

Moduli space of compact Lagrangian submanifold

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April 2, 2017

Abstract

We describe the deformations of moduli space \mathcal{M} of Special Lagrangian submanifolds in the compact case and we give a characterization of the topology of \mathcal{M} by using McLean theorem. By constructing Banach spaces on bundle sections and by elliptical operators, we are able to use Hodge theory to study the topology of the manifold. Starting from McLean results, for which moduli spaces of compact special Lagrangian submanifolds is smooth and its tangent space can be identified with harmonic 1-forms on the special Lagrangian submanifolds, we can analyze deformation theory. Then we introduce a Riemannian metric on \mathcal{M} , from which we obtain other important properties.

1 Sobolev and Hölder space of section

Let \mathbb{k} be the field of the real number R or the field of complex number C and let E be a vector bundle of rank r on \mathbb{k} , on a compact n -manifold X . We can define Banach spaces on the sections of E starting from a trivialization atlas. Let $\mathcal{U} = \{U_1, \dots, U_\ell\}$ an open finite cover of X by coordinate charts which are also trivialization open sets. We call ξ_1^v, \dots, ξ_r^v the fiber vector components of ξ on a point of \mathcal{U}_v in the assigned trivialization $E|_{U_v} \simeq U_v \times \mathbb{R}^r$. Let's fix a partition $\{\rho_1, \dots, \rho_\ell\}$ of unity χ on X subordinate to \mathcal{U} . We can now use coordinates to identify U_v with a domain in \mathbb{R}^n .

In this way we can consider a section u of E on U_v like a function u^v defined on an open set of \mathbb{R}^n which takes values in \mathbb{k}^r . More specifically, if $u \in \chi_0(U_v, E)$

for every real number $p \geq 1$ and every integer $k \geq 0$ we can define its *Sobolev norm*

$$\|u\|_{W_k^p(E)} = \left[\sum_{|\alpha| \leq k} \int_{U_\nu} \|\partial^\alpha u^\nu\|^p dx \right]^{1/p},$$

where $\|\cdot\|$ is the standard norm on \mathbb{k}^r .

If $u \in \chi(X, E)$, using the partition of unity, we have

$$\|u\|_{W_k^p(E)} = \left[\sum_{\nu=1}^{\ell} \sum_{|\alpha| \leq k} \int_{U_\nu} \|\partial^\alpha (\rho_\nu u)^\nu\|^p dx \right]^{1/p}. \quad (1.1)$$

This definition of Sobolev norm depends on the choice of the trivialization atlas, but it can be seen that different choices gives equivalent norms. We can now invariantly define the Sobolev space $W_k^p(E)$ as the *completion of $\chi(X, E)$ for the norm (1.1)*. It is the space of the sections of E on X that are p -power summable with their derivatives on X till the k th order. If $k = 0$, we call them simply the sections of E p -power summable on X .

Let's give another way to build Sobolev spaces $W_k^p(E)$ starting from a Riemannian metric g on X , an Hermitian metric $(\cdot | \cdot)_E$ of class χ on the fiber of E and a linear connection ∇E on E . We canonically define χ metrics on the fiber of tensorial products $E \otimes_X T^{p,q}X$. Being $\|\cdot\|$ the corresponding norms, we obtain a scalar product equivalent to (1.1) by considering

$$\|u\|'_{W_k^p(E)} = \left[\sum_{h=0}^k \|\nabla_E^h u\|^p dV_g \right]^{1/p}, \quad (1.2)$$

where ∇_E is the linear connection on the tensorial products $E \otimes_X T^{0,q}X$ which are products of the assigned connection on E and connections on tensorial spaces coming from Levi-Civita connection on X . It's easy to verify that (1.1) is equivalent to (1.2). More specifically, a different choice of g and ∇_E gives equivalent norms.

The space $W_0^2(E)$ is the Hilbert space $L^2(E)$. Its inner product, defined by (1.1) or (1.2), restricted to $\chi(E) \times \chi(E)$, gives a map, bilinear if $\mathbb{k} = \mathbb{R}$, sesquilinear if $\mathbb{k} = \mathbb{C}$, that is continuous on $L^p(E) \times L^{p'}(E)$ if $p, p' > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. So we have a duality:

$$\langle | \rangle : L^p(E) \times L^{p'}(E) \rightarrow \mathbb{R}, \quad \forall 1 < p, p' < +\infty, \quad \text{con } \frac{1}{p} + \frac{1}{p'} = 1. \quad (1.3)$$

This *pairing* brings an isomorphism between Banach spaces

$$\Phi : L^p(E) \rightarrow [L^{p'}(E)]^*, \quad (1.4)$$

which, on the regular sections, is given by $\Phi(\xi)(\eta) := \langle \xi | \eta \rangle_{L^2(E)}$. More generally,

Proposition 1.1. *If p is a real number > 1 and k an integer ≥ 0 , then the Banach space $W_k^p(E)$ is reflexive.*

The hypothesis that $1 < p < +\infty$ is necessary for the validity of Proposition 1.1: Banach spaces $W_k^1(E)$ are not reflexive.

The dual space of W_k^p , for $k \geq 1$, can be identified with a *distribution* space, and it's indicated by $W_{-k}^{p'}(E)$. It's a Sobolev space with *negative integer exponent*.

On the space $C^k(E)$ of sections of E on X , that are continuous and differentiable till the order $k \geq 0$, is normal to consider, starting from the trivialization atlas \mathcal{U} , the norm

$$\|\xi\|_{C^k(E)} := \sum_{\nu=1}^n \sum_{0 \leq |\alpha| \leq k} \sup |\partial^\alpha(\rho_\nu \xi)^\nu|. \quad (1.5)$$

With this norm, $C_0^k(E)$ is a Banach space.

Remembering the definition of Hölder seminorm of order a , with $0 < a \leq 1$, for functions u defined on R^n and which take values in \mathbb{k} :

$$[u]_a = \sup_{\substack{x, y \in R^n, \\ x \neq y}} \frac{|u(x) - u(y)|}{\|x - y\|^a} \in [0, +\infty].$$

A function with finite Hölder seminorm is continuous, and a function of class C^1 and with compact support has Hölder seminorm finite. Let's define the Hölder norm of type $\mathcal{C}^{k,a}$ (or \mathcal{C}^{k+a}), with k integer ≥ 0 and a real with $0 < a \leq 1$, on $\chi(E)$, by

$$\|\xi\|_{C^{k,a}(E)} = \|\xi\|_{\mathcal{C}^k(E)} + \sum_{\nu=1}^\ell \sum_{|\alpha|=k} \sum_{i=1}^r [\partial^\alpha(\rho_\nu \xi)_i^\nu]_a. \quad (1.6)$$

The completion $\mathcal{C}^{k,a}(E)$ of $\chi(E)$ respect to the norm (1.6) is a Banach space, that can be identified with the subspace of sections ξ of $\mathcal{C}^k(E)$ for which the partial derivatives $\partial^\alpha(\rho_\nu \xi)_i^\nu$, con $|\alpha| = k$ have Hölder seminorm of finite order. We call $C^{k,a}(E)$ the *Hölder space of order (k, a)* of sections of E . When $a = 1$, the sections are called *Lipschitz*.

As it can be seen, a different choice of trivialization atlas defines equivalent Hölder norms. Spaces $C^{k,a}(E)$ are intrinsically defined, whereas their norms are not.

As in the case of Sobolev norms, we can obtain equivalent norms on $C^{k,a}(E)$ from a Riemannian metric g on X , a metric χ on the fiber of E and a linear connection ∇_E on E .

Given $\xi \in C^k(E)$ e $0 \leq j \leq k$, the j -th covariant derivative $\nabla_E^j(\xi)$ is a continuous section of the j -th tensorial power of the cotangent bundle T^*X . The metric on

the fiber of E defines a metric on the fiber of $T^{*\otimes j}X$ and we can then find a norm equivalent to (1.5) by setting

$$\|\xi\|_{(k)} = \sum_{j=0}^k \sup_X \|\nabla_E^j \xi\|_{T^{*\otimes j}}. \quad (1.7)$$

To define an equivalent norm on Hölder spaces starting from $g, \|\cdot\|_E, \nabla_E$, there are many equivalent ways. One of them is by remembering that a linear connection on E allows us to break in a unique way¹ a vector \vec{v} , tangent to the total space of the bundle, in the sum of an horizontal vector \vec{v}_h and a vertical vector \vec{v}_v . Since the fibers are vector spaces, the vector \vec{v}_v is naturally identified with an element of E . So we can define a Riemannian metric g_E on E by setting

$$g_E(\vec{v}, \vec{v}) = g(\pi_*(\vec{v}), \pi_*(\vec{v})) + \|\vec{v}_v\|_E^2.$$

Let d_E be the distance on E defined by the Riemannian metric g_E and d_X be the distance on X defined by g . It is natural to introduce the *vertical component of distance* $d_v(\xi_1, \xi_2) \geq 0$, between two elements $\xi_1, \xi_2 \in E$ by putting²

$$d_v^2(\xi_1, \xi_2) = d_E^2(\xi_1, \xi_2) - d_X^2(\pi(\xi_1), \pi(\xi_2)).$$

By doing this we can define Hölder seminorm of order a of a section of ξ , with $0 < a \leq 1$, by the following

$$[\xi]_{(a)} = \sup_{\substack{x,y \in X \\ x \neq y}} \frac{d_v(\xi(x), \xi(y))}{d_X^a(x, y)}.$$

In the same way we can consider vertical distances and Hölder seminorms on sections, for the bundles $E \otimes T^{*\otimes j}$. From these seminorms we can define Hölder norms of order (k, a) by

$$\|\xi\|_{(k,a)} = \|\xi\|_{(k)} + \sum_{j=0}^k [\nabla_E^j \xi]_{(a)}. \quad (1.8)$$

Let's now describe another classic construction of Hölder norms from $g, \|\cdot\|_E, \nabla_E$, in which we use the notion of parallel transport.

