DISCRETE MELLIN CONVOLUTION AND ITS EXTENSIONS, PERRON FORMULA AND EXPLICIT FORMULAE

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ABSTRACT: In this paper we define a new Mellin discrete convolution, which is related to Perron's formula. Also we introduce new explicit formulae for arithmetic function which generalize the explicit formulae of Weil.

MELLIN DISCRETE CONVOLUTION:

We define the Mellin discrete convolution in the form

$$\sum_{n=1}^{\infty} a(n) f\left(\frac{n}{x}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)G(s)x^{s} \qquad (1)$$

Where $\sum_{n=1}^{\infty}\frac{a(n)}{n^{s}} = G(s)$ is the Dirichlet generating functio of the coefficients a(n) and $F(s) = \int_{0}^{\infty} dx f(x) x^{s-1}$

The proof is quite easy, first we apply the integral operator $\int_{0}^{\infty}\frac{dx}{x^{s+1}} f(x)$ to the left of (1) so if the series involving a(n) is completely convergent , so we can switch between the series and the integral then , we have

$$\int_{0}^{\infty}\frac{dx}{x^{s+1}}\left(\sum_{n=1}^{\infty} a(n) f\left(\frac{n}{x}\right)\right) = \sum_{n=1}^{\infty} a(n)\int_{0}^{\infty} t^{s-1} f\left(nt\right)dt = \sum_{n=1}^{\infty}\frac{a(n)}{n^{s}}\int_{0}^{\infty} u^{s-1} f\left(u\right) ds = G(s)F(s) \quad (2)$$

If we apply the inverse operator of $\int_{0}^{\infty}\frac{dx}{x^{s+1}} f(x)$ which is to both sides $\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\int_{0}^{\infty}\frac{dx}{x^{s+1}} f(x)\right)x^{s} = f(x)$ then we have proved (1) .

this kind of discrete transform is a discrete analogue to the Mellin Convolution theorem defined for Mellin transforms

$$\int_0^\infty \frac{dt}{t} f\left(\frac{x}{t}\right) g(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)G(s)x^{-s} \quad F(s) = \int_0^\infty dx f(x) x^{s-1} \quad \mathrm{G}(s) = \int_0^\infty dx g(x) x^{s-1} \quad (3)$$

Now, if we set $f\left(\frac{1}{t}\right) = H(t-1) = \begin{cases} 1 & \mathrm{t{>}1} \\ 0 & t<1 \end{cases}$ we recover Perron's formula [5] for the Coefficients of the Dirichlet series

$$\sum_{n=1}^{\infty} a(n) H\left(\frac{x}{n}-1\right) = \sum_{n\le x} a(n) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(s)\frac{x^s}{s} \quad \text{since} \quad F(s) = \frac{1}{s} = \int_1^\infty \frac{dx}{x^{s+1}} \quad (4)$$

But one of the best applications of our Mellin convolution is related to several Dirichlet series(see [4]) in the form $\sum_{n=1}^{\infty}\frac{a(n)}{n^s} = G(s)$, Where G(s) includes powers or quotients of the Riemann zeta function for example

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} \qquad -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s} \qquad \frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s} \quad (5)$$

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^s} \qquad \frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\varphi(n)}{n^s} \qquad (6)$$

The definition of the functions inside () and () is as follows

- The Möbius function, $\mu(n)=1$ if the number 'n' is square-free (not divisible by an square) with an even number of prime factors , $\mu(n)=0$ if n is not squarefree and if the number 'n' is square-free with an odd number of prime factors.
- The Von Mangoldt function $\Lambda(n) = \log p$, in case 'n' is a prime or a prime power and takes the value 0 otherwise
- The Liouville function $\lambda(n) = (-1)^{\Omega(n)}$ $\Omega(n)$ is the number of prime factors of the number 'n'
- $|\mu(n)|$ is 1 if the number is square-free and 0 otherwise
- $\varphi(n) = n\prod_{p|n}\left(1-\frac{1}{p}\right)$, the meaning of $p\,|\,n$ is that the product is taken only over the primes p that divide 'n'.

