Multifractal Analysis and the Dynamics of Effective Field Theories

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Abstract

We show that the flow from the ultraviolet to the infrared sector of any multidimensional nonlinear field theory approaches chaotic dynamics in a universal way. This result stems from several independent routes to aperiodic behavior and implies that the infrared attractor of effective field theories is likely to replicate the geometry of *multifractal* sets. In particular, we find that the Einstein-Hilbert Lagrangian is characterized by a *single* generalized dimension ($D_{GR} = 4$), while the Standard Model (SM) Lagrangian is defined by a *triplet* of generalized dimensions ($D_{SM} = 2, 3$ and 4). On the one hand, this finding disfavors any naïve field-theoretic unification of SM and General Relativity (GR). On the other, it hints that the continuous spectrum of generalized dimensions lying between $D = 2$ and $D = 4$ may naturally account for the existence of non-baryonic Dark Matter.

1. Introduction

Few theorists would dispute the compelling success enjoyed by the two pillars of contemporary science, the Standard Model of high-energy physics (SM) and General Relativity (GR). As *effective* field-theories, SM and GR describe remarkably well a wealth of phenomena, from sub-nuclear physics to the realm of astronomical scales and cosmology. However, several long-standing issues hint that either new physics *or* a deeper conceptual structure is required for a complete account of Nature beyond SM and GR [1-3, 26-27].

Recently, H. Nicolai has summarized the main foundational challenges confronting both the SM and GR [4]. His critique targets the vastly uncharted territory lying beyond *perturbative quantum field theory*, as well as the inherent singularities of the strong gravity regime in GR:

"*But the real problem with the SM is theoretical: it is not clear whether it makes sense at all as a theory beyond perturbation theory, and these doubts extend to the whole framework of quantum field theory (QFT) (with perturbation theory as the main tool to extract quantitative predictions). The occurrence of "ultraviolet" (UV) divergences in Feynman diagrams, and the need for an elaborate mathematical procedure called renormalisation to remove these infinities and make testable predictions order-by-order in perturbation theory, strongly point to the necessity of some other and more complete theory of elementary particles.*

On the GR side, we are faced with a similar dilemma. Like the SM, GR works extremely well in its domain of applicability and has so far passed all experimental tests with flying colours, most recently and impressively with the direct detection of gravitational waves (see "General relativity at 100"). Nevertheless, the need for a theory beyond Einstein is plainly evident from the existence of space–time singularities such as those occurring inside black holes or at the moment of the Big Bang. Such singularities are an unavoidable consequence of Einstein's equations, and the failure of GR to provide an answer calls into question the very conceptual foundations of the theory."

In this work, we do not proceed along the path of Quantum Gravity, as suggested by Nicolai and pursued by many other researchers in the field. Working in the context of the emergence paradigm, we model the flow from the ultraviolet to the infrared regime of field theory starting from the universal behavior of far-from-equilibrium nonlinear dynamical systems. The bottom line of this approach is that globally stable *strange attractors,* along with their description in terms of *multifractals*, define the fabric of *effective field theories (EFT).*

The paper is organized as follows: section two covers the long-term approach to chaos in nonlinear dynamics and section three presents a brief overview of multifractal analysis. This enables development of next sections, which reveal surprising connections between multifractal analysis and effective field theories, in particular the SM and GR. A brief discussion and concluding remarks form the topic of the last section.

The reader is cautioned upfront on the preliminary nature of our investigation. Concurrent research is needed to support, expand or debunk these tentative ideas.

2. The long-term approach to chaos in nonlinear dynamics

Generic Quantum Field Theories (QFT) are known to become scale-invariant at large distances. Viewed in the context of conformal field theory, this property is customarily associated with the *fixed-point* structure of the Renormalization Group (RG) flow [5-6]. Starting from this observation, we conjecture below that all field theories evaluated at sufficiently low-energy scales emerge from an underlying system of coupled high-energy entities called *primary variables*. Let the ultraviolet (UV) sector of field theory be described by a large set of such variables $x \equiv \{x_i\}$, $i = 1, 2, ..., n$, $n >> 1$, whose dynamics is *far-from-equilibrium*. The specific nature of the UV variables is irrelevant to our context, as they can take the form of irreducible objects such as, but not limited to, strings, branes, loops, knots, bits of information and so on.