¹See p.87 of: [?]

²We apply Pythagoras's theorem to an ideal right triangle in which the length of hypotenuse is equal to the distance between the two points ξ_1, ξ_2 in E , and the length of a cathetus is equal to the distance between the projections of ξ_1, ξ_2 in X .

If V is a vector bundle over a manifold Y on which we fixed a connection ∇_V then, for every piecewise C^1 curve γ that joins two points $x, y \in Y$, we can associate an isomorphism $\tau_\gamma : V_x \rightarrow V_y$, by the means of the parallel transport along γ . Moreover, if the bundle metric is compatible with the connection, the parallel transport is an isometry.

We provide Y with a Riemannian metric h . If Y is compact, we can fix a real positive number $\epsilon > 0$ such that the open ball of radius ϵ of the distance d_h defined by h will be convexes, Let a Riemannian metric h be assigned on Y . If Y is compact, we can fix a positive real number $\epsilon > 0$ such that the open balls of radius ϵ , measured with the distance d_h defined by h , will be convexes, that is such that every couple of points of a ball B of Y which have radius less or equal than ϵ are ends of a geodetic of length less than ϵ , all contained in B . More specifically, if $x, y \in Y$ and $d_h(x, y) < \epsilon$, there is a unique geodetic $\gamma_{x,y} : [0, 1] \rightarrow Y$, of length less than ϵ , that join this points, and so a unique isometry $\tau_{x,y} : V_x \rightarrow V_y$. We can now define Hölder seminorm of order $a \in (0, 1]$ on A by:

$$[v]_{a;A}^h := \sup_{\substack{x,y \in A, \\ 0 < d_h(x,y) < \epsilon}} \frac{|v_y - \tau_{x,y}(v_x)|_V}{|d_h(x,y)|^a}, \quad \forall \text{ section } v : A \rightarrow V. \quad (1.9)$$

We can apply this to the compact Riemannian manifold (X, g) and to the bundles $E \otimes T^{*\otimes l} X$ and so we define a norm

$$\|\xi\| := \left(\sum_{j=0}^k \sup |\nabla_E^j \xi|_E \right) + [\nabla_E^k \xi]_{a;X}^g \quad (1.10)$$

for every ξ in $C^{k,a}(E) = \{\xi \in C^k(E) : [\nabla_E^k \xi]_{a;X}^g < \infty\}$. Sobolev and Hölder spaces are particularly useful for the theory of elliptic PDEs. They satisfy important property of inclusion and compactness. In the following results, p, q are real numbers with $1 < p, q < +\infty$, a, b real numbers with $0 < a, b < 1$ and k, ℓ not negative integers.

Theorem 1.2 (Embedding theorems). (1) *If $k \geq \ell \geq 0$ e $k - \frac{n}{p} \geq \ell - \frac{n}{p}$, then there is an embedding $W_k^p(E) \hookrightarrow W_\ell^q(E)$.*

(2) *If $k + a \geq \ell + b$, then there are embeddings:*

$$C^{k+1}(E) \hookrightarrow C^{k,a}(E) \hookrightarrow C^{\ell,b}(E) \hookrightarrow C^\ell(E) \quad e \quad C^k(E) \hookrightarrow C^\ell(E).$$

(3) *If $k - \frac{n}{p} \geq \ell + a$, then there are embeddings:*

$$W_k^p(E) \hookrightarrow C^{\ell,a}(E) \hookrightarrow C^\ell(E) \hookrightarrow W_\ell^q(E).$$

Moreover we have:

Theorem 1.3 (Compactness theorems). (1) *The embedding $W_k^p(E) \hookrightarrow W_\ell^q(E)$ is compact if $k > \ell \geq 0$ e $k - \frac{n}{p} > \ell - \frac{n}{q}$.*

(2) *The embedding $C^{k,a}(E) \hookrightarrow C^k$ is compact.*

(3) *The embedding $W_k^p(E) \hookrightarrow C^{\ell,a}(E)$ is compact when $k - \frac{n}{p} > \ell + a$.*

1.1 Elliptical operator

Sobolev and Hölder spaces are the classical environment for developing elliptic operator theory. Let X be a differentiable manifold of class \mathcal{C}^∞ . For a theorem by Peetre³, we can characterize a linear differential operator which takes values χ on X between the sections of two real vector bundles⁴ E, F on X as a linear function $P : \chi(E) \rightarrow \chi(F)$ which doesn't increase the supports, such that

$$\text{supp } Pu \subset \text{supp } u, \quad \forall u \in \chi(E).$$

In a trivialization chart U for E and F which is relatively compact in X , a differential operator P is written

$$Pu = \sum_{|\alpha| \leq \ell} A_\alpha \partial^\alpha u, \quad \forall u \in \mathcal{C}^\infty(U, R^{r_E}), \quad \text{con } A_\alpha \in \mathcal{C}^\infty(U, R^{r_F \times r_E}),$$

where we have called r_E, r_F the ranks of the bundles E and F respectively. If $A_\alpha \neq 0$ on U for some $|\alpha| = \ell$, we call the not negative integer ℓ the *order* of P on U . We say that P has order less or equal than ℓ on X if it has order less or equal than ℓ in every coordinate chart of local trivialization.

If $P : C^\infty(E) \rightarrow C^\infty(F)$ is a linear differential operator of order $\ell \geq 0$, then, for every not negative integer k , every real number $p > 1$ and every real number $a \in (0, 1]$, P can be extended in a unique way to linear and continuous functions

$$P : W_{k+\ell}^p(E) \rightarrow W_k^p(F), \quad (1.11)$$

$$P : C^{k+\ell,a}(E) \rightarrow C^{k,a}(F). \quad (1.12)$$

Let's suppose that X is a compact (without border) Riemannian manifold equipped with metric g . On the fiber of E and F we can fix scalar products of class \mathcal{C}^∞ ,

³see [10] 211-218.

⁴We can consider complex vector bundles as real by restriction to the scalar field.

which allows us to identify every bundle to its dual. We define the *formal adjoint* $P^* : \mathcal{C}^\infty(F) \rightarrow \mathcal{C}^\infty(E)$ using integration by parts:

$$\langle \xi | P^* \eta \rangle_{L^2(E)} = \langle P \xi | \eta \rangle_{L^2(F)}, \quad \forall \xi \in \mathcal{C}^\infty(E), \forall \eta \in \mathcal{C}^\infty(F). \quad (1.13)$$

The formal adjoint P^* has the same order than P .

If $P : \mathcal{C}^\infty(E) \rightarrow \mathcal{C}^\infty(F)$ is a linear differential operator of order less or equal than m , then, for every $\xi \in \mathcal{C}^\infty(E)$ and every $\phi \in \mathcal{C}^\infty(X, R)$ the function $R \ni \lambda \rightarrow \exp(-\lambda\phi)P(\exp(\lambda\phi)\xi)$ is a polynomial of degree less or equal than ℓ which take values in $\mathcal{C}^\infty(F)$. In such a way we obtain

$$\exp(-\lambda\phi)P(\exp(\lambda\phi)\xi) = \sum_{h=0}^{\ell} \lambda^h P_{\ell-h}(d\phi)(\xi),$$

where, for every $h = 0, \dots, \ell$, we call $P_h(d\phi)$ the differential operator of order less or equal than h from $\mathcal{C}^\infty(E)$ to $\mathcal{C}^\infty(F)$. More specifically, $P_0(d\phi)$ is an operator of order 0, that is a morphism of real vector bundles from E to F . Pointwise the transformation defined by $P_0(d\phi)$ between the fibers E_x and F_x over the same point x depends only on the value $d\phi(x) \in T_x^*X$. So we have, for every $x \in X$, a function

$$P_0 : T_x^*X \ni \xi \rightarrow P_m(\xi) \in \text{Hom}_R(E_x, F_x),$$

which is a polynomial homogeneous of degree ℓ called the *principal symbol of degree ℓ* of P in x .

The principal symbol is an important geometric invariant. There are class of operators that are characterized by property of their principal symbol.

Definition 1.4 (ellipticity). The operator P is called *elliptic* of order ℓ if it has order less or equal than ℓ and, for every $x \in X$, its principal symbol of order ℓ is injective for every $\xi \in T_x^*X \setminus \{0\}$.

If an elliptic operator from E to F exists, the rank of E has to be less o equal than the rank of F . An example of elliptic operator is the differential from the tautological line bundle to the cotangent bundle on X .

Definition 1.5 (ellipticity). The operator P is called *fixed elliptic* of order ℓ if it has order less or equal than ℓ and, for every $x \in X$, its principal symbol of order ℓ is an isomorphism for every $\xi \in T_x^*X \setminus \{0\}$.

