UNREALISTIC ASSUMPTIONS INHERENT IN MAXIMAL EXTENSION

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ABSTRACT. I argue that maximal extension makes improbable assumptions about future conditions. I start by looking at the Schwarzschild metric, and showing that it does not quite represent the exterior of a collapsed star, although it is easy to argue that the mismatch is immaterial. I then look at the collapse of a cloud of dust using the Robinson Walker metric, which might seem to justify using the Schwarzschild metric to describe the exterior of a black hole. I then show how the Schwarzschild metric is modified when the interior is a collapsed dust cloud, and finally show how the maximal extension of a Schwarzschild black hole makes unrealistic assumptions about the future.

I apologise to the reader who feels that much of the argument is a statement of the obvious.

1. THE SCHWARZSCHILD METRIC

The Schwarzschild metric describes the empty space surrounding a star. It assumes the star is isolated in asymptotically flat space and deduces the line element:

$$ds^2 = (1 - 2m/r)dt^2 - dr^2/(1 - 2m/r) - r^2(d\theta^2 + sin^2(\theta)d\phi^2)$$

In this coordinate system the surface described by a constant value of r at some time t is a sphere of area $4\pi r^2$ positioned symmetrically about the star. This encourages us to think of r as a radius.

The coordinate system fails to describe space inside the sphere corresponding to r=2m, but we can describe the space if we swap to (for example) an isotropic coordinate system[1]with line element:

$$ds^2 = \frac{(2r - m)^2}{(2r + m)^2}dt^2 - \frac{(2r + m)^4}{(2r)^4}(dr^2 + r^2(d\theta^2 + sin^2(\theta)d\phi^2)$$

The r coordinate now continues down to r=0, but r=0 is not the centre of a star, nor is it a curvature singularity. The Schwarzschild coordinate system describes one half of an Einstein Rosen Bridge: the isometric coordinate system describes both halves, one in the space covered by $r \geq 2m$, and the other mapped into the space covered by $0 < r \leq 2m$.

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[1]See for example Ray d'Inverno "Introducing Einstein's Relativity" page 189-190, available at `http://documents.mx/documents/ray-dinverno-introducing-einsteins-relativitypdf.html`

The Schwarzschild metric does not represent a collapsed star, it represents a space/time that contains no mass whatsoever. In either coordinate system, a constant value of t represents the whole of space at a coordinate defined instant of time, and that space is empty by construction. The singularity encountered when this space/time is maximally extended (in, say, Kruskal coordinates) is not formed by a concentration of the Einstein tensor, but by a concentration of the Weyl tensor.

Nevertheless the Schwarzschild coordinates or some modification thereof are used to represent the exterior of a black hole.

2. A collapsing dust cloud

The Schwarzschild metric describes only the empty space surrounding a star, and cannot follow the progress of a star as it collapses. The Robertson Walker metric was developed to describes a universe filled with dust as it collapses from infinity to a singularity, but it also describes a sphere of dust as it collapses into a singularity.

The following is a simplified Robertson Walker line element:

$$ds^2 = dt^2 - S(t)^2(dr^2 - r^2(d\theta^2 + sin^2(\theta)d\phi^2)) \qquad \text{where } dS/dt = -S^{-0.5}.$$

It uses co-moving coordinates, meaning that (for example) the sphere of dust at radius r=1 remains at r=1, S goes from plus infinity at time t=minus infinity, when the cloud is infinitely big, to S=0 at t=0 when the cloud forms a singularity.

The statement that $dS/dt = -S^{-0.5}$ is a simplification of Friedmann's equation, chosen so that if the outermost radius of the dust cloud is r=1, then the dust cloud becomes a black hole at S=1, and so the co-moving mass (the 'mass at infinity') of the dust cloud is 0.5.

3. An external view of a collapsing dust cloud

I want to develop a line element for the dust cloud which includes a stationary region where an observer can stay at a 'safe' distance from the dust cloud. To this end I replace the inner portion of the Schwarzschild metric with a remapped Robertson Walker metric.

Any spherically symmetric space can be described by a global coordinate system whose line element has the form:

$$ds^2 = A^2dT^2 - B^2dR^2 - R^2(d\theta^2 + sin^2(\theta)d\phi^2). \tag{1}$$

Outside the cloud the metric becomes the Schwarzschild metric for a star of mass 0.5, so $A^2 = 1 - 1/R$ and $B^2 = 1/(1 - 1/R)$. Inside the dust cloud, where $R \leq S$, the space/time is also described by co-moving coordinates with line element:

$$ds^2 = SdS^2 - S^2dr^2 - R^2(d\theta^2 + sin^2(\theta)d\phi 2) \qquad where\ R = Sr. \tag{2}$$

I have used $dS/dt = -S^{-0.5}$ to replace t with S as the time coordinate: S better reflects the state of the cloud at a point.

