DISCRETE MELLIN CONVOLUTION AND ITS EXTENSIONS, PERRON FORMULA AND EXPLICIT FORMULAE

Jose Javier Garcia Moreta
Graduate student of Physics at the UPV/EHU (University of Basque country)
In Solid State Physics
Address: Address: Practicantes Adan y Grijalba 2 5 G
P.O 644 48920 Portugalete Vizcaya (Spain)
E-mail: josegarc2002@yahoo.es

ABSTRACT: In this paper we define a new Mellin discrete convolution, which is related to Perron's formula. Also we introduce new explicit formulae for arithmetic function which generalize the explicit formulae of Weil.

MELLIN DISCRETE CONVOLUTION:

We define the Mellin discrete convolution in the form

$$\sum_{n=1}^{\infty} a(n) f\left(\frac{n}{x}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)G(s)x^{s} \qquad (1)$$

Where $\sum_{n=1}^{\infty}\frac{a(n)}{n^{s}} = G(s)$ is the Dirichlet generating functio of the coefficients a(n) and $F(s) = \int_0^{\infty} dx f(x) x^{s-1}$

The proof is quite easy, first we apply the integral operator $\int_0^{\infty}\frac{dx}{x^{s+1}} f(x)$ to the left of (1) so if the series involving a(n) is completely convergent , so we can switch between the series and the integral then , we have

$$\int_0^{\infty}\frac{dx}{x^{s+1}}\left(\sum_{n=1}^{\infty} a(n) f\left(\frac{n}{x}\right)\right) = \sum_{n=1}^{\infty} a(n)\int_0^{\infty} t^{s-1} f\left(nt\right)dt = \sum_{n=1}^{\infty}\frac{a(n)}{n^{s}}\int_0^{\infty} u^{s-1} f\left(u\right)ds = G(s)F(s) \quad (2)$$

If we apply the inverse operator of $\int_0^{\infty}\frac{dx}{x^{s+1}} f(x)$ which is to both sides $\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\int_0^{\infty}\frac{dx}{x^{s+1}} f(x)\right)x^{s} = f(x)$ then we have proved (1) .

this kind of discrete transform is a discrete analogue to the Mellin Convolution theorem defined for Mellin transforms

$$\int_0^\infty \frac{dt}{t} f\left(\frac{x}{t}\right) g(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)G(s)x^{-s} \quad F(s) = \int_0^\infty dx f(x) x^{s-1} \quad \mathrm{G}(s) = \int_0^\infty dx g(x) x^{s-1} \quad (3)$$

Now, if we set $f\left(\frac{1}{t}\right) = H(t-1) = \begin{cases} 1 & \mathrm{t>1} \\ 0 & t<1 \end{cases}$ we recover Perron's formula [5] for the Coefficients of the Dirichlet series

$$\sum_{n=1}^{\infty} a(n) H\left(\frac{x}{n} - 1\right) = \sum_{n \le x} a(n) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(s)\frac{x^s}{s} \quad \text{since} \quad F(s) = \frac{1}{s} = \int_1^\infty \frac{dx}{x^{s+1}} \quad (4)$$

But one of the best applications of our Mellin convolution is related to several Dirichlet series(see [4]) in the form $\sum_{n=1}^{\infty}\frac{a(n)}{n^s} = G(s)$, Where G(s) includes powers or quotients of the Riemann zeta function for example

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} \qquad -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s} \qquad \frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s} \quad (5)$$

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^s} \qquad \frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\varphi(n)}{n^s} \qquad (6)$$

The definition of the functions inside () and () is as follows

- The Möbius function, $\mu(n) = 1$ if the number 'n' is square-free (not divisible by an square) with an even number of prime factors , $\mu(n) = 0$ if n is not squarefree and if the number 'n' is square-free with an odd number of prime factors.
- The Von Mangoldt function $\Lambda(n) = \log p$, in case 'n' is a prime or a prime power and takes the value 0 otherwise
- The Liouville function $\lambda(n) = (-1)^{\Omega(n)}$ $\Omega(n)$ is the number of prime factors of the number 'n'
- $|\mu(n)|$ is 1 if the number is square-free and 0 otherwise
- $\varphi(n) = n\prod_{p|n}\left(1 - \frac{1}{p}\right)$, the meaning of $p\,|\,n$ is that the product is taken only over the primes p that divide 'n'.