To allow us to have fixed elliptic operator from E to F , they must have the same rank. An example of fixed elliptic operator is the classical Laplace operator in Euclidean spaces R^n , considered as operator between tautological line bundles.

As we will see, elliptic operators are very useful in the analysis of compact manifolds. We show now a fundamental result for the theory of elliptic operators.

Theorem 1.6. *Let X be a compact manifold, E, F vector bundles on X and $P : C^\infty(E) \rightarrow C^\infty(F)$ a linear elliptic differential operator with regular coefficients of class \mathcal{C}^∞ , of order $\ell \geq 1$.*

Let k be a not negative integer, p and a real numbers, with $p > 1$ and $0 < a < 1$. Then, maps $P : W_{k+l}^p(E) \rightarrow W_k^p(F)$ and $P : C^{k+l,a}(E) \rightarrow C^{k,a}(F)$ have both finite kernels and finite images.

If $\xi \in W_\ell^1(E)$, then

$$P(\xi) \in W_k^p(F) \Rightarrow \xi \in W_{k+l}^p(E), \quad P(\xi) \in \mathcal{C}^{k,a}(F) \Rightarrow \xi \in \mathcal{C}^{k+l,a}(E).$$

Proof. This result is a consequence of Compactness theorems (Theorems 1.3) for Sobolev and Hölder spaces and of prior estimates of elliptic operators: if k is a positive integer greater or equal than ℓ , p, a real numbers with $p > 1$ and $0 < a < 1$, we can find a positive constant C such that

$$\begin{aligned} \|\xi\|_{C^{k,a}(E)} &\leq C \left(\|P(\xi)\|_{C^{k-\ell,a}(F)} + \|\xi\|_{C^0(E)} \right), & \forall \xi \in C^{k,a}(E), \\ \|\xi\|_{W_k^p(E)} &\leq C \left(\|P(\xi)\|_{W_{k-\ell}^p(F)} + \|\xi\|_{L^2(E)} \right), & \forall \xi \in W_k^p(E). \end{aligned}$$

From these it follows immediately that the norm $\|\xi\|_{C^0(E)}$ is equivalent to the norm $\|\xi\|_{C^{k,a}(E)}$ and the norm $\|\xi\|_{L^2(E)}$ to the $\|\xi\|_{W_k^p(E)}$ on $\ker P$, which is of finite dimension for the Theorem 1.3, because, for the norms equivalence, is a Banach space with compact unit ball.

Moreover, Sobolev and Hölder spaces are reflexive, so, in both of them, $\ker P$ has closed completions $\mathcal{W}'^{k,a}$, \mathcal{W}_k^p , in which we have increases of type:

$$\begin{aligned} \|\xi\|_{C^{k,a}(E)} &\leq C' \|P(\xi)\|_{C^{k-\ell,a}(F)}, & \forall \xi \in \mathcal{W}'^{k,a}, \\ \|\xi\|_{W_k^p(E)} &\leq C' \|P(\xi)\|_{W_{k-\ell}^p(F)}, & \forall \xi \in \mathcal{W}_k^p. \end{aligned}$$

From that it follows the theorem of images closure.

A Banach space has finite dimension when its closed unit ball is compact. Now, $\ker P$ is a closed subspace of $C^{k+l,a}(E)$, so it's Banach. Let \mathcal{B} be the unit closed ball in $\ker P$; let's suppose that $(\xi_j) \subseteq \mathcal{B}$ is any sequence. Then (ξ_j) is bounded for the norm $C^{k+l,a}$ and for the theorem 1.2 there exists a subsequence (ξ_{j_r}) which is C^{k+l} -Cauchy and, consequently, C^0 -Cauchy. We know that, if P is an elliptic operator and $\xi \in C^0(E)$ with $\eta \in C^{k,a}(F)$, then $\xi \in C^{k+l,a}(E)$ with $P\xi = \eta$ and

$$\|\xi\|_{C^{k+l,a}(E)} \leq C(\|P\xi\|_{C^{k,a}(F)} + \|\xi\|_{C^0(E)}) \quad (1.14)$$

From this, we have that $C > 0$, so:

$$\|\xi_{j_r} - \xi_{j_s}\|_{C^{k+l,a}(E)} \leq C \|\xi_{j_r} - \xi_{j_s}\|_{C^0(E)} \quad (1.15)$$

for every $r, s \geq 1$, so that (ξ_{j_r}) is $C^{k+l,a}$ -Cauchy and has a limit ξ in \mathcal{B} . Then, \mathcal{B} is compact and $\ker P$ is finite. To show that $\text{Im}P \subset C^{k,a}$ is closed we have to define the closed subspace $\mathcal{A} := (\ker P)^\perp \subset C^{k+l,a}(E)$. We use the L^2 inner product on $C^{k+l,a}(E)$ to build \mathcal{A} . It's useful to remember that $C^{k+l,a}(E) = \ker P \oplus \mathcal{A}$, as one can see by taking a basis in L^2 orthonormal to $\ker P$. Suppose for the sake of contradiction that there exists a sequence $(\xi_j) \subseteq \mathcal{A}$ with:

$$\|\xi_j\|_{C^{k+l,a}(E)} = 1 \quad \text{for every } j \geq 1 \quad (1.16)$$

$$\|P\xi_j\|_{C^{k,a}(F)} \rightarrow 0 \quad \text{se } j \rightarrow 0 \quad (1.17)$$

From the equation 1.16 and from theorem 1.3, there exists a subsequence (ξ_{j_r}) which is $C^{k+l,a}$ -Cauchy and so C^0 -Cauchy. From the equation 1.17 and from the equation 1.14 we can see that (ξ_{j_r}) is $C^{k+l,a}$ -Cauchy and so converges to some ξ in \mathcal{A} . Now, from equation 1.17, we have that $P\xi = 0$ and so necessarily $\xi = 0$. But this contradicts the equation 1.16. From this contradiction follows that there exists a constant $C_1 > 0$ such that:

$$\|\xi\|_{C^{k+l,a}(E)} \leq C_1 \|P\xi\|_{C^{k,a}(E)} \quad (1.18)$$

for every $\xi \in \mathcal{A}$. So we can finally show that $\text{Im}P \subset C^{k,a}(F)$ is closed. Let's take a sequence $(\eta_j) \subseteq \text{Im}P$ that converges to $\eta \in C^{k,a}(F)$. Then we put $\eta_j = P\xi_j$ for $j \geq 1$ with $\xi_j \in \mathcal{A}$. From equation 1.18, the sequence $(\xi_j) \subseteq \mathcal{A}$ is $C^{k+l,a}$ -Cauchy and so it converges to $\xi \in \mathcal{A}$; moreover $P\xi = \eta$. This shows that $\text{Im}P$ is a closed subspace of $C^{k,a}(F)$. \square

The theorem 1.6 can help us proving the following result, which gives us the characterization of the image of an elliptic operator between Sobolev spaces. We'll prove the theorem using M. Cantor method, following [1].

Theorem 1.7. *Let X be a compact manifold and E, F vector bundles on X . Let $P : C^\infty(E) \rightarrow C^\infty(F)$ be a smooth linear elliptic differential operator of order $l \geq 1$ with formal adjoint P^* . Then for the extension $P : W_{k+l}^p(E) \rightarrow W_k^p(F)$ we have:*

$$\text{Im}P = \left\{ \eta \in W_k^p(F) : \langle \eta | h \rangle_{L^2(F)} = 0 \text{ for every } h \in \ker P^* \right\} \quad (1.19)$$

Proof. First of all, we know that $\ker P^*$ is a subspace of $C^\infty(F)$ and is contained in $W_0^{p'}(F)$, giving sense to the right part of the equation 1.19. We note that

$$\text{Im}P \subset \left\{ \eta \in W_k^p(F) : \langle \eta|h \rangle_{L^2(F)} = 0 \text{ for every } h \in \ker P^* \right\} \quad (1.20)$$

follows from integration by parts.

Let's consider the case $k = 0$. We call $P' : W_0^p(F)^* \rightarrow W_l^p(E)^*$ the adjoint Banach space of P map, to distinguish it from the formal adjoint P^* of P . Now, using theorem 1.3 we can identify $W_0^p(F)^* \cong W_0^{p'}(F)$. An integration by parts gives us

$$\ker P^* \subset \ker P' \quad (1.21)$$

in $W_0^p(F)^*$. We point out that

$$\ker P' \subset \ker P^* \quad (1.22)$$

because, if $\eta \in W_0^{p'}(F)$ with $\langle P\phi|\eta \rangle_{L^2(F)} = 0$ for every $\phi \in W_l^p(E)$, then the equation $P^*\eta = 0$ is weakly valid. Now we take $\eta \in W_0^p(F)$ such that $\langle \eta|h \rangle_{L^2(F)} = 0$ for every $h \in \ker P^*$. Then $\eta \in W_0^{p'}(F)$ is in

$$(\ker P^*)^\circ = (\ker P')^\circ = \text{Im}P \quad (1.23)$$

as we requested. In this way we prove the theorem in the case $k = 0$.

