A Pair of Smarandachely Isotopic Quasigroups and Loops Of The Same Variety*†

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Abstract

The isotopic invariance or universality of types and varieties of quasigroups and loops described by one or more equivalent identities has been of interest to researchers in loop theory in the recent past. A variety of quasigroups(loops) that are not universal have been found to be isotopic invariant relative to a special type of isotopism or the other. Presently, there are two outstanding open problems on universality of loops: semi automorphic inverse property loops(1999) and Osborn loops(2005). Smarandache isotopism(S-isotopism) was originally introduced by Vasantha Kandasamy in 2002. But in this work, the concept is re-restructured in order to make it more explorable. As a result of this, the theory of Smarandache isotopy inherits the open problems as highlighted above for isotopy. In this short note, the question 'Under what type of S-isotopism will a pair of S-quasigroups(S-loops) form any variety?' is answered by presenting a pair of specially S-isotopic S-quasigroups(loops) that both belong to the same variety of S-quasigroups(S-loops). This is important because pairs of specially S-isotopic S-quasigroups(e.g Smarandache cross inverse property quasigroups) that are of the same variety are useful for applications(e.g cryptography).

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1 Introduction

1.1 Isotopy Theory Of Quasigroups And Loops

The isotopic invariance of types and varieties of quasigroups and loops described by one or more equivalent identities, especially those that fall in the class of Bol-Moufang type loops as first named by Fenyves [17] and [16] in the 1960s and later on in this 21st century by Phillips and Vojtěchovský [35], [36] and [29] have been of interest to researchers in loop theory in the recent past. Among such is Etta Falconer's Ph.D [14] and her paper [15] which investigated isotopy invariants in quasigroups. Loops such as Bol loops, Moufang loops, central loops and extra loops are the most popular loops of Bol-Moufang type whose isotopic invariance have been considered. For more on loops and their properties, readers should check [34], [8],[11], [13], [18] and [40].

Bol-Moufang type of quasigroups(loops) are not the only quasigroups(loops) that are isomorphic invariant and whose universality have been considered. Some others are flexible loops, F-quasigroups, totally symmetric quasigroups(TSQ), distributive quasigroups, weak inverse property loops(WIPLs), cross inverse property loops(CIPLs), semi-automorphic inverse property loops(SAIPLs) and inverse property loops(IPLs). As shown in Bruck [34], a left(right) inverse property loop is universal if and only if it is a left(right) Bol loop, so an IPL is universal if and only if it is a Moufang loop. Jaíyéọlá [20] investigated the universality of central loops. Recently, Michael Kinyon et. al. [25], [26], [27] solved the Belousov problem concerning the universality of F-quasigroup which has been open since 1967. The universality of WIPLs and CIPLs have been addressed by OSborn [32] and Artzy [2] respectively while the universality of elasticity(flexibility) was studied by Syrbu [39]. In 1970, Basarab [4] later continued the work of J. M. Osborn of 1961 on universal WIPLs by studying isotopes of WIPLs that are also WIPLs after he had studied a class of WIPLs([3]) in 1967. The universality of SAIPLs is still an open problem to be solved as stated by Michael Kinyon during the LOOPs'99 conference. After the consideration of universal AIPLs by Karklinsh and Klin [24], Basarab [6] obtained a sufficient condition for which a universal AIPL is a G-loop. Although Basarab [5] and [7] considered universal Osborn loops but the universality of Osborn loops was raised as an open problem by Michael Kinyon [28]. Up to the present moment, the problem is still open.

Interestingly, Adeniran [1] and Robinson [37], Oyebo and Adeniran [33], Chiboka and Solarin [12], Bruck [9], Bruck and Paige [10], Robinson [38], Huthnance [19] and Adeniran [1] have respectively studied the holomorphs of Bol loops, central loops, conjugacy closed loops, inverse property loops, A-loops, extra loops, weak inverse property loops, Osborn loops and Bruck loops. Huthnance [19] showed that if $(L,\cdot)$ is a loop with holomorph $(H,\circ)$, $(L,\cdot)$ is a WIPL if and only if $(H,\circ)$ is a WIPL. The holomorphs of an AIPL and a CIPL are yet to be studied.

1.2 Isotopy Theory Of Smarandache Quasigroups And Loops

The study of Smarandache loops was initiated by W.B. Vasantha Kandasamy in 2002. In her book [40], she defined a Smarandache loop(S-loop) as a loop with at least a subloop which forms a subgroup under the binary operation of the loop. In her book, she introduced over 75 Smarandache concepts on loops. In her first paper [41], she introduced Smarandache : left(right) alternative loops, Bol loops, Moufang loops, and Bruck loops. But in Jaíyéọlá [21], Smarandache : inverse property loops (IPL), weak inverse property loops (WIPL), G-loops, conjugacy closed loops (CC-loop), central loops, extra loops, A-loops, K-loops, Bruck loops, Kikkawa loops, Burn loops and homogeneous loops were introduced and studied relative to the holomorphs of loops. It is particularly established that a loop is a Smarandache loop if and only if its holomorph is a Smarandache loop. This statement was also shown to be true for some weak Smarandache loops(inverse property, weak inverse property) but false for others(conjugacy closed, Bol, central, extra, Burn, A-, homogeneous) except if their holomorphs are nuclear or central. The study of Smarandache quasigroups was carried out in Jaíyéọlá [22] