$\text{ind}(O_{i_1 i_2 \dots i_m}) - \text{ind}(O_{l_1 l_2 \dots l_m}) = 1,$$

Then

$$W^u(O_{i_1 i_2 \dots i_m}) \cap W^s(O_{i_1 i_2 \dots (i_k-1) \dots i_m}) = \tilde{U}_{i_1 i_2 \dots i_m}(x_{k(i_k-1)}; x_{k(i_k-1)} \neq 0)$$

*Proof.* By Lemma 8.,  $\forall p \in W^u(O_{i_1 i_2 \dots i_m}) \cap W^s(O_{i_1 i_2 \dots (i_k-1) \dots i_m})$ , the coordinate in  $\tilde{U}_{i_1 i_2 \dots i_m}$  is

$$\begin{aligned} & (x_{11}(p), x_{12}(p), \dots, x_{1(i_1-1)}(p), 1, \\ & x_{21}(p), x_{22}(p), \dots, \widehat{x_{2i_1}(p)}, \dots, x_{2(i_2-1)}(p), 1, \\ & x_{31}(p), x_{32}(p), \dots, \widehat{x_{3i_1}(p)}, \dots, \widehat{x_{3i_2}(p)}, \dots, x_{3(i_3-1)}(p), 1, \\ & \dots, \\ & x_{k1}(p), x_{k2}(p), \dots, \widehat{x_{ki_1}(p)}, \dots, \widehat{x_{ki_2}(p)}, \dots, \widehat{x_{ki_3}(p)}, \dots, \widehat{x_{ki_{k-1}}(p)}, \dots, x_{k(i_k-1)}(p), 1) \\ & \dots, \\ & x_{m1}(p), x_{m2}(p), \dots, \widehat{x_{mi_1}(p)}, \dots, \widehat{x_{mi_2}(p)}, \dots, \widehat{x_{mi_3}(p)}, \dots, \widehat{x_{mi_{m-1}}(p)}, \dots, x_{m(i_m-1)}(p), 1); \end{aligned}$$

the coordinate in  $\tilde{U}_{i_1 i_2 \dots (i_k-1) \dots i_m}$  is

$$\begin{aligned} & (1, y_{1(i_1+1)}(p), y_{1(i_1+2)}(p), \dots, \widehat{y_{1i_2}(p)}, \dots, \widehat{y_{1i_3}(p)}, \dots, \widehat{y_{1i_m}(p)}, \dots, y_{1n}(p), \\ & 1, y_{2(i_2+1)}(p), y_{2(i_2+2)}(p), \dots, \widehat{y_{2i_3}(p)}, \dots, \widehat{y_{2i_m}(p)}, \dots, y_{2n}(p), \\ & 1, y_{3(i_3+1)}(p), y_{3(i_3+2)}(p), \dots, \widehat{y_{3i_4}(p)}, \dots, \widehat{y_{3i_m}(p)}, \dots, y_{3n}(p), \\ & \dots, \\ & 1, y_{k(i_k+1)}(p), y_{k(i_k+2)}(p), \dots, \widehat{y_{ki_{k+1}}(p)}, \dots, \widehat{y_{ki_m}(p)}, \dots, y_{kn}(p), \\ & \dots, \\ & 1, y_{m(i_m+1)}(p), y_{m(i_m+2)}(p), \dots, y_{mn}(p)). \end{aligned}$$

So the change of the coordinate is

$$x_{k(i_k-1)}(p) = \frac{1}{y_{ki_k}(p)} \neq 0, \quad 1 = \frac{y_{ki_k}(p)}{y_{ki_k}(p)};$$

Then we get

$$W^u(O_{i_1 i_2 \dots i_m}) \cap W^s(O_{i_1 i_2 \dots (i_k-1) \dots i_m}) = \tilde{U}_{i_1 i_2 \dots i_m}(x_{k(i_k-1)}; x_{k(i_k-1)} \neq 0)$$