Let's suppose now that $k \geq 1$ and that $\eta \in W_k^p(F)$ with $\langle \eta|h \rangle_{L^2(F)} = 0$ for every $h \in \ker P^*$. A consequence of the proof for $k = 0$ is that there exists $\xi \in W_l^p(E)$ such that $P\xi = \eta$. From elliptic operator theory we know that, if $\eta \in W_k^p(F)$, then $\eta \in W_{k+l}^p(E)$ with $P\xi = \eta$ (here we see that $\eta \in L^1(F)$ and $\xi \in L^1(E)$ are weak solution of equation $P\xi = \eta$ and P linear smooth elliptic differential operator). Then in this case we have that $\xi \in W_{k+l}^p(E)$ and we can end the proof. \square

The theorem 1.7 is important because we can generalize it to the not-compact case. We can also use it to prove the next useful result.

Theorem 1.8. *Let X be a compact manifold and E, F vector bundles on X . Let $P : C^\infty(E) \rightarrow C^\infty(F)$ be a linear smooth elliptic differential operator of order $l \geq 1$, with formal adjoint P^* . Then there are L^2 -orthogonal decompositions*

$$W_k^p(F) = P(W_{k+l}^p(E)) \oplus \ker P^* \quad (1.24)$$

$$C^{k,a}(F) = P(C^{k+l,a}(E)) \oplus \ker P^* \quad (1.25)$$

Proof. Let's prove the Sobolev decomposition 1.24, which can be used to prove the Hölder decomposition 1.25

From theorem 1.6 we have that $\ker P^*$ is a finite dimensional subspace of $W_k^p(F)$ contained in $C^\infty(F)$. We choose a L^2 -orthonormal basis $\{h_1, \dots, h_n\}$ of $\ker P^*$. Given $\eta \in W_k^p(F)$, we can write

$$\eta = \left(\eta - \sum_{j=1}^N \langle \eta | h_j \rangle_{L^2(F)} h_j \right) + \left(\sum_{j=1}^N \langle \eta | h_j \rangle_{L^2(F)} h_j \right) \quad (1.26)$$

and this leads to

$$W_k^p(F) = \{ \eta \in W_k^p(F) : \langle \eta | h_j \rangle_{L^2(F)} = 0 \text{ per ogni } h \in \ker P^* \} \oplus \ker P^* \quad (1.27)$$

and Sobolev decomposition follows easily from theorem 1.7. We see that $C^{k,a}(F) \subset W_k^p(F)$. If we intersect the decomposition 1.24 with $C^{k,a}(F)$ we obtain

$$C^{k,a}(F) = \{ \eta \in C^{k,a}(F) : \eta = P\xi \text{ for some } \xi \in W_{k+l}^p(E) \} \oplus \ker P^*. \quad (1.28)$$

Now let's choose $P > 1$ so $k + l - a \geq \frac{n}{p}$, thus from theorem 1.2 we have that $W_{k+l}^p(E) \subset C^{0,a}(E)$. In this way, using elliptic operator theory we obtain

$$\{ \eta \in C^{k,a}(F) : \eta = P\xi \text{ per qualche } \xi \in W_{k+l}^p(E) \} = P(C^{k+l,a}(E)) \quad (1.29)$$

and this complete the proof. \square

Given the hypothesis of theorem 1.8 we can deduce immediately that the linear map $P : C^\infty(E) \rightarrow C^\infty(F)$ admits a L^2 -orthogonal decomposition

$$C^\infty(F) = P(C^\infty(E)) \oplus \ker P^*. \quad (1.30)$$

with similar orthogonal property. This fact follows from the intersection of both sides of the equation 1.25 with $C^\infty(F)$ and using elliptic regularity results which comes from the operator and that we have pointed out earlier.

1.2 Hodge Theory

Hodge theory is an important analytical instrument to study the topology of a differentiable manifold. With this theory we can represent De Rham cohomology class on a compact manifold (which are topological objects) as harmonic forms (solutions of PDEs that depends on Riemannian metric on the manifold).

Hölder and Sobolev spaces are suitable functional classes to study elliptic differential operator theory. Here we consider Hodge theory in Hölder spaces, keeping in mind that analogous results are true even in Sobolev classes.

On a compact Riemannian manifold (X, g) we can define the formal adjoint d_g^* of the exterior derivative d and we take a linear self-adjoint elliptic operator with coefficients \mathcal{C}^∞ , of order 1, on Grassman bundle Λ^*T^*X , by

$$d_g^* + d : \mathcal{C}^\infty(\Lambda^*T^*X) \rightarrow \mathcal{C}^\infty(\Lambda^*T^*X).$$

Its square $(d_g^* + d)^2 = \Delta_g$ is a self-adjoint elliptic differential operator of order 2 on $C^\infty(\Lambda^*T^*X)$. For every integer $k \geq 1$ and every real number $a \in (0, 1)$, the operator $d_g^* + d$ defines a linear continuous function between Banach spaces:

$$d_g^* + d : \mathcal{C}^{k+1,a}(\Lambda^*T^*X) \rightarrow \mathcal{C}^{k,a}(\Lambda^*T^*X). \quad (1.31)$$

From elliptic theory we know that the subspace $\text{Im}(d_g^* + d)$ is closed in $C^{k,a}(\Lambda^*T^*X)$ and $\ker(d_g^* + d)$ is a finite real vector subspace of $\mathcal{C}^\infty(\Lambda^*T^*X)$. Moreover, from theorem 1.8 and from the fact that $d_g^* + d$ is self-adjoint, we have the direct sum decomposition:

$$\mathcal{C}^{k,a}(\Lambda^*T^*X) = \text{Im}(d_g^* + d) \oplus \ker(d_g^* + d), \quad (1.32)$$

which is orthogonal in L^2 . It's necessary now to give a better description of kernel and image of this operator with the following

Proposition 1.9. *Let $k \geq 2$. The application (1.31) has kernel*

$$\ker(d_g^* + d) = \{\xi \in \mathcal{C}^\infty(\Lambda^*T^*X) : d^*\xi = 0, \quad d\xi = 0\}$$

and image

$$\begin{aligned} \text{Im}(d_g^* + d) &= \{d_g^*\theta_1 + d\theta_2 : \theta_1, \theta_2 \in \mathcal{C}^{k+1,a}(\Lambda^*T^*X)\} \\ &= d_g^*(\mathcal{C}^{k+1,a}(\Lambda^*T^*X)) \oplus d(\mathcal{C}^{k+1,a}(\Lambda^*T^*X)), \end{aligned}$$

with the direct sum terms all orthogonal in L^2 .

Proof. A closed and co-closed form is in the kernel of $d_g^* + d$. Vice versa, if $\xi \in \mathcal{C}^{k,a}(\Lambda^*T^*X)$ and $dg = -d_g^*\xi$, then

$$\|d_g^*\xi\|_{L^2}^2 = \langle d_g^*\xi | d_g^*\xi \rangle_{L^2} = \langle \xi | dd_g^*\xi \rangle_{L^2} = -\langle \xi | dd\xi \rangle_{L^2} = 0.$$

So we obtain that $d_g^* \xi = 0$ and $d\xi = -d_g^* \xi = 0$. Moreover $\ker(d_g^* + d) \subset \mathcal{C}^\infty(\Lambda^* T^* X)$ from the regularity properties of the solutions of linear elliptic systems with \mathcal{C}^∞ homogeneous coefficients.

All that's left to do is verify the image characterization. Clearly $\text{Im}(d_g^* + d)$ is contained in

$$\mathcal{A} = \{d_g^* \theta_1 + d\theta_2 : \theta_1, \theta_2 \in C^{k+1,a}(\Lambda^* T^* X)\}. \quad (1.33)$$

We have to prove the inverse inclusion. Let $\theta_1, \theta_2 \in C^{k+1,a}(\Lambda^* T^* X)$ and we apply the decomposition (1.32) to the form $d_g^* \theta_1 + d\theta_2 \in C^{k,a}(\Lambda^* T^* X)$. We can find $h \in \ker(d_g^* + d)$ and $\theta_3 \in \mathcal{C}^{k+1,a}(\Lambda^* T^* X)$ such that

$$d_g^* \theta_1 + d\theta_2 = (d_g^* + d)\theta_3 + h. \quad (1.34)$$

Since the forms in $\ker(d_g^* + d)$ are closed and co-closed, we obtain:

$$\begin{aligned} d_g^* d(\theta_2 - \theta_3) &= 0, \\ dd_g^*(\theta_1 - \theta_3) &= 0, \end{aligned}$$

by applying d_g^* and d to the members of (1.34). From

$$\|d(\theta_2 - \theta_3)\|_{L^2}^2 = \langle d(\theta_2 - \theta_3) | d(\theta_2 - \theta_3) \rangle_{L^2} = \langle \theta_2 - \theta_3 | d_g^* d(\theta_2 - \theta_3) \rangle_{L^2} = 0$$

we obtain that $d\theta_2 = d\theta_3$, and, by similarity, that $d_g^* \theta_1 = d_g^* \theta_3$. So $h = 0$ and $(d_g^* + d)\theta_3 = d_g^* \theta_1 + d\theta_2$. We prove that $\text{Im}(d_g^* + d) = \mathcal{A}$. Finally,

$$\mathcal{A} = d_g^*(C^{k+1,a}(\Lambda^* T^* X)) \oplus d(C^{k+1,a}(\Lambda^* T^* X)) \quad (1.35)$$

because, if θ_1, θ_2 are two forms in $C^{k+1,a}(\Lambda^* T^* X)$, then

$$\langle d_g^* \theta_1 | d\theta_2 \rangle_{L^2} = \langle \theta_1 | dd\theta_2 \rangle_{L^2} = 0,$$

this shows that both the subspaces at the second member of (1.35) are orthogonal in L^2 . The proof is complete. \square

Now let's consider the Laplace operator $\Delta = (d_g^* + d)^2 = d_g^* d + dd_g^*$. For every integer $k \geq 2$ and every real number $a \in (0, 1)$, it defines a linear continuous application between Banach spaces

$$\Delta : \mathcal{C}^{k+2,a}(\Lambda^* T^* X) \rightarrow \mathcal{C}^{k,a}(\Lambda^* T^* X).$$

The operator Δ is linear elliptic, with \mathcal{C}^∞ -coefficients, and *maintains the forms degrees*. From general results on elliptic operators, $\text{Im}\Delta \subset C^{k,a}(\Lambda^* T^* X)$ is closed

and $\ker \Delta$ is a finite real vector subspace of $\mathcal{C}^\infty(\Lambda^*T^*X) \subset C^{k+,a}(\Lambda^*T^*X)$. Moreover, for the theorem 1.8 and the fact that Δ is self-adjoint, we have the following decomposition:

$$C^{k,a}(\Lambda^*T^*X) = \text{Im } \Delta \oplus \ker \Delta, \quad (1.36)$$

which is orthogonal in L^2 .