To obtain the coefficients of the Dirichlet series we can use the Perron formula

$$\sum_{n=1}^{\infty}\frac{a(n)}{n^s} = G(s) = s\int_1^\infty \frac{A(x)}{x^{s+1}} \qquad A(x) = \sum_{n \le x} a(n) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{s} G(s) ds \quad (7)$$

If the function G(s) includes powers and quotients of the Riemann zeta function we can use Cauchy's theorem to obtain the explicit formulae for example

$$M(x)=\sum_{n\le x}\mu(n)=-2+\sum_{\rho}\frac{x^{\rho}}{\rho\zeta'(\rho)}+\sum_{n=1}^{\infty}\frac{x^{-2n}}{\zeta'(-2n)(-2n)} \quad (8)$$

$$\Psi(x)=\sum_{n\le x}\Lambda(n)=x-\frac{\zeta'(0)}{\zeta(0)}-\sum_{\rho}\frac{x^{\rho}}{\rho}+\sum_{n=1}^{\infty}\frac{x^{-2n}}{(-2n)} \quad (9)$$

$$L(x)=\sum_{n\le x}\lambda(n)=1+\frac{\sqrt{x}}{\zeta(1/2)}+\sum_{\rho}\frac{x^{\rho}\zeta(2\rho)}{\rho\zeta'(\rho)} \quad (10)$$

$$Q(x)=\sum_{n\le x}|\mu(n)|=1+\frac{6x}{\pi^2}+\sum_{\rho}\frac{x^{\frac{\rho}{2}}\zeta\left(\frac{\rho}{2}\right)}{\rho\zeta'(\rho)}+\sum_{n=1}^{\infty}\frac{x^{-n}\zeta(-n)}{(-2n)\zeta'(-2n)} \quad (11)$$

$$\Phi(x)=\sum_{n\le x}\varphi(n)=\frac{1}{6}+\frac{3x^2}{\pi^2}+\sum_{\rho}\frac{x^{\rho}\zeta(\rho-1)}{\rho\zeta'(\rho)}+\sum_{n=1}^{\infty}\frac{x^{-2n}\zeta(-2n-1)}{(-2n)\zeta'(-2n)} \quad (12)$$

Under the assumption that all the Riemann Non-trivial zeros are simple.

Also we have for the Riemann zeta function and its derivatives

$$\zeta'(-2n)=\frac{(-1)^n\zeta(2n+1)(2n)!}{2^{2n+1}\pi^{2n}} \qquad \zeta'(0)=-\frac{1}{2}\log(2\pi) \qquad \zeta(0)=-\frac{1}{2} \quad (13)$$

The reader will remember the relation between Perron's formula and our discrete convolution , using the work of Baillie [ ] we will give different explicit formulae, to do so we need to use Cauchy's theorem on complex integration and evaluate the closed mellin inverse transform by using the residue theorem $\frac{1}{2\pi i}\oint_C F(s)G(s)x^s$ where 'C' is a closed circuit including all the poles of the Dirichlet series G(s) , we can do this assuming all the Riemann zeros are simple and that the Melliin transform F(s) has no poles inside 'C' , in this case we have the 'explicit formulae'

$$\sum_{n=1}^{\infty}\Lambda(n)f\left(\frac{n}{x}\right)=xF(1)-\sum_{\rho}x^{\rho}F(\rho)-\sum_{n=1}^{\infty}F(-2n)\frac{1}{x^{2n}} \quad (14)$$