The downward flow of $x \equiv \{x_i\}$ may be mapped to a system of ordinary differential equations having the universal form

$$x'_\tau = f(x(\tau), \lambda(\tau), \tau, D(\tau)) \tag{1}$$

Here, λ, τ, D denote, respectively, the control parameters vector $\lambda = \{\lambda_j\}$, $j = 1, 2, \ldots m$, the evolution parameter and the dimension of the embedding space. If the dimension of the embedding space is taken to be independent variable or control parameter, the system (1) further reduces to

$$x'_\tau = f(x(\tau), \lambda(\tau), \tau) \tag{2}$$

It is reasonable to assume that the flow (1) and (2) occurs in the presence of non-vanishing *perturbations* induced by far-from-equilibrium conditions. These may arise, for example, from primordial density fluctuations in the early Universe or from unbalanced vacuum fluctuations in the high-energy regime of QFT.

To make explicit the effect of perturbations, we resolve $x(\tau)$ into a reference stable state $x_s(\tau)$ and a deviation generated by perturbations, i.e.,

$$x(\tau) = x_s(\tau) + y(\tau) \tag{3}$$

Direct substitution in (1) yields the set of homogeneous equations

$$y'_\tau = f(\{x_s + y\}, \lambda) - f(\{x_s\}, \lambda) \tag{4}$$

Further expanding around the reference state leads to

$$y'_\tau = \sum_j L_{ij}(x_s, \lambda) y_j + h_i(\{y_j\}, \lambda) \quad (5)$$

where L_{ij} and h_i denote, respectively, the coefficients of the linear and nonlinear contributions induced by the deviations from the reference state. Here, L_{ij} represents a $n \times n$ matrix dependent on the reference state and on the control parameters vector. Under the assumption that parameters λ stay close to their critical values $(\lambda = \lambda_c)$, it can be shown that either (1) or (2) undergoes bifurcations and it can be mapped to a closed set of universal equations referred to as *normal forms* [7]. If, at $\lambda = \lambda_c$ perturbations are non-oscillatory (steady-state), the normal form equations are

$$z'_\tau = (\lambda - \lambda_c) - uz^2 \quad (6)$$

$$z'_\tau = (\lambda - \lambda_c) z - uz^3 \quad (7)$$

$$z'_\tau = (\lambda - \lambda_c) z - uz^2 \quad (8)$$

If perturbations are oscillatory (periodic) at $\lambda = \lambda_c$, the normal form equation is instead given by

$$z'_\tau = [(\lambda - \lambda_c) + i\omega_0] z - uz|z|^2 \quad (9)$$

which relates to the *complex Ginzburg-Landau equation* [8-9]. Either way, the outcome of this brief analysis is that the multivariable dynamics contained in (1) and (2) reduces to a lower dimensional system of equations, with the emerging variable z playing the role of an effective *order parameter*.

It can be further shown that, as the control parameter scans a sequence of critical values defined through

$$\lambda_l - \lambda_c \approx K\delta^{-l} \tag{10}$$

where K and δ are constants and $l = 1, 2, \ldots, N$ ($N >> 1$), (6)–(8) undergo universal transition to chaos via a cascade of period-doubling bifurcations called the *Feigenbaum scenario* [10-11]. In general, the transition to chaos in multivariable systems of nonlinear equations either confines trajectories to a *strange attractor* or drives them away from a *strange repeller* [12-13]. Likewise, (9) can be shown to lead to *spatiotemporal chaos* through collective bifurcations and pattern formation outside equilibrium [14].

It has been long known that the asymptotic onset of chaos in (1) or (2) can develop through alternative mechanisms that do not involve reduction to normal forms. Among them we mention quasi-periodicity, intermittency and crises, as well as chaotic transients and homoclinic orbits [15].

Taken together, these considerations suggest that the flow from the ultraviolet to the infrared sector of any high-dimensional nonlinear field theory is prone to transition to chaos, regardless of the specific content of (1) and (2) or their boundary conditions. Since *multifractals* are the natural descriptors of strange attractors, it follows that the infrared attractor (or repeller) of effective field theories is likely to replicate the geometry of multifractal sets.

It is important to also recall that strange attractors, as fingerprints of chaos, share deep roots with equilibrium statistical mechanics via ergodicity, global stability and invariant probability distributions [16]. As a result, the connection between equilibrium statistical mechanics and field theory [17] reinforces the view that effective field theories emerge as the most likely infrared endpoint of either (1) or (2).