Proposition 1.10. *In the equation (1.36), we have*

$$\begin{aligned} \ker \Delta &= \ker(d_g^* + d), \\ \text{Im } \Delta &= \text{Im}(d_g^* + d). \end{aligned}$$

Proof. It's $\ker(d_g^* + d) \subset \ker(d_g^* + d)^2 = \ker \Delta$. The opposite inclusion can be obtained integrating by parts. The identity

$$\|(d_g^* + d)\xi\|_{L^2}^2 = \langle (d_g^* + d)\xi | (d_g^* + d)\xi \rangle_{L^2} = \langle \xi | (d_g^* + d)^2 \xi \rangle_{L^2} = \langle \xi | \Delta \xi \rangle_{L^2},$$

valid for every form $\xi \in \mathcal{C}^{k,a}(\Lambda^*T^*X)$ with $k \geq 2$, shows that $\ker(d_g^* + d) \subset \ker \Delta$, and so we have the equivalence.

Then we have

$$\text{Im } \Delta \subset \{d_g^* \theta_1 + d\theta_2 : \theta_1, \theta_2 \in C^{k+1,a}(\Lambda^*T^*X)\} = \text{Im}(d_g^* + d).$$

We ave now to prove the opposite inclusion. Given $\xi_1, \xi_2 \in \mathcal{C}^{k-1,a}(\Lambda^*T^*X)$, we use (1.36) to break down $d_g^* \xi_1 + d\xi_2$ in the sum

$$d_g^* \xi_1 + d\xi_2 = \Delta \eta_1 + \eta_2 = d_g^* d\eta_1 + dd_g^* \eta_1 + \eta_2,$$

with $\eta_1, \eta_2 \in \mathcal{C}^{k,a}(\Lambda^*T^*X)$, $\eta_2 \in \ker \Delta$. By applying d , and then d_g^* to the members of these equalities, we have that

$$\begin{cases} dd_g^* \xi_1 = dd_g^* d\eta_1, \\ d_g^* d\xi_2 = d_g^* dd_g^* \eta_1. \end{cases}$$

As we have done in the previous proposition, we integrate by parts:

$$\begin{cases} d_g^* \xi_1 = d_g^* d\eta_1, \\ d\xi_2 = dd_g^* \eta_1. \end{cases}$$

From this we have the inclusion $\text{Im}(d_g^* + d) \subset \text{Im } \Delta$, which gives us the equality. The proof is complete. \square

Definition 1.11. The elements of $\mathcal{H}(X) := \ker \Delta$ are called *harmonic forms*.

Since the operator Δ preserve the degrees of the differential forms , the space of harmonic forms can be broken down into the direct sum

$$\mathcal{H}(X) = \sum_{r \geq 0} \mathcal{H}^r(X), \quad \text{con } \mathcal{H}^r(X) = \mathcal{H}(X) \cap \Omega^r(X). \quad (1.37)$$

Proposition 1.12. For every couple of integer numbers $r \geq 0$, $k \geq 1$ and every real number $a \in (0, 1)$ we have the orthogonal decomposition with respect to the inner product in L^2 :

$$\mathcal{C}^{k,a}(\Lambda^r T^* X) = \mathcal{H}^r(X) \oplus d_g^*(\mathcal{C}^{k+1,a}(\Lambda^{r+1} T^* X)) \oplus d(\mathcal{C}^{k+1,a}(\Lambda^{r-1} T^* X)). \quad (1.38)$$

Proof. The thesis follows from (1.32) and from Proposition 1.9, we have the orthogonal decomposition in L^2 because Δ preserves the degrees of the differential forms. \square

Corollary 1.13. If $\eta \in \mathcal{C}^1(\Lambda^{r-1} T^* X)$ and $d\eta \in C^{k,a}(\Lambda^r T^* X)$ for an integer $k > 0$ and a real number $a \in (0, 1)$, then there exists $\xi \in C^{k+1,a}(\Lambda^{r-1} T^* X)$ with $d\eta = d\xi$.

Proof. For the (1.38) we can write $d\eta = h + d_g^*\theta + d\xi$ with $h \in \mathcal{H}^r(X)$, $\theta \in \mathcal{C}^{k+1,a}(\Lambda^{r+1} T^* X)$ and $\xi \in \mathcal{C}^{k+1,a}(\Lambda^{r-1} T^* X)$. Integrating by parts as the proof of Proposition 1.9, we can show that $h = 0$ and $d_g^*\theta = 0$. \square

The decomposition in L^2 -orthogonal direct sums

$$\Omega^*(X) = \bigoplus_{r=0}^n \Omega^r(X), \quad (1.39)$$

$$\Omega^r(X) = \mathcal{H}^r(X) \oplus d_g^*\Omega^{r+1}(X) \oplus d\Omega^{r-1}(X), \quad (1.40)$$

$$0 \leq r \leq n = \dim_{\mathbb{R}} X.$$

are valid even for forms \mathcal{C}^∞ (in this case we will write $\mathcal{C}^\infty(\Lambda^* T^* X) = \Omega^*(X)$, $\mathcal{C}^\infty(\Lambda^r T^* X) = \Omega^r(X)$). For every $0 \leq r \leq n = \dim_{\mathbb{R}} X$ the application induced by the quotient

$$\mathcal{H}^r(X) \ni h \longrightarrow [h] \in H^r(X) \quad (1.41)$$

is a linear isomorphism from the linear harmonic r -forms to the r th de Rham cohomology group of X .

Proof. Let's prove that (1.41) is injective. If $h \in \mathcal{H}^r(X)$ ed $h = df$ per una $f \in \Omega^{r-1}(X)$, is $h = df = 0$, because the (1.40) is a direct sum. For the inverse implication, if we break down a closed form $f \in \Omega^r(X)$ into the sum

$$f = h + d\theta + d_g^* \xi, \quad \text{con } h \in \mathcal{H}^r(X), \theta \in \Omega^{r-1}(X), \xi \in \Omega^{r+1}(X),$$

from $dd_g^* \xi = 0$ we obtain, by integration by parts, that $d_g^* \xi = 0$. So f is cohomologous to h . This prove that (1.41) is also surjective and so is a linear isomorphism. \square

2 Deformation of special Lagrangian submanifolds

In this section we describe the local structure of the moduli space of special Lagrangian submanifolds of a Calabi-Yau manifold.

This structure can be analyzed using a fundamental theorem on normal deformations proved by McLean in [13]. McLean results show that, if L is a compact special Lagrangian submanifold, the moduli space in a neighbourhood of L is a finite differential manifold whose tangent space in L can be canonically identified with the harmonic 1-forms space on L . The Riemannian metric on \mathcal{M} on the fiber becomes the L^2 -inner product on the harmonic forms.

Let's define the *orthonormal coreference system bundle* on a manifold (M, g) as:

$$\mathcal{F}(M) = \{u : (T_x M, g_x) \rightarrow (\mathbb{R}^n, g_0) : u \text{ is a linear isometry}\}. \quad (2.1)$$

Let ω be the 1-form on $\mathcal{F}(M)$ which take values in \mathbb{R}^n and defined as $\omega(V) = u(\pi_*(V))$ in which $V \in T_x \mathcal{F}(M)$ and π is the projection of $\mathcal{F}(M)$ on M . Let ϕ be the Levi-Civita connection form that satisfy $d\omega = -\phi \wedge \omega$. We can now fix indexes $1 \leq a, b, c, d \leq n$.

By choosing an orthonormal basis e_a of \mathbb{R}^n , we have that $\omega = \omega^a e_a$, $\phi = \varphi_b^a e_a \otimes (e_b)^*$ and so $d\omega^a = -\varphi_b^a \wedge \omega^b$. If I have a 1-parameter family of immersion $f_t = F : L^s \times I \rightarrow M$ such that $f = f_0 : L \rightarrow M$ is calibrated from ϑ . We can do a re-parametrization to make the immersion family be normal and we can adapt the fibers restricting them to the subbundle $\mathcal{F}^{(1)}$ of $\mathcal{F}^{(0)} = F^*(\mathcal{F}(M))$ such that $\omega^a = V^a dt$, $\frac{\partial}{\partial t} \lrcorner \omega^i = 0$.