If the boundary of the cloud is at $R = R_T$ at time T, then inside the cloud

$$R = rR_T + (r - r^3)/2.$$

Supporting calculations are given in an appendix, but to interpret, when the cloud has collapsed, then in the Schwarzschild coordinate system the majority of the cloud particles will be clustered near the edge of the cloud. When $R_T = 1.01$, meaning that R is just 1% bigger than the asymptotic value, 22% of the dust particles have still not fallen through R=1.

In the limit, when $R_T = 1$, 23% of the particles have still not fallen through R=0.99 and the co-moving density of the cloud $(1/S^3)$ is over 3 times as great at edge than at the centre. To a distant observer the entire mass is concentrated in the outer skin of the dust cloud, since the particles there are moving at the speed of light.

4. A black hole in context

Suppose we now wish to study the evolution of a collection of stars, including one or more nominal black holes. In such a system, the internal evolution of the stars we call black holes will be suspended, but they may still interact with other stars. In the Autumn of 2016, gravity waves were detected from what is believed to be the merger of two black holes. That merger must have occurred long before the formation of a singularity, and the internal evolution of the star would be utterly unlike the evolution predicted by the maximal extension of an isolated black hole. The maximal extension is, in effect, making extremely improbable assumptions about the future.

5. Appendix: a Robertson Walker interior and a Schwarzschild exterior

Since the θ and ϕ coordinates are common to both metrics, we need only consider the surface formed by a radial line over time. The two metrics reduce to:

for $R \leq S$:

$$ds^2 = SdS^2 - S^2dr^2, \qquad where\ R = rS \tag{3}$$

and everywhere, including for $R \leq S$:

$$ds^2 = A^2dT^2 - B^2dR^2. \tag{4}$$

Let T^a, R^a, S^a, r^a be the tangent vectors for the various coordinates, and T_a, R_a, S_a, r_a be the corresponding cotangents. So, given the simplicity of the line elements:

$$\begin{aligned} T^aT_a &= 1/A^2 \\ R^aR_a &= -1/B^2 \\ S^aS_a &= 1/S \qquad\qquad \text{and} \\ r^ar_a &= -1/S^2. \end{aligned}$$

$R = Sr \quad \Longrightarrow \quad dR = rdS + Sdr \quad \Longrightarrow \quad R_a = rS_a + Sr_a \quad$ so:

$$R_aR^a = r^2S_aS^a + S^2r_ar^a$$
$$(5) \qquad -1/B^2 = (r^2)/S - (S^2)/S^2$$
$$B^2 = 1/(1 - r^2/S)$$
$$= 1/(1 - r^3/R)$$

r^3 is proportional to the co-moving mass contained in the sphere of radius r. At the boundary of the cloud, and hence everywhere, the locally measured 'mass at infinity'[2]contained in the sphere of area $4\pi R^2$ is $r^3/2$.

We have two coordinate systems which at any given point are in relative motion v, where $v = d(rS)/dt = -r/S^{0.5}$. If (for example) a vector $aS^a + br^a$ is parallel to R^a, then the Lorentz transform tells us that $a^2S^aS_a = -v^2b^2r^ar_a$. Equation (5) conforms to this.

If a vector aS^a+br^a is parallel to T^a, then the Lorentz transform tells us that $v^2a^2S^aS_a = -b^2r^ar_a$, so $S^a + rr^a$ is parallel to T^a and orthogonal to R^a. Thus $S^aR_a + rr^aR_a = 0$.

So working with a line at constant T:

$$\frac{dS}{dR} + r\frac{dr}{dR} = 0$$

and integrating:

$$S + r^2/2 = f(T)$$

where f(T) is the constant of integration. If the size of the cloud at time T is R_T, then at $r = 1, S = R_T$, so:

$$f(T) = R_T + 0.5.$$
$$(6) \qquad S + r^2/2 = R_T + 0.5$$

At the centre of the cloud, at r=0, S is 0.5 greater than at the edge, so in the limit the particles are $(3/2)^3 = 3.375$ more dense around the edge of the cloud than at the centre.

From equation (6) together with $R = Sr$:

$$R = rR_T + (r - r^3)/2$$

So (for example) when the edge of the cloud is at $R = R_T = 1.1$, 65% of the particles have still not fallen through R=1.

When the cloud has collapsed further, and $R_T = 1.01$, 22% of the particles have still not fallen through R=1.

In the limit, when $R_T = 1$, 23% of the particles lie outside R=0.99.

[2]The term 'mass at infinity' is a way of describing the curvature on the surface of a stationary sphere which makes assumptions about the space outside the sphere. By 'locally measured mass at infinity', I mean that the curvature is that which could be so described if the surrounding space satisfied the implied assumptions.