3. Multifractals: a concise overview

As it is known, the *box-counting dimension* defines the main scaling property of fractal structures and is a measure of their self-similarity. Multifractals are global mixtures of fractal structures, each characterized by its local box-counting dimension. Self-similarity of multifractals is accordingly defined in terms of a multifractal spectrum describing the overall distribution of dimensions. In the language of chaos and complexity theory, multifractal analysis is the study of *invariant sets* and is a powerful tool for the characterization of generic dynamical systems.

In the recursive construction of multifractal sets from $i = 1, 2..., N$ local scales r_i with probabilities p_i, the definition of the box-counting dimension leads to [18-19]

$$\sum_{i=1}^{N} p_i^q r_i^{\tau(q)} = 1 \tag{11}$$

in which

$$\sum_{i=1}^{N} p_i = 1 \tag{12}$$

Here, q and $\tau(q)$ are two arbitrary exponents and the latter is typically presented as

$$\tau(q) = (1-q)D_q \tag{13}$$

where D_q plays the role of a *generalized dimension*.

The closure relationship (11) may be extended to a continuous distribution of scales in D - dimensional space time. It reads

$$\boxed{\int p^{q}(x)\, r^{\tau(q)}(x)\, d^{D}x = 1} \tag{14}$$

4. GR as topological analogue of SM

Consider now the field makeup of the SM, formed by 16 *independent* “flavors”: two massive gauge bosons (W, Z), gluon (g), the Higgs scalar (H), neutrinos, charged leptons and quarks. The SM structure can be conveniently organized in the 4×4 matrix

$$SM = \begin{pmatrix} g & \nu_e & \nu_\mu & \nu_\tau \\ W & e & \mu & \tau \\ Z & u & c & b \\ H & d & s & t \end{pmatrix} \tag{15}$$

The photon (γ) is absent from (15) as it is built from the underlying components of the electroweak sector, whereby $\gamma = \gamma(W_\mu^3, B_\mu)$ and $B_\mu = B_\mu(W_\mu^3, Z)$ [19].

It was shown in [20-21] that, when evaluated at the global electroweak scale M_{EW}, the spectrum of particle masses m_i entering the SM satisfies the “closure” relation

$$\sum_{i=1}^{16} \left(\frac{m_i}{M_{EW}}\right)^2 = 1 \tag{16}$$

It is apparent that (15) shares the same formal structure with the metric tensor of GR, that is,

$$GR = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} \tag{17}$$

where there are only 10 independent entries under the standard assumption $g_{\mu\nu} = g_{\nu\mu}$.

Starting from the GR definitions of interval and proper time leads to ($c = 1$)

$$\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = 1 \tag{18}$$

subject to the constraint

$$\sum_{\nu=0}^{3} g^{\mu\nu}g_{\nu\rho} = \delta^{\mu}_{\rho} = \begin{cases} 1, & \mu = \rho \\ 0, & \mu \neq \rho \end{cases} \tag{19}$$

Comparing (11), (12) to (16) and (18), (19) reveals the following mapping

$$\boxed{GR : (\, p_i \Rightarrow g^{\mu\nu}g_{\nu\rho},\; q = 1/2\;,\; D_q = 4\,,\; \tau(q) = 2\,)}$$

$$\boxed{SM : (\, p_i \Rightarrow 1,\; q = 0,\; D_q = \tau(q) = 2\,)} \tag{20}$$

It is instructive to note that $D_0 = 2$ coincides with the fractal dimension of quantum mechanical paths in free space [22], whereas $D_{1/2} = 4$ recovers the four-dimensionality of geodesic paths in classical spacetime.

A couple of conclusions may be drawn from (20):

1) GR may be viewed as topological analogue of the SM, defined by a half-unitary exponent q and a dimension that is twice the SM dimension (that is, $D_{1/2} = 2D_0$).

2) The spectrum of particle *mass scales* (m_i/M_{EW}) and the *four-vector of differential coordinates* ($dx^\mu/d\tau$) form the basis for the multifractal description of SM and GR, respectively.