Let's now show that the moduli space \mathcal{M} of special Lagrangian submanifolds locally has the structure of Lagrangian submanifold and is *special* in the appropriate way.

We will for now consider only compact manifolds L for three fundamental reasons. The first one is that in this case we can apply Hodge theory: in particular, for the Hodge-De Rham theorem, we can identify harmonic forms on manifold with closed and co-closed forms and so with singular cohomology classes (that are topological invariants). Moreover we can describe a submanifold deformation $f_t : L \rightarrow L_t \subset \mathcal{M}$ like a *normal deformation*, that is such that $\frac{\partial}{\partial t} f_t(p)$ is normal to L_t for fixed $p \in L_t$. This can be done because, by using the compactness of L , we can re-parametrize with a diffeomorphism on L depending on time. In this construction, the existence of a tubular neighbourhood of L in \mathcal{M} has an important role because it can be identified (through the exponential map) with a neighbourhood $N_\epsilon = \{V \in N(L) \mid \|V\| \leq \epsilon\}$ of the null section of $N(L)$. By the mean of the pullback, this fact allows us to move structures from ambient manifold \mathcal{M} to N_ϵ and to identify \mathcal{M} with one of its submanifolds.

From now on we consider the indexes $1 \leq i, j, k, \ell \leq n$ and the indexes with superscripts, for example i' to point to indexes with value $i + n$. For example on \mathbb{C}^n we consider coordinates

$$z^1 = x^1 + \dots + ix^{n+1}, \dots, z^n = x^n + \dots + ix^{2n}$$

in which $h_0 = \sum dz_i \otimes d\bar{z}^i$ is a standard Hermitian metric. If we consider \mathbb{C}^n as \mathbb{R}^{2n} , then there exists a complex structure J_0 that is given by the multiplication by $\sqrt{-1}$ which gives $J_0(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^{i'}}$ and $J_0(\frac{\partial}{\partial x^{i'}}) = -\frac{\partial}{\partial x^i}$.

The isomorphism $\flat : T(L) \rightarrow T^*(L)$ induced by the metric allows to build an isomorphism between $N(L)$ and $T^*(L)$. From the relation $\kappa(\cdot, \cdot) = g(J\cdot, \cdot)$ which defines Kähler forms and from the fact that the restriction of κ to L is null, it follows that complex structure J restrict to an isomorphism of TL on the normal bundle NL . By composing J with the isomorphism \flat , we obtain a bundle isomorphism between $N(L)$ and $T^*(L)$. In this way normal vector fields on L can be identified to differential forms: the normal vector field $V = V^{i'} \frac{\partial}{\partial \omega^{i'}}$ is identified with $v = V_i \omega^i$, in which $V_i = V^{i'}$.

We can now enunciate McLean theorem.

Theorem 2.1 (McLean). *Let L be a compact special Lagrangian submanifold. A normal vector field V on L define a normal deformation of special Lagrangian submanifold if and only if the equivalent 1-form $(JV)^\flat$ is closed and co-closed, that is to say if it is harmonic.*

So the moduli space of special Lagrangian submanifolds in a neighbourhood of L is a differential manifold whose tangent space in L can be identified with the space $\mathcal{H}^1(L)$, of the harmonic 1-forms on L .

Proof. Let's fix a tubular neighbourhood U on L in \mathcal{M} and correspondently a neighbourhood \mathcal{U} of the null section in $\mathcal{C}^\infty(L, NL)$, such that, for every $V \in \mathcal{U}$ and $t \in [0, 1]$, the application \exp_{tV} is a diffeomorphism of L on a submanifold L_{tV} of \mathcal{M} contained in U . Let κ be the Kähler form, ω the section of the canonical bundle in which $\omega \wedge \bar{\omega}$ is the volume element κ^n of \mathcal{M} , and $\beta = \text{Im } \omega = \frac{1}{2i}(\omega - \bar{\omega})$. Let's consider the application

$$F : \mathcal{U} \ni V \longrightarrow ((\exp_V)^*(\beta), -(\exp_V)^*(\kappa)) \in \Omega^n(L) \oplus \Omega^2(L). \quad (2.2)$$

F is Fréchet differentiable and $F^{-1}(0, 0)$ is the set of the normal fields V that deform L in a special Lagrangian submanifold contained in U . From the implicit function theorem in Fréchet spaces we can obtain some information about the local moduli space of normal deformations of L in special Lagrangian by studying the derivative $F'(0)$ of F in the null section. It is suitable to use the identification between $N(L)$ and T^*L , given by $V \leftrightarrow (JV)^b$. This identification gives a relation between \mathcal{U} and an open set of $\Omega^1(L)$ and so between $\Omega^1(L)$ and the tangent space to \mathcal{U} in 0. By doing this we obtain

$$F'(0) : \Omega^1(L) \longrightarrow \Omega^n(L) \oplus \Omega^2(L).$$

We call L the Lie derivative and we use the formula that gives Lie derivative of an alternate form by differential and exterior product and we obtain:

$$\left\{ \begin{array}{l} F'(0) = \frac{\partial}{\partial t} F(tV) \Big|_{t=0} = (L_V(\beta) |_L, -L_V(\kappa) |_L) \\ = ((V] d\beta + d(V] \beta) |_L, -(V] d\kappa + d(V] \kappa) |_L) \\ = (d(V] \beta) |_L, -d(V] \kappa) |_L . \end{array} \right\}. \quad (2.3)$$

By putting $v = (JV)^b$, we have:

$$-V] \kappa = -(V^{i'} \frac{\partial}{\partial \omega^{i'}}] (\omega^i \wedge \omega^{i'}) = V^{i'} \omega^i = V_i \omega^i = v, \quad (2.4)$$

$$\left\{ \begin{array}{l} (V] \beta |_L = (V^{i'} \frac{\partial}{\partial \omega^{i'}}] \text{Im} [(\omega^1 + i\omega^{1'}) \wedge \cdots \wedge (\omega^n + i\omega^{n'})] \Big|_L \\ = V_1 * \omega^1 + \cdots + V_n * \omega^n = *v \end{array} \right\} \quad (2.5)$$

By substituting 2.4 and 2.5 in 2.3, we see that

$$F'(0)(V) = (*d^*v, dv), \quad \text{ove } v = (JV)^b, \quad \forall V \in \mathcal{C}^\infty(L, N(L)). \quad (2.6)$$

The condition that L_{tV} are special Lagrangian manifolds for t in a neighbourhood of 0 is that $F'(0)(V) = 0$. So, the special Lagrangian deformation at the first order (kernel of $F'(0)$) corresponds to the closed and co-closed forms, that is to the harmonic 1-form. \square

According to McLean theorem, the moduli space \mathcal{M} of special Lagrangian submanifolds is (in a neighbourhood of L) a smooth manifold, whose dimension is the first Betti number $[\dim H^1(L)]$ of L . From the theorem we also have that the theory of deformation of a Lagrangian submanifold is nonobstructed. To show this fact, let's consider the F defined in (2.2) as an application from the space $\mathcal{C}^{1,\alpha}(T^*L)$ of the 1-form on L , which are Hölder forms of order α with their first derivatives, to the space $\mathcal{C}^{0,\alpha}(\Lambda^n T^*L) \oplus \mathcal{C}^{0,\alpha}(\Lambda^2 T^*L)$ of the couple of Hölder n -forms and 2-forms of order α .

Lemma 2.2. *The F transforms fields of $\Gamma(N(L))$ in exact n -forms and 2-forms.*

Proof. For the sake of simplicity we write $F(V) = (\beta_V, -\kappa_V)$. For every V , the forms β_V and κ_V are closed, because are pullback of closed forms. Moreover, $\exp_V : L \rightarrow \mathcal{M}$ is homotopous to the inclusion $i : L \rightarrow \mathcal{M}$, by means of $[0, 1] \ni t \rightarrow L_{tV} \subset \mathcal{M}$. Then, \exp_V^* and i^* induce the same application in cohomology. So, using the brackets $[\cdot]$ to point to the related cohomology class, we have $[\exp_V^*(\beta)] = [i^*(\beta)] = [\beta|_L] = 0$, since L is special Lagrangian. Similarly we also have that $[\exp_V^*(\kappa)] = 0$. So the forms β_V, κ_V in the image of F are cohomologous to zero, and so exact. \square

With this lemma it's easy to see that the deformation theory is not obstructed. In fact, let's consider F as a map from the 1-forms of class $C^{1,\alpha}$ to the couple of exact n -forms and 2-forms of class $C^{0,\alpha}$. According to Hodge theory, $F'(0) = (-d^*) \oplus d$ is surjective on the spaces of the couple of exact forms.

It's useful now to remember the implicit function theorem in Banach spaces.