□

By the theorem 2., we can get the following corollary

**Corollary 1.** Let  $O_{i_1 i_2 \dots i_m}, O_{l_1 l_2 \dots l_m}$  be any two different critical points of  $f$ , with

$$\text{ind}(O_{i_1 i_2 \dots i_m}) - \text{ind}(O_{l_1 l_2 \dots l_m}) = 1.$$

Let  $\Gamma(O_{i_1 i_2 \dots i_m}, O_{l_1 l_2 \dots l_m})$  be the numbers of trajectories connecting  $O_{i_1 i_2 \dots i_m}$  to  $O_{l_1 l_2 \dots l_m}$ , then

$$\Gamma(O_{i_1 i_2 \dots i_m}, O_{l_1 l_2 \dots l_m}) = \begin{cases} 2, & i_1 = l_1, i_2 = l_2, \dots, i_{k-1} = l_{k-1}, i_k = l_k + 1, i_{k+1} = l_{k+1}, \dots, i_m = l_m; \\ 0, & \text{otherwise.} \end{cases}$$

where  $k \in 1, 2, \dots, m$ .

## 7 The homology groups of $G_{n,m}(\mathbb{R})$

Now we choose an orientation of the vector space  $E^u(O_{i_1 i_2 \dots i_m}) = T_{O_{i_1 i_2 \dots i_m}} W^u(O_{i_1 i_2 \dots i_m})$  for every critical point  $O_{i_1 i_2 \dots i_m}$  and denote by  $\langle O_{i_1 i_2 \dots i_m} \rangle$  the pair consisting of the critical point  $O_{i_1 i_2 \dots i_m}$  and the orientation.

$$\begin{aligned} \langle O_{i_1 i_2 \dots i_m} \rangle = \{ & O_{i_1 i_2 \dots i_m}; (\partial_{11}, \partial_{12}, \dots, \partial_{1(i_1-1)}, \\ & \partial_{21}, \partial_{22}, \dots, \widehat{\partial_{2i_1}}, \dots, \partial_{2(i_2-1)}, \\ & \partial_{31}, \partial_{32}, \dots, \widehat{\partial_{3i_1}}, \dots, \widehat{\partial_{3i_2}}, \dots, \partial_{3(i_3-1)}, \\ & \dots, \\ & \partial_{m1}, \partial_{m2}, \dots, \widehat{\partial_{mi_1}}, \dots, \widehat{\partial_{mi_2}}, \dots, \widehat{\partial_{mi_3}}, \dots, \widehat{\partial_{mi_{m-1}}}, \dots, \partial_{m(i_m-1)}) \} \end{aligned}$$

By theorem 2., between the critical points  $O_{i_1 i_2 \dots i_m}$  and  $O_{i_1 i_2 \dots (i_k-1) \dots i_m}$ , we have two orbits  $\varphi^+(t)$  and  $\varphi^-(t)$  as following in the coordinate system  $\widetilde{U}_{i_1 i_2 \dots i_m}$

$$\varphi^+(t) = (0, 0, 0, \dots, 0, x_{k(i_k-1)}(t), 0, \dots, 0, 0, 0), \quad x_{k(i_k-1)}(t) > 0$$

where  $\lim_{t \rightarrow -\infty} x_{k(i_k-1)}(t) = 0$ ,  $\lim_{t \rightarrow -\infty} \varphi^+(t) = O_{i_1 i_2 \dots i_m}$ ;

$$\varphi^-(t) = (0, 0, 0, \dots, 0, x_{k(i_k-1)}(t), 0, \dots, 0, 0, 0), \quad x_{k(i_k-1)}(t) < 0$$

where  $\lim_{t \rightarrow -\infty} x_{k(i_k-1)}(t) = 0$ ,  $\lim_{t \rightarrow -\infty} \varphi^-(t) = O_{i_1 i_2 \dots i_m}$ ; and in the coordinate system  $\widetilde{U}_{i_1 i_2 \dots (i_k-1) \dots i_m}$

$$\varphi^+(t) = (0, 0, 0, \dots, 0, \frac{1}{x_{k(i_k-1)}(t)}, 0, \dots, 0, 0, 0), \quad x_{k(i_k-1)}(t) > 0$$

where  $\lim_{t \rightarrow +\infty} x_{k(i_k-1)}(t) = +\infty$ ,  $\lim_{t \rightarrow +\infty} \varphi^+(t) = O_{i_1 i_2 \dots (i_k-1) \dots i_m}$ ;

$$\varphi^-(t) = (0, 0, 0, \dots, 0, \frac{1}{x_{k(i_k-1)}(t)}, 0, \dots, 0, 0, 0), \quad x_{k(i_k-1)}(t) < 0$$

where  $\lim_{t \rightarrow +\infty} x_{k(i_k-1)}(t) = -\infty$ ,  $\lim_{t \rightarrow +\infty} \varphi^-(t) = O_{i_1 i_2 \dots (i_k-1) \dots i_m}$ .