5. Multifractal formulation of effective field theories

Effective Lagrangians in field theory may be represented as sums of polynomial terms having the generic form

$$L(\varphi,\partial\varphi) = \sum_{i,k,l,m,n} [c_{i,i}(\partial\varphi)^k + c_{i,i+1}(\partial\varphi)^l(\varphi)^m + c_{i+1,i+1}(\varphi)^n] \tag{21}$$

To simplify notation, we focus below on the basic unit entering (21), namely on

$$L_u(\varphi,\partial\varphi) = c_{11}(\partial\varphi)^k + c_{12}(\partial\varphi)^l(\varphi)^m + c_{22}(\varphi)^n \Rightarrow c_{11}z_1^k + c_{12}z_1^l z_2^m + c_{22}z_2^n = 1 \tag{22}$$

in which

$$z_1^k = \frac{(\partial\varphi)^k}{L(\varphi,\partial\varphi)},\ z_1^l = \frac{(\partial\varphi)^l}{\sqrt{L(\varphi,\partial\varphi)}},\ z_2^m = \frac{\varphi^m}{\sqrt{L(\varphi,\partial\varphi)}},\ z_2^n = \frac{\varphi^n}{L(\varphi,\partial\varphi)} \tag{23}$$

$c_{1,2,3}$ are constants at given setting, for example, at a given energy scale. Therefore,

$$r_{11}^k + r_{12}^l r_{21}^m + r_{22}^n = 1 \tag{24}$$

where

$$r_{11}^{k} = c_{11} z_1^{k},\ r_{12}^{l} = \sqrt{c_{12}}\, z_1^{l},\ r_{21}^{m} = \sqrt{c_{12}}\, z_2^{m},\ r_{22}^{n} = c_{22} z_2^{n}$$

If $c_{1,2,3}$ depend on the field content or their derivatives, (24) assumes the general form

$$c_{11}^{q_1}(r_{11})\, r_{11}^{k} + c_{12}^{q_2}(r_{12}, r_{21}) r_{12}^{l} r_{21}^{m} + c_{22}^{q_3}(r_{22}) r_{22}^{n} = 1 \tag{25}$$

where $q_{1,2,3}$ are non-vanishing exponents and

$$c_{11} + c_{12} + c_{22} = 1 \tag{26}$$

Comparing (25) and (26) to (11) and (12) leads to the conclusion that effective field theories may be formally cast in the language of *multifractals*. Next two sections apply this formalism to GR and the SM, respectively.

6. GR as multifractal set

Four-dimensional GR is characterized by the gravitational action [23]

$$S = \int R\sqrt{-g}\, d^4x \tag{27}$$

Einstein's field equations follow upon applying small variations $\delta g_{\mu\nu}$ to the metric and holding $g_{\mu\nu}$ and their first derivatives constant on the boundary of the four-dimensional volume. The scalar curvature is given by

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu}(\Gamma^{\sigma}_{\mu\sigma,\nu} - \Gamma^{\sigma}_{\mu\nu,\sigma} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\sigma}_{\sigma\rho} + \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho}) \tag{28}$$

and the gravitational Lagrangian by

$$L_G = g^{\mu\nu}(\Gamma^{\sigma}_{\mu\nu}\Gamma^{\sigma}_{\sigma\rho} - \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho}) \tag{29}$$

where $\Gamma^{\sigma}_{\mu\nu}$ represent the Christoffel symbols of the second kind. Combined use of (28) and (29) leads to

$$L_G = \frac{dS}{\sqrt{-g}d^4x} = g^{\mu\nu}(\Gamma^{\sigma}_{\mu\nu}\Gamma^{\sigma}_{\sigma\rho} - \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho}) \rightarrow g^{\mu\nu}\frac{\Gamma^{\sigma}_{\mu\nu}\Gamma^{\sigma}_{\sigma\rho} - \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho}}{L_G} = 1 \tag{30}$$

It is seen that the gravitational Lagrangian contains terms having the *symbolic* form

$$GR \Rightarrow g \cdot (g \cdot \partial g) \cdot (g \cdot \partial g) \tag{31}$$

On account of (19), the components of the metric tensor g act as *probability amplitudes* in (11) and (12). Comparing (30) with (11), (12), as well as with (25), (26), enables one to retrieve the generalized dimension of GR found in (20), namely $D_{1/2} = 4$. In short,

$$\boxed{GR : q = \tfrac{1}{2}, \tau(q) = 2, D_{\frac{1}{2}} = 4} \tag{32}$$

Finally, a glance at (11) and (25) shows that the role of local scales in (30) is being played by the product of Christoffels normalized to the gravitational Lagrangian L_G.