Theorem 2.3. *Let \mathcal{Z} and \mathcal{Y} be Banach spaces and $\mathcal{U} \subseteq \mathcal{Z}$ an open neighbourhood of 0. Let $F : \mathcal{U} \rightarrow \mathcal{Y}$ be a map of class C^k , with $F(0) = 0$. We suppose that the linear bounded map $F'(0) : \mathcal{Z} \rightarrow \mathcal{Y}$ is surjective and its kernel $\mathcal{K} \leq \mathcal{Z}$ separates \mathcal{Z} from a complementary space \mathcal{A} . Then, there are open sets $\mathcal{W}_1 \subseteq \mathcal{K}$ and $\mathcal{W}_2 \subseteq \mathcal{A}$ that contain 0 with $\mathcal{W}_1 \times \mathcal{W}_2 \subseteq \mathcal{U}$ and a unique map $\xi : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ such that*

$$F^{-1}(0) \cap (\mathcal{W}_1 \times \mathcal{W}_2) = \{x, \xi(x) : x \in \mathcal{W}_1\} \quad (2.7)$$

in $\mathcal{Z} = \mathcal{K} \oplus \mathcal{A}$. Moreover, the map is of class C^k .

This theorem, combined with the result of the previous lemma, allows us to conclude that $F^{-1}(0, 0)$ is a manifold with tangent space in 0 equal to the kernel of $F'(0) \simeq \mathcal{H}^1(L)$. For the elliptic regularity results, $F^{-1}(0, 0)$ is smooth.

3 Induced metric on moduli space

We have seen that infinitesimal deformations of a closed special Lagrangian submanifold L in a Calabi-Yau space can be identified with $H^1(L)$. Moreover these deformations are nonobstructed and the local moduli space is a smooth manifold diffeomorphic to an open neighbourhood of the origin in $H^1(L)$.

Let's show that this manifold admits a natural Riemmanian metric induced by the L^2 product on harmonic 1-form.

Let's give some preliminary definitions. From now on we call κ the Kähler form and K_1 and K_2 the real and imaginary parts of the constant covariant n -form

We call L_t the one-parameter family of Lagrangian submanifolds, considered as maps $f : \mathcal{L} \rightarrow U$, in which $\mathcal{L} = L \times U$ in M Calabi-Yau, with $U \subset \mathbb{R}$ open interval. Furthermore, $f(L, t) = L_t$.

$\tilde{\theta}$ is the 1-form on \mathcal{L} such that

$$f^*\kappa = dt \wedge \tilde{\theta}.$$

The restriction of $\tilde{\theta}$ to the fiber $L \times \{t\}$ is called θ and, knowing that $d\kappa = 0$, then

$$d\theta = 0.$$

Let's now consider the local coordinates t_1, \dots, t_m on the moduli space \mathcal{M} of the deformation of $L = L_0$. From McLean theorem, we have that $m = b_1(L) = \dim H^1(L)$. For every tangent vector $\frac{\partial}{\partial t_j}$ we define (like we did for $\tilde{\theta}$) a closed 1-form θ_j on L_t for every $t \in \mathcal{M}$:

$$\iota\left(\frac{\partial}{\partial t_j}\right)\kappa = \theta_j,$$

(with a little abuse of notation).

Let A_1, \dots, A_m be a basis of $H_1(L, \mathbb{Z})$, we evaluate closed forms θ_j on the homology class A_i to obtain a period matrix λ_{ij} , that is a function on the moduli space:

$$\lambda_{ij} = \int_{A_i} \theta_j$$

Since, for McLean theorem, harmonic forms are linearly independent, the matrix λ_{ij} is invertible. We can identify the tangent space to \mathcal{M} with the cohomology group $H^1(L, \mathbb{R})$. Let $\alpha_1, \dots, \alpha_m \in H^1(L, \mathbb{Z})$ be the dual base to A_1, \dots, A_m . The correspondence

$$\frac{\partial}{\partial t_j} \mapsto [u(\frac{\partial}{\partial t_j})\kappa] = \sum \lambda_{ij}\alpha_i \quad (3.1)$$

identifies $T_{t'}\mathcal{M}$ with $H^1(L, \mathbb{R})$.

Let's now see some properties of the period matrix.

Proposition 3.1. *The 1-forms $\xi_i = \sum \lambda_{ij}dt_j$ on \mathcal{M} are closed*

Proof. We represent the local family of deformation with a map $f : \mathcal{G} \rightarrow M$ in which $\mathcal{G} = L \times M$ with projection $p : \mathcal{G} \rightarrow \mathcal{M}$. Let's choose, in every fiber of p , a loop which stands for A_i to give a fibration $\mathcal{G}_i \subseteq \mathcal{G}$ on \mathcal{M} . We define the 1-form ξ on \mathcal{M} as

$$\xi = p_*f^*\kappa$$

The map p_* (integration on fibers) send closed forms to closed forms and, since $d\kappa = 0$, $df^*\kappa = 0$, is also $d\xi = 0$.

Now, in coordinates, $\kappa = \sum_j dt_j \wedge \tilde{\theta}_j$ and $\tilde{\theta}_j$ restricts to θ_j on every fiber. Since θ_j is closed, the integration along the fibers of \mathcal{G}_i is only the evaluation of homology class A_i . So $\xi_i = \xi$ and ξ_i is closed. \square

Following from this proposition, on \mathcal{M} we can find local functions u_1, \dots, u_m , defined up to a constant, such that:

$$du_i = \xi_i = \sum_j \lambda_{ij}dt_j \quad (3.2)$$

Since λ_{ij} is invertible, u_1, \dots, u_m are local coordinates on \mathcal{M} . We have a coordinate chart

$$u : \mathcal{M} \rightarrow H^1(L, \mathbb{R}) \quad (3.3)$$

defined as $u(t) = \sum_i u_i\alpha_i$ that is independent from the choice of the basis.

Let's now do the same for K . The basis $\alpha_1, \dots, \alpha_m$ defines a basis B_1, \dots, B_m of $H_{n-1}(L, \mathbb{Z})$ and so we have a period matrix

$$\mu_{ij} = \int_{B_i} \varphi_j$$

As we've done for κ , we find local coordinates v_1, \dots, v_m on \mathcal{M} such that

$$dv_i = \sum_j \mu_{ij} dt_j \quad (3.4)$$

and a map

$$v : \mathcal{M} \rightarrow H^{n-1}(L, \mathbb{R}) \quad (3.5)$$

given (using the basis β_1, \dots, β_m of $H^{n-1}(L, \mathbb{R})$ dual to B_1, \dots, B_m) by $v(t) = \sum_i v_i \beta_i$.

From u and v we obtain a map

$$F : \mathcal{M} \rightarrow H^1(L, \mathbb{R}) \times H^{n-1}(L, \mathbb{R}),$$

defined by $F(t) = (u(t), v(t))$.

Now we have to introduce the metric induced by L^2 on \mathcal{M} . L is oriented; for Poincaré duality, $H^1(L)$ and $H^{n-1}(L)$ are in canonical duality. For every vector space W there is a symmetric form on $W \oplus W^*$ associated to the quadratic form

$$B((v, \alpha), (v, \alpha)) = \langle v, \alpha \rangle.$$

This method allows us to define a metric G on $H^1(L) \oplus H^{n-1}(L)$.

Proposition 3.2. *The L^2 metric g on \mathcal{M} is F^*G*

Proof. From (3.1) we have

$$dF\left(\frac{\partial}{\partial t_j}\right) = \left(\sum_i \lambda_{ij} \alpha_i, \sum_i \mu_{ij} \beta_i\right)$$

So

$$F^*G\left(\sum_j a_j \frac{\partial}{\partial t_j}, \sum_j a_j \frac{\partial}{\partial t_j}\right) = \sum_{i,j,k,\ell} a_j a_k \lambda_{ij} \mu_{k\ell} \langle \alpha_i, \beta_\ell \rangle = \sum_{i,j,k,\ell} a_j a_k \lambda_{ij} \mu_{k\ell} \int_L \alpha_i \wedge \beta_\ell \quad (3.6)$$

But

$$\int_L \left(\sum_i a_i \theta_i\right) \wedge * \left(\sum_i a_i \theta_i\right) = \int_L \sum_{j,k} a_j a_k \theta_j \wedge \varphi_k$$

and, using $\theta_j = \sum_i \lambda_{ij} \alpha_i$, $\varphi_k = \sum_i \mu_{ik} \beta_i$ we obtain the same expression of (3.6) \square

On the moduli space \mathcal{M} is defined a Riemannian metric $g_{\mathcal{M}}$ in the following way: given two tangent vectors $v_1, v_2 \in T_L(\mathcal{M})$, I can identify them with the correspondent harmonic 1-forms θ_1, θ_2 on L and define $g_{\mathcal{M}}(v_1, v_2)$ as $\int_L \langle \theta_1, \theta_2 \rangle d\text{vol}_L$, in which $\langle \cdot, \cdot \rangle$ is the scalar product on the cotangent fibers.

The last step is studying the embedding of the moduli space \mathcal{M} in $H^1(L) \times H^{n-1}(L)$.

If W is a real vector space, the direct sum with its dual $W \oplus W^*$ has a natural symplectic form defined by:

$$w((v, \alpha), (v', \alpha')) = \langle v, \alpha' \rangle - \langle v', \alpha \rangle$$

We use this fact to consider $H^1(L) \times H^{n-1}(L)$ as a symplectic space.