To obtain the coefficients of the Dirichlet series we can use the Perron formula

$$\sum_{n=1}^{\infty}\frac{a(n)}{n^s} = G(s) = s\int_1^\infty \frac{A(x)}{x^{s+1}} \qquad A(x) = \sum_{n\le x} a(n) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{s}G(s)ds \quad (7)$$

If the function G(s) includes powers and quotients of the Riemann zeta function we can use Cauchy's theorem to obtain the explicit formulae for example

$$M(x)=\sum_{n\le x}\mu(n)=-2+\sum_{\rho}\frac{x^{\rho}}{\rho\zeta'(\rho)}+\sum_{n=1}^{\infty}\frac{x^{-2n}}{\zeta'(-2n)(-2n)} \quad (8)$$

$$\Psi(x)=\sum_{n\le x}\Lambda(n)=x-\frac{\zeta'(0)}{\zeta(0)}-\sum_{\rho}\frac{x^{\rho}}{\rho}+\sum_{n=1}^{\infty}\frac{x^{-2n}}{(-2n)} \quad (9)$$

$$L(x)=\sum_{n\le x}\lambda(n)=1+\frac{\sqrt{x}}{\zeta(1/2)}+\sum_{\rho}\frac{x^{\rho}\zeta(2\rho)}{\rho\zeta'(\rho)} \quad (10)$$

$$Q(x)=\sum_{n\le x}|\mu(n)|=1+\frac{6x}{\pi^2}+\sum_{\rho}\frac{x^{\frac{\rho}{2}}\zeta\left(\frac{\rho}{2}\right)}{\rho\zeta'(\rho)}+\sum_{n=1}^{\infty}\frac{x^{-n}\zeta(-n)}{(-2n)\zeta'(-2n)} \quad (11)$$

$$\Phi(x)=\sum_{n\le x}\varphi(n)=\frac{1}{6}+\frac{3x^2}{\pi^2}+\sum_{\rho}\frac{x^{\rho}\zeta(\rho-1)}{\rho\zeta'(\rho)}+\sum_{n=1}^{\infty}\frac{x^{-2n}\zeta(-2n-1)}{(-2n)\zeta'(-2n)} \quad (12)$$

Under the assumption that all the Riemann Non-trivial zeros are simple.

Also we have for the Riemann zeta function and its derivatives

$$\zeta'(-2n)=\frac{(-1)^n\zeta(2n+1)(2n)!}{2^{2n+1}\pi^{2n}} \qquad \zeta'(0)=-\frac{1}{2}\log(2\pi) \qquad \zeta(0)=-\frac{1}{2} \quad (13)$$

The reader will remember the relation between Perron's formula and our discrete convolution , using the work of Baillie [ ] we will give different explicit formulae, to do so we need to use Cauchy's theorem on complex integration and evaluate the closed mellin inverse transform by using the residue theorem $\frac{1}{2\pi i}\oint_C F(s)G(s)x^s$ where 'C' is a closed circuit including all the poles of the Dirichlet series G(s) , we can do this assuming all the Riemann zeros are simple and that the Melliin transform F(s) has no poles inside 'C' , in this case we have the 'explicit formulae'

$$\sum_{n=1}^{\infty}\Lambda(n)f\left(\frac{n}{x}\right)=xF(1)-\sum_{\rho}x^{\rho}F(\rho)-\sum_{n=1}^{\infty}F(-2n)\frac{1}{x^{2n}} \quad (14)$$

$$\sum_{n=1}^{\infty}\mu(n)f\left(\frac{n}{x}\right)=\sum_{\rho}x^{\rho}\frac{F(\rho)}{\zeta'(\rho)}+\sum_{n=1}^{\infty}\frac{F(-2n)}{\zeta'(-2n)}\frac{1}{x^{2n}} \quad (15)$$

$$\sum_{n=1}^{\infty}\lambda(n)f\left(\frac{n}{x}\right)=\frac{\sqrt{x}}{2\zeta\left(\frac{1}{2}\right)}F\left(\frac{1}{2}\right)+\sum_{\rho}x^{\rho}\frac{\zeta(2\rho)F(\rho)}{\zeta'(\rho)} \quad (16)$$

$$\sum_{n=1}^{\infty}\varphi(n) f\left(\frac{n}{x}\right)=\frac{6}{\pi^2}F(2)x^2+\sum_{\rho}x^{\rho}\frac{\zeta(\rho-1)F(\rho)}{\zeta'(\rho)}+\sum_{n=1}^{\infty}\frac{F(-2n)}{x^{2n}}\frac{\zeta(-2n-1)}{\zeta'(-2n)} \quad (17)$$

$$\sum_{n=1}^{\infty}|\mu(n)| f\left(\frac{n}{x}\right)=\frac{6}{\pi^2}F(1)x+\sum_{\rho}x^{\frac{\rho}{2}}\frac{\zeta\left(\frac{\rho}{2}\right)F\left(\frac{\rho}{2}\right)}{2\zeta'(\rho)}+\sum_{n=1}^{\infty}\frac{F(-n)}{x^{n}}\frac{\zeta(-n)}{2\zeta'(-2n)} \quad (18)$$

If the Mellin transform has poles inside the closed circuit 'C' $\oint_C F(s)G(s)x^s$, then this poles will contribute with a remainder term due to the Residue theorem [1] in this case we have the extra term

$$r(x)=\sum_{k}\mathrm{Re}s\left\{F(s)G(s)x^s\right\}_{s=k} \quad \text{with} \quad F(k)=\int_0^{\infty}dxf(x)x^{k-1}=\infty \quad (19)$$

this is what happens in Perron formula , due to the step function $H(x-1)$ in this case its Mellin transform has a pole at $s=0$ since $F(s)=\frac{1}{s}$ this is why in formulae (8-12) there is a constant term.