7. <u>SM as multifractal set</u>

Consider now the SM Lagrangian [24]

$$L_{SM} = -\frac{1}{4}\sum_{V} V^{a}_{\mu\nu}V^{a\mu\nu} + \overline{f^i_L} i\gamma^{\mu}D_{\mu} f^i_L + \overline{f^i_R} i\gamma^{\mu}D_{\mu} f^i_R + (Y_{ij}\overline{f^i_L} H f^j_R + h.c.) + (D^{\mu}H)^{\dagger}(D_{\mu}H) - V(H) \tag{33}$$

Here, the summation convention over repeated indices is assumed, with $(i, j) = 1, 2, 3$ extending over the three fermion families. The vector fields V corresponds to the three gauge groups of the SM, namely $U(1)_Y$, $SU(2)_L$ and $SU(3)_C$,

$$V = \left\{B, W^{a=1,2,3}, G^{a=1....8}\right\} \tag{34}$$

to which we associate the field-strength tensors

$$V_{\mu\nu}^{a} = \partial_\mu V_\nu^a - \partial_\nu V_\mu^a + g\, f_{abc} V_\mu^b V_\nu^c \tag{35}$$

and covariant derivative operators

$$D_\mu = \partial_\mu - i \sum_V g_V t_V^a V_\mu^a \tag{36}$$

The last couple of terms denote the kinetic and potential contributions of the Higgs field,

$$\mathrm{V}(H) = -m_H^2\, H^\dagger H + \lambda (H^\dagger H)^2 \tag{37}$$

In 3+1 dimensions, all mass and coupling charges act as *free scalar parameters* that are independent of fields or their derivatives.

In *symbolic* form, it is seen that the SM Lagrangian (33) contains terms including only field derivatives ($\partial\varphi$), a mix of fields and their derivatives ($\varphi\partial\varphi$) and terms with only fields (φ). The composition of the (33) is shown in Tab. 1 below, side by side with the corresponding composition of the gravitational Lagrangian (30). Since (33) contains terms that are quadratic, trilinear and quadrilinear in fields and their derivatives, SM is characterized by a *triplet* of generalized dimensions, that is,

$$\boxed{SM: \begin{cases} q=0,\, \tau(q)=2,\, D_0^1=2 \\ q=0,\, \tau(q)=3,\, D_0^2=3 \\ q=0,\, \tau(q)=4,\, D_0^3=4 \end{cases}} \tag{38}$$

Comparing (38) with (20) reveals that the SM Lagrangian displays a much richer multifractal structure than its overall field makeup embodied in (15).

EFT	**($\partial\varphi$) term**	**($\varphi\partial\varphi$) term**	**(φ) term**
GR	-	$g\cdot(g\cdot\partial g)\cdot(g\cdot\partial g)$	-
SM	$\partial V\cdot\partial V$ $\partial H\cdot\partial H$	$\partial V\cdot V\cdot V$; $f\cdot\partial f$ $\partial H\cdot V$	$V\cdot V$; $(V\cdot V)^2$ $f\cdot f\cdot V$; $H\cdot V$ $f\cdot f\cdot H$; $H\cdot H$ $(H\cdot H)^2$

Tab 1: The symbolic structure of GR and SM in terms of fields and their derivatives

8. Discussion and concluding remarks

Our study points out that effective field theories (EFT) replicate the properties of strange attractors. In particular, the geometric structure of both SM and GR can be formulated using the language of multifractals. It is found that the GR Lagrangian is characterized by a single generalized dimension ($D_{GR}=4$), whereas the SM Lagrangian requires a triplet of generalized dimensions ($D_{SM}=2,3$ and 4). The only generalized dimension where GR and SM overlap is $D=4$, which suggests a deeper topological connection between gauge bosons, the Higgs scalar and classical gravity. For example, the Higgs scalar may be regarded as a *short-range* condensate of gauge bosons [25, 29-30], whereas classical gravity may emerge as ultra-weak and *long-range* excitation of the Higgs scalar [25]. Our findings also suggest that the continuous spectrum of generalized dimensions lying between $D = 2$ and $D = 4$ may naturally account for the existence of non-baryonic Dark Matter, viz. the long-range manifestation of *Cantor Dust* [28].

Parameter/Theory	**GR**		**SM**		
D_q	$D_{1/2} = 4$	$D_{1/2} = 4$	$D_0^1 = 2$	$D_0^2 = 3$	$D_0^3 = 4$
Source	$(\varphi\partial\varphi)$ Tab.1	$\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 1$ Eq. (18)	$(\partial\varphi)$ Tab. 1	$(\varphi\partial\varphi)$ Tab.1	(φ) Tab. 1

Tab. 2: Generalized GR and SM dimensions along with their sources

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