Theorem 3.3. *The map F embeds \mathcal{M} in $H^1(L) \times H^{n-1}(L)$ as a Lagrangian submanifold*

Proof. The Kähler form κ and the real and imaginary parts K_1 and K_2 of the constant canonical n -form K of a Calabi-Yau manifold satisfy the relation $K_1 \wedge \kappa = K_2 \wedge \kappa = 0$. Therefore

$$\kappa \wedge K = 0. \quad (*)$$

Let Y and Z be two vector fields. The inner product of Y and Z for the two members of $(*)$ gives

$$0 = (\iota(Z)\iota(Y)\kappa) \wedge K - \iota(Y)\kappa \wedge \iota(Z)K + \iota(Z)\kappa \wedge \iota(Y)K + \kappa \wedge (\iota(Z)\iota(Y)K)$$

and, by restriction to the special Lagrangian submanifold L , in which κ and K vanish, we have:

$$\iota(Y)\kappa \wedge \iota(Z)K = \iota(Z)\kappa \wedge \iota(Y)K.$$

By substituting to Y and Z the extensions of vector fields $\frac{\partial}{\partial t_i}$ e $\frac{\partial}{\partial t_j}$, we obtain

$$\theta_i \wedge \varphi_j = \theta_j \wedge \varphi_i$$

and, integrating,

$$\int_L \theta_i \wedge \varphi_j = \int_L \theta_j \wedge \varphi_i.$$

By using $\theta_j = \sum_i \lambda_{ij} \alpha_i$, $\varphi_k = \sum_i \mu_{ik} \beta_i$, we have that

$$\sum_i \lambda_{ik} \mu_{ij} = \sum_i \lambda_{ij} \mu_{ik}. \quad (3.7)$$

From the coordinates definition previously given (which satisfies (3.2)), we have:

$$\lambda_{ij} = \frac{\partial u_i}{\partial t_j}, \quad \mu_{ij} = \frac{\partial v_i}{\partial t_j}$$

so that (3.7) becomes

$$\sum_i \frac{\partial u_i}{\partial t_k} \frac{\partial v_i}{\partial t_j} = \sum_i \frac{\partial u_i}{\partial t_j} \frac{\partial v_i}{\partial t_k}$$

and so

$$F^*\left(\sum_i du_i \wedge dv_i\right) = 0.$$

□

We have seen that the graphic in the cotangent bundle of a differential form is a Lagrangian submanifold if and only if the form is closed. If the graphic is obtained by deforming the null section, the form must be exact, that is equal to the differential of a function. In our case, we can write:

$$v_j = \frac{\partial \phi}{\partial u_j} \quad (3.8)$$

for a $\phi(u_1, \dots, u_m)$. The metric on \mathcal{M} can be written in coordinates as

$$g = F^*G = \sum_i du_i dv_i = \sum_{i,j} \frac{\partial^2 \phi}{\partial u_i \partial u_j} du_i du_j. \quad (3.9)$$

In a similar way, taking $N = H^{n-1}(L)$ and using a function $\psi(u_1, \dots, u_m)$, we can represent the metric by

$$g = \sum_{i,j} \frac{\partial^2 \psi}{\partial v_i \partial v_j} dv_i dv_j.$$

Such metrics are called *Hessian*. Now, starting from the fact that \mathcal{M} parametrizes special Lagrangian submanifolds, we search for the special condition that \mathcal{M} can inherit from the embedding F .

If V is a normal vector field, the generators of $\Lambda^n V$ and of $\Lambda^n V^*$ define two constant n -forms R_1 and R_2 on the manifold $V \times V^*$. We say that a Lagrangian submanifold $V \times V^*$ is *special* if on it a linear combination of the two forms vanishes. Therefore,

Proposition 3.4. *The map F embeds \mathcal{M} as a special Lagrangian submanifold if and only if all the following conditions are verified:*

- ϕ satisfies Monge-Ampère equation $\det(\partial^2\phi/\partial u_i\partial u_j) = c$.
- ψ satisfies Monge-Ampère equation $\det(\partial^2\psi/\partial v_i\partial v_j) = c^{-1}$.
- The volume of the torus $H^1(L_t, \mathbb{R}/\mathbb{Z})$ is independent from $t \in \mathcal{M}$.
- The volume of the torus $H^{n-1}(L_t, \mathbb{R}/\mathbb{Z})$ is independent from $t \in \mathcal{M}$.

Proof. For the first part, using the coordinates u_1, \dots, u_m , the condition for which the m -form $c_1R_1 + c_2R_2$ vanishes is precisely the

$$c_1 du_1 \wedge \dots \wedge du_m + c_2 \det(\partial^2\phi/\partial u_i\partial u_j) du_1 \wedge \dots \wedge du_m = 0,$$

that gives

$$\det(\partial^2\phi/\partial u_i\partial u_j) = -c_1/c_2 = c.$$

Exchanging the roles of $H^1(L)$ and $H^{n-1}(L)$ we find the second condition.

To find the volume of the torus $H^1(L_t, \mathbb{R}/\mathbb{Z})$ we take the basis a_1, \dots, a_m of harmonic 1-forms, normalized by

$$\int_{A_i} a_j = \delta_{ij}$$

and so we will have the volume $\sqrt{\det(a_i, a_j)}$ using the scalar product on the harmonic forms. From the definition of the coefficients λ_{ij} , the normalized harmonic forms are

$$a_j = \sum_k (\lambda_{kj}^{-1}) \theta_k.$$

The scalar product is

$$(\theta_j, \theta_k) = \int_L \theta_j \wedge * \theta_k = \sum_i \lambda_{ij} \mu_{ik}.$$

Then the volume is

$$\sqrt{\det(\mu\lambda^{-1})}.$$

Now, in the coordinates t_1, \dots, t_m that we introduced on the moduli space, the form $c_1R_1 + c_2R_2$ restricted to $F(\mathcal{M})$ is

$$(c_1 \det \lambda + c_2 \det \mu) dt_1 \wedge \dots \wedge dt_m$$

and this form vanishes if and only if $\det(\mu\lambda^{-1}) = -c_1/c_2$. The last condition is analogous, but in that case the volume is $\sqrt{\det(\lambda\mu^{-1})}$. \square

4 Kähler metric

We can now consider a more general moduli space, that contains not only special Lagrangian submanifolds, but manifolds with flat unitary linear bundle.

Since a unitary linear bundle is classified by an element of the group $H^1(L, \mathbb{R}/\mathbb{Z})$, for homotopic invariance the moduli space becomes

$$\mathcal{M}^C = \mathcal{M} \times H^1(L, \mathbb{R}/\mathbb{Z}).$$

The tangent space to a point of this space is a complex vector space with an almost complex structure

$$T_m \cong H^1(L, \mathbb{R}) \oplus H^1(L, \mathbb{R}) \cong H^1(L, \mathbb{R}) \otimes \mathbb{C}.$$

Given that, for every real vector space W , a scalar product on W defines an Hermitian form on $W \otimes \mathbb{C}$, then \mathcal{M}^C has an Hermitian metric.

Proposition 4.1. *is integrable and the scalar product on $H^1(L, \mathbb{R})$ defines a Kähler metric on \mathcal{M}^C .*

Proof. Let's use a basis $\alpha_1, \dots, \alpha_m$ of $H^1(L, \mathbb{R})$ to define coordinates x_1, \dots, x_m on the universal covering of the torus $H^1(L, \mathbb{R}/\mathbb{Z})$. The $(t_1, \dots, t_m, x_1, \dots, x_m)$ are local coordinates for \mathcal{M}^C . From the equation (3.1), the almost complex structure is defined by:

$$I\left(\frac{\partial}{\partial t_j}\right) = \sum_i \lambda_{ij} \frac{\partial}{\partial x_i},$$

$$I\left(\sum_i \lambda_{ij} \frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial t_j}.$$

Let's now define the complex vector fields

$$W_j = \frac{\partial}{\partial t_j} - iI\left(\frac{\partial}{\partial t_j}\right) = \frac{\partial}{\partial t_j} - i \sum \lambda_{jk} \frac{\partial}{\partial x_k}.$$

They satisfy $IW_j = iW_j$ and form a basis for $(1, 0)$ -vector fields. The forms $\theta_j = \sum \lambda_{jk} dt_k - idx_j$ make W_j vanish and form a basis for the $(0, 1)$ -forms. But $\theta_j = d(u_j - ix_j)$, and so $\kappa_j = u_j + ix_j$ are complex coordinates and the complex structure is integrable.

The 2-form $\tilde{\kappa}$ associated to the Hermitian metric ids defined by

$$\tilde{\kappa}\left(\frac{\partial}{\partial t_j}, \frac{\partial}{\partial x_k}\right) = g\left(\frac{\partial}{\partial t_j}, I \frac{\partial}{\partial x_k}\right)$$

and, for the definition of I ,

$$\tilde{\kappa}\left(\frac{\partial}{\partial t_j}, \frac{\partial}{\partial x_k}\right) = - \sum_l \lambda_{lk}^{-1} g_{jl}.$$

We know that the metric is F^*G and so, in local coordinates t_1, \dots, t_m

$$g_{ij} = \sum_k \frac{\partial u_k}{\partial t_i} \frac{\partial v_k}{\partial t_j} = \sum_k \lambda_{ki} \mu_{kj}.$$

Then,

$$\tilde{\kappa} = - \sum_{j,k} \mu_{kj} dt_j \wedge dx_k = - \sum_k dv_k \wedge dx_k,$$

Since $dv_i = \sum_j \mu_{ij} dt_j$, with v_i local coordinates on \mathcal{M} and μ_{ij} period matrix on $H_{n-1}(L, \mathbb{Z})$. So $\tilde{\kappa}$ is closed and the metric is Kähler. \square

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