As a curious final example of our Mellin discrete convolution , if we use the Dirichlet generating function $G(s)=\zeta(s-k)$ and the floor function as a test function so $\int_0^{\infty}\frac{dx}{x^{s+1}}[x]=\frac{\zeta(s)}{s}$, then our Mellin discrete convolution becomes the identity for the k-th order sum of the divisor function

$$\sum_{n\le x}\sigma_k(n)=\sum_{n=1}^{\infty}n^k\left[\frac{x}{n}\right]=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{ds}{s}x^s\zeta(s-k)\zeta(s) \quad (20)$$

We have previously investigated this kind of explicit formula [3] but instead of the Mellin transform we used the Fourier transform and Fourier convolution theorem for test functions g(x) and h(x) related by a dualFourier transform , so the integral $h(c)=\int_{-\infty}^{\infty}dxg(x)e^{icx}$ exists and is finite for every real number (positive or negative) 'c', and $g(\alpha)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dxh(x)e^{-i\alpha x}$ or $g(\alpha)=\frac{1}{\pi}\int_0^{\infty}dxh(x)\cos(\alpha x)$ depending on if the test function are even or not $h(x)=h(-x)$.

For the case of the Liouville function, there is no contribution due to the nontrivial Riemann zeroes -2,-4,-6,... since the Dirichlet generating functions for this case $\frac{\zeta(2s)}{\zeta(s)}$ is Holomorphic on the region of the complex plane $\mathrm{Re}(s)<0$

In our previous work [3] we have stablished similar formulae to (14-18) but in terms only of the imaginary part of the Riemann zeros

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{\sqrt{n}}g(\log n)=\sum_{\gamma}\frac{h(\gamma)}{\zeta'(\rho)}+\sum_{n=1}^{\infty}\frac{1}{\zeta'(-2n)}\int_{-\infty}^{\infty}dxg(x)e^{-x\left(2n+\frac{1}{2}\right)} \quad (21)$$

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{\sqrt{n}}g(\log n)=\frac{1}{2\zeta(1/2)}\int_{-\infty}^{\infty}dxg(x)+\sum_{\gamma}\frac{\zeta(2\rho)h(\gamma)}{\zeta'(\rho)} \quad (22)$$

$$\sum_{n=1}^{\infty}\frac{\varphi(n)}{\sqrt{n}}g(\log n)=\frac{6}{\pi^2}\int_{-\infty}^{\infty}dxg(x)e^{\frac{3}{2}x}+\sum_{\gamma}\frac{h(\gamma)}{\zeta'(\rho)}\zeta(\rho-1)+\sum_{n=1}^{\infty}\frac{\zeta(-2n-1)}{\zeta'(-2n)}\int_{-\infty}^{\infty}dxg(x)e^{-x\left(2n+\frac{1}{2}\right)} \quad (23)$$

$$\sum_{n=1}^{\infty}\frac{|\mu(n)|}{n^{\frac{1}{4}}}g(\log n)=\frac{6}{\pi^2}\int_{-\infty}^{\infty}dxg(x)e^{\frac{3}{4}x}+\sum_{\gamma}\frac{h\left(\frac{\gamma}{2}\right)}{2\zeta'(\rho)}\zeta\left(\frac{\rho}{2}\right)+\frac{1}{2}\sum_{n=1}^{\infty}\frac{\zeta(-n)}{\zeta'(-2n)}\int_{-\infty}^{\infty}dxg(x)e^{-x\left(n+\frac{1}{4}\right)} \quad (24)$$

And finally the explcit formula for the divisor function $\sigma(n)$ which is the sum of divisors of 'n' $\sigma(12)=1+2+3+4+6+12=28$, given by

$$\sum_{n=1}^{\infty}\frac{\sigma_1(n^2)}{n^{\frac{5}{4}}}g(\log n)=-\frac{1}{12}\int_{-\infty}^{\infty}g(x)e^{\frac{-x}{4}}dx-\frac{1}{2}\int_{-\infty}^{\infty}g(x)e^{\frac{3x}{4}}dx+\frac{15\zeta(3)}{\pi^2}\int_{-\infty}^{\infty}g(x)e^{\frac{7x}{4}}+\sum_{\gamma}\frac{h\left(\frac{\gamma}{2}\right)}{2\zeta'(\rho)}\zeta\left(\frac{\rho}{2}\right)\zeta\left(\frac{\rho}{2}+1\right)\zeta\left(\frac{\rho}{2}-1\right) \quad (25)$$

Where the sum inside (21-25) are over the imaginary part of the zeros of the Riemann zeta function on the critical line, and $\rho=\frac{1}{2}+i\gamma$.