**Theorem 3.**

$$\begin{aligned} n_{\varphi^+(O_{i_1 i_2 \dots i_m}, O_{i_1 i_2 \dots (i_k-1) \dots i_m})} &= 1, \\ n_{\varphi^-(O_{i_1 i_2 \dots i_m}, O_{i_1 i_2 \dots (i_k-1) \dots i_m})} &= (-1)^{i_k - k}. \quad (1 \leq k \leq m) \end{aligned}$$

*Proof.* Because the tangent vector of  $\varphi^\pm(O_{i_1 i_2 \dots i_m}, O_{i_1 i_2 \dots (i_k-1) \dots i_m})$  at  $O_{i_1 i_2 \dots i_m}$  is  $\pm \partial_{k(i_k-1)}$ , then  $\langle O_{i_1 i_2 \dots i_m} \rangle$  induces an orientation on the orthogonal complement  $E_{\varphi^\pm}^u(O_{i_1 i_2 \dots i_m})$  of  $\pm \partial_{k(i_k-1)}$  in  $E^u(O_{i_1 i_2 \dots i_m})$  as follow

$$\begin{aligned} & (\pm \partial_{11}, \partial_{12}, \dots, \partial_{1(i_1-1)}, \\ & \partial_{21}, \partial_{22}, \dots, \widehat{\partial_{2i_1}}, \dots, \partial_{2(i_2-1)}, \\ & \partial_{31}, \partial_{32}, \dots, \widehat{\partial_{3i_1}}, \dots, \widehat{\partial_{3i_2}}, \dots, \partial_{3(i_3-1)}, \\ & \dots, \\ & \partial_{k1}, \partial_{k2}, \dots, \widehat{\partial_{ki_1}}, \dots, \widehat{\partial_{ki_2}}, \dots, \partial_{k(i_k-2)}, \end{aligned}$$

$$\begin{aligned} & \cdots, \\ & \partial_{m1}, \partial_{m2}, \cdots, \widehat{\partial_{mi_1}}, \cdots, \widehat{\partial_{mi_2}}, \cdots, \widehat{\partial_{mi_3}}, \cdots, \widehat{\partial_{mi_{m-1}}}, \cdots, \partial_{m(i_{m-1})}. \end{aligned}$$

Because

$$\begin{aligned} \langle O_{i_1 i_2 \cdots (i_k-1) \cdots i_m} \rangle = & \{ O_{i_1 i_2 \cdots (i_k-1) \cdots i_m}; (\partial_{11}, \partial_{12}, \cdots, \partial_{1(i_1-1)}, \\ & \partial_{21}, \partial_{22}, \cdots, \widehat{\partial_{2i_1}}, \cdots, \partial_{2(i_2-1)}, \\ & \partial_{31}, \partial_{32}, \cdots, \widehat{\partial_{3i_1}}, \cdots, \widehat{\partial_{3i_2}}, \cdots, \partial_{3(i_3-1)}, \\ & \cdots, \\ & \partial_{k1}, \partial_{k2}, \cdots, \widehat{\partial_{ki_1}}, \cdots, \widehat{\partial_{ki_2}}, \cdots, \partial_{k(i_k-2)}, \\ & \cdots, \\ & \partial_{m1}, \partial_{m2}, \cdots, \widehat{\partial_{mi_1}}, \cdots, \widehat{\partial_{mi_2}}, \cdots, \widehat{\partial_{mi_3}}, \cdots, \widehat{\partial_{mi_{m-1}}}, \cdots, \partial_{m(i_{m-1})} \} \end{aligned}$$

to compute  $n_{\varphi^\pm(O_{i_1 i_2 \cdots i_m}, O_{i_1 i_2 \cdots (i_k-1) \cdots i_m})}$ , we compute the Jacobian of the coordinate transformation from  $\widetilde{U}_{i_1 i_2 \cdots i_m}$  to  $\widetilde{U}_{i_1 i_2 \cdots (i_k-1) \cdots i_m}$ , we denote it by  $J_{n_{\varphi^\pm(O_{i_1 i_2 \cdots i_m}, O_{i_1 i_2 \cdots (i_k-1) \cdots i_m})}}$ ,