$$\sum_{n=1}^{\infty}\mu(n)f\left(\frac{n}{x}\right)=\sum_{\rho}x^{\rho}\frac{F(\rho)}{\zeta'(\rho)}+\sum_{n=1}^{\infty}\frac{F(-2n)}{\zeta'(-2n)}\frac{1}{x^{2n}} \quad (15)$$

$$\sum_{n=1}^{\infty}\lambda(n)f\left(\frac{n}{x}\right)=\frac{\sqrt{x}}{2\zeta\left(\frac{1}{2}\right)}F\left(\frac{1}{2}\right)+\sum_{\rho}x^{\rho}\frac{\zeta(2\rho)F(\rho)}{\zeta'(\rho)} \quad (16)$$

$$\sum_{n=1}^{\infty}\varphi(n)f\left(\frac{n}{x}\right)=\frac{6}{\pi^2}F(2)x^2+\sum_{\rho}x^{\rho}\frac{\zeta(\rho-1)F(\rho)}{\zeta'(\rho)}+\sum_{n=1}^{\infty}\frac{F(-2n)}{x^{2n}}\frac{\zeta(-2n-1)}{\zeta'(-2n)} \quad (17)$$

$$\sum_{n=1}^{\infty}|\mu(n)|f\left(\frac{n}{x}\right)=\frac{6}{\pi^2}F(1)x+\sum_{\rho}x^{\frac{\rho}{2}}\frac{\zeta\left(\frac{\rho}{2}\right)F\left(\frac{\rho}{2}\right)}{2\zeta'(\rho)}+\sum_{n=1}^{\infty}\frac{F(-n)}{x^{n}}\frac{\zeta(-n)}{2\zeta'(-2n)} \quad (18)$$

If the Mellin transform has poles inside the closed circuit 'C' $\oint_C F(s)G(s)x^s$, then this poles will contribute with a remainder term due to the Residue theorem [1] in this case we have the extra term

$$r(x)=\sum_k \mathrm{Res}\left\{F(s)G(s)x^s\right\}_{s=k} \quad \text{with} \quad F(k)=\int_0^{\infty}dxf(x)x^{k-1}=\infty \quad (19)$$

this is what happens in Perron formula , due to the step function $H(x-1)$ in this case its Mellin transform has a pole at $s=0$ since $F(s)=\frac{1}{s}$ this is why in formulae (8-12) there is a constant term.

As a curious final example of our Mellin discrete convolution , if we use the Dirichlet generating function $G(s)=\zeta(s-k)$ and the floor function as a test function so $\int_0^{\infty}\frac{dx}{x^{s+1}}[x]=\frac{\zeta(s)}{s}$, then our Mellin discrete convolution becomes the identity for the k-th order sum of the divisor function

$$\sum_{n\le x}\sigma_k(n)=\sum_{n=1}^{\infty}n^k\left[\frac{x}{n}\right]=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{ds}{s}x^s\zeta(s-k)\zeta(s) \quad (20)$$

We have previously investigated this kind of explicit formula [3] but instead of the Mellin transform we used the Fourier transform and Fourier convolution theorem for test functions g(x) and h(x) related by a dualFourier transform , so the integral $h(c)=\int_{-\infty}^{\infty}dxg(x)e^{icx}$ exists and is finite for every real number (positive or negative) ‘c’, and $g(\alpha)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dxh(x)e^{-i\alpha x}$ or $g(\alpha)=\frac{1}{\pi}\int_0^{\infty}dxh(x)\cos(\alpha x)$ depending on if the test function are even or not $h(x)=h(-x)$.

For the case of the Liouville function, there is no contribution due to the nontrivial Riemann zeroes -2,-4,-6,... since the Dirichlet generating functions for this case $\frac{\zeta(2s)}{\zeta(s)}$ is Holomorphic on the region of the complex plane $\mathrm{Re}(s)<0$

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