Equations (14-18) are equivalent to the equations (21-25) but in one hand we use the Mellin transform and in the other hand we use the Fourier transform $g(\alpha)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dxh(x)e^{-i\alpha x}$, the use of the Fourier transform is in analogy to the Riemann-Weil explicit formula for the Von Mangoldt function

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}g(\log n)=\frac{1}{4\pi}\int_{-\infty}^{\infty}dr\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{ir}{2}\right)+h\left(\frac{i}{2}\right)-\frac{h(0)}{2}\log\pi-\sum_{k=0}^{\infty}h(\gamma_k) \quad (26)$$

In the formula (26) the sum is over the positive imaginary parts of the Riemann zeros . For the case of the explicit formulae which involve the test function g(x) the Laplace Bilateral transform of this function defined by

$$L\{g(t)\}(s)=\int_{-\infty}^{\infty}dtg(t)e^{-st} \qquad s\in(0,\infty) \quad (27)$$

must be finite , or at least regularizable,

Formulae (14-18) and (21-24) are related if we use the test function $e^{au}\phi(xe^{u})$ for some real positive number 'a' ($a=\frac{1}{2},\frac{1}{4},\frac{5}{4},....$)and a nonzero number 'x' , in this case using 2 change of variables

$$\int_{-\infty}^{\infty}due^{au}\phi(xe^{u})e^{iu\omega} \quad \to \int_{0}^{\infty}dtt^{a}\phi(xt)t^{i\omega-1} \quad \to \frac{1}{x^{s}}\int_{0}^{\infty}dtt^{s-1}\phi(t) \qquad (28)$$

With $s=a+i\omega$, then the Fourier transform turns into a Mellin transform

An easier derivation of our explicit formulae

There is an easier derivation for our explicit formulae, in general after Perron's formula is applied we find the following identity

$$\sum_{n<x}a_{n}=P+Qx^{d}+\sum_{\rho}h(\rho)x^{\rho}+\sum_{n=1}^{\infty}c_{2n}x^{-2nr} \quad (29)$$

For some real constants P,Q,d,c_{2n},r and a function $h(\rho)$,which includes the Riemann zeta function and its first derivative

Taking the distributional derivative for an step function of the form $\sum_{n<x}a_{n}$

$$\frac{d}{dx}\left(\sum_{n<x}a_{n}\right)=\sum_{n=1}^{\infty}a_{n}\delta(x-n)=dQx^{d-1}+\sum_{\rho}h(\rho)\rho x^{\rho-1}+\sum_{n=1}^{\infty}c_{2n}(-2nr)x^{-2nr-1} \quad (30)$$

So if we apply a certain test function with a parameter 'x' $f\left(\frac{t}{x}\right)$ and its Mellin transform defined by

$$\int_{0}^{\infty}dtf\left(\frac{t}{x}\right)t^{s-1}\to x^{s}\int_{0}^{\infty}dtf\left(t\right)t^{s-1}=x^{s}F(s) \quad (31)$$

Then we find the desired explicit formula

$$\sum_{n=1}^{\infty} a_n f\left(\frac{n}{x}\right) = dQF(d) + \sum_{\rho} h(\rho)\rho F(\rho) + \sum_{n=1}^{\infty} c_{2n}(-2nr)F(-2nr) \quad (32)$$

In general for Dirichlet series generating functions $\sum_{n=1}^{\infty} \frac{a_n}{n^s} = G(s)$ the zeroeth coefficient is $a_0 = 0$ so the series inside (31) begins at $n = 1$

A similar method can be applied to derive the Poisson summation formula, let be the Floor function $[x]$, then we have a formula valid on the whole real line

$$[x] = x - \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n} \quad (33)$$

Taking the distributional derivative of (33) and using the Euler's formula for the cosine function

$$\frac{d}{dx}[x] = \sum_{n=-\infty}^{\infty} \delta(x-n) = 1 + 2\sum_{n=1}^{\infty} \cos(2\pi nx) \qquad \cos(x) = \frac{e^{ix} + e^{-ix}}{2} \quad (34)$$

Now if we use a test function inside (33) we have the Poisson summation formula

$$\sum_{n=-\infty}^{\infty} f(n) = \int_{-\infty}^{\infty} f(x) \sum_{n=-\infty}^{\infty} e^{2\pi inx} \quad (35)$$

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