$$\begin{aligned} & J_{n_{\varphi^\pm(O_{i_1 i_2 \cdots i_m}, O_{i_1 i_2 \cdots (i_k-1) \cdots i_m})}} = \\ & \text{diag}(1, \cdots, 1, \frac{1}{x_{k(i_k-1)}}, \cdots, \frac{1}{x_{k(i_k-1)}}, -\frac{1}{x_{k(i_k-1)}^2}, \frac{1}{x_{k(i_k-1)}}, \cdots, \frac{1}{x_{k(i_k-1)}}, 1, \cdots, 1) \\ & J_{n_{\varphi^\pm(O_{i_1 i_2 \cdots i_m}, O_{i_1 i_2 \cdots (i_k-1) \cdots i_m})}}(\pm \partial_{11}, \partial_{12}, \cdots, \partial_{1(i_1-1)}, \\ & \partial_{21}, \partial_{22}, \cdots, \widehat{\partial_{2i_1}}, \cdots, \partial_{2(i_2-1)}, \\ & \partial_{31}, \partial_{32}, \cdots, \widehat{\partial_{3i_1}}, \cdots, \widehat{\partial_{3i_2}}, \cdots, \partial_{3(i_3-1)}, \\ & \cdots, \\ & \partial_{k1}, \partial_{k2}, \cdots, \widehat{\partial_{ki_1}}, \cdots, \widehat{\partial_{ki_2}}, \cdots, \partial_{k(i_k-2)}, \\ & \cdots, \\ & \partial_{m1}, \partial_{m2}, \cdots, \widehat{\partial_{mi_1}}, \cdots, \widehat{\partial_{mi_2}}, \cdots, \widehat{\partial_{mi_3}}, \cdots, \widehat{\partial_{mi_{m-1}}}, \cdots, \partial_{m(i_{m-1})}) \\ & = (\pm \partial_{11}, \partial_{12}, \cdots, \partial_{1(i_1-1)}, \\ & \partial_{21}, \partial_{22}, \cdots, \widehat{\partial_{2i_1}}, \cdots, \partial_{2(i_2-1)}, \\ & \partial_{31}, \partial_{32}, \cdots, \widehat{\partial_{3i_1}}, \cdots, \widehat{\partial_{3i_2}}, \cdots, \partial_{3(i_3-1)}, \\ & \cdots, \\ & \frac{\partial_{k1}}{x_{k(i_k-1)}}, \frac{\partial_{k2}}{x_{k(i_k-1)}}, \cdots, \widehat{\partial_{ki_1}}, \cdots, \widehat{\partial_{ki_2}}, \cdots, \frac{\partial_{k(i_k-2)}}{x_{k(i_k-1)}}, \\ & \cdots, \\ & \partial_{m1}, \partial_{m2}, \cdots, \widehat{\partial_{mi_1}}, \cdots, \widehat{\partial_{mi_2}}, \cdots, \widehat{\partial_{mi_3}}, \cdots, \widehat{\partial_{mi_{m-1}}}, \cdots, \partial_{m(i_{m-1})}) \end{aligned}$$

So we get, if  $x_{k(i_k-1)} > 0$  then  $n_{\varphi^+(O_{i_1 i_2 \cdots i_m}, O_{i_1 i_2 \cdots (i_k-1) \cdots i_m})} = 1$ ; if  $x_{k(i_k-1)} < 0$  then  $n_{\varphi^-(O_{i_1 i_2 \cdots i_m}, O_{i_1 i_2 \cdots (i_k-1) \cdots i_m})} = (-1)^{i_k-k}$ .  $\square$

By Theorem 3., we have

$$n(O_{i_1 i_2 \dots i_m}, O_{i_1 i_2 \dots (i_k-1) \dots i_m}) = n_{\varphi^+}(O_{i_1 i_2 \dots i_m}, O_{i_1 i_2 \dots (i_k-1) \dots i_m}) + n_{\varphi^-}(O_{i_1 i_2 \dots i_m}, O_{i_1 i_2 \dots (i_k-1) \dots i_m})$$

Then we can get the result about Witten's boundary operator

**Lemma 11.**

$$\partial \langle O_{i_1 i_2 \dots i_m} \rangle = \sum_k (1 + (-1)^{i_k - k}) \langle O_{i_1 i_2 \dots (i_k-1) \dots i_m} \rangle, \quad (1 \leq k \leq m)$$

and if  $i_1 - 1, i_2 - 2, \dots, i_m - m$  all is odd, then

$$\partial \langle O_{i_1 i_2 \dots i_m} \rangle = 0$$

*Proof.* Because  $n_{\varphi^+}(O_{i_1 i_2 \dots i_m}, O_{i_1 i_2 \dots (i_k-1) \dots i_m}) + n_{\varphi^-}(O_{i_1 i_2 \dots i_m}, O_{i_1 i_2 \dots (i_k-1) \dots i_m}) = 1 + (-1)^{i_k - k}$  so  $n(O_{i_1 i_2 \dots i_m}, O_{i_1 i_2 \dots (i_k-1) \dots i_m}) = 1 + (-1)^{i_k - k}$ , and by the definition of Witten's boundary operator, we get

$$\partial \langle O_{i_1 i_2 \dots i_m} \rangle = \sum_k (1 + (-1)^{i_k - k}) \langle O_{i_1 i_2 \dots (i_k-1) \dots i_m} \rangle.$$

When  $i_1 - 1, i_2 - 2, \dots, i_m - m$  all is odd, then  $1 + (-1)^{i_k - k} = 0$  so get the result.  $\square$

Now we can give the most important result in this article,

**Theorem 4.** *The homology groups of real Grassmann manifold  $G_{n,m}(\mathbb{R})$  with integral coefficients by Witten complex is*

$$H_r(G_{n,m}(\mathbb{R}), \mathbb{Z}) = \bigoplus_k \left( \bigoplus_{\substack{j_1+j_2+\dots+j_m=r \\ j_k+1-k=\text{odd}}} \mathbb{Z}_2 \oplus \bigoplus_{\substack{j_1+j_2+\dots+j_m=r \\ j_k+1-k=\text{even}}} \mathbb{Z} \right)$$

where  $k = 1, 2, \dots, m$ .

*Proof.* Since Witten chain complex

$$0 \rightarrow C_{m(n-m)} \rightarrow C_{m(n-m)-1} \rightarrow \dots \rightarrow C_1 \rightarrow C_0 \rightarrow 0$$

where

$$C_r = \bigoplus_{i_1+i_2+\dots+i_m=r} \mathbb{Z} \langle O_{i_1 i_2 \dots i_m} \rangle.$$

By lemma 11., we have

$$\begin{aligned} \partial(C_{r+1}) &= \bigoplus_{i_1+i_2+\dots+i_m=r+1} \mathbb{Z} \partial \langle O_{i_1 i_2 \dots i_m} \rangle \\ &= \bigoplus_{i_1+i_2+\dots+i_m=r+1} \mathbb{Z} \left( \sum_k (1 + (-1)^{i_k - k}) \langle O_{i_1 i_2 \dots (i_k-1) \dots i_m} \rangle \right); \end{aligned}$$

$\ker\{\partial : C_r \rightarrow C_{r-1}\} = \bigoplus_{j_1+j_2+\dots+j_m=r} \mathbb{Z} \langle O_{j_1 j_2 \dots j_m} \rangle$ , where  $j_1 - 1, j_2 - 2, \dots, j_m - m$  all is odd. By the definition of homology groups

$$H_r^W(M, \mathbb{Z}) = \frac{\ker\{\partial : C_r \rightarrow C_{r-1}\}}{\partial(C_{r+1})},$$

then

$$H_r^W(G_{n,m}(\mathbb{R}), \mathbb{Z}) = \bigoplus_k \left( \bigoplus_{\substack{j_1+j_2+\dots+j_m=r \\ j_k+1-k=\text{odd}}} \mathbb{Z}_2 \oplus \bigoplus_{\substack{j_1+j_2+\dots+j_m=r \\ j_k+1-k=\text{even}}} \mathbb{Z} \right)$$

where  $k = 1, 2, \dots, m$ . And because  $H_r^W(G_{n,m}(\mathbb{R}), \mathbb{Z}) = H_r(G_{n,m}(\mathbb{R}), \mathbb{Z})$ , so we get the result.  $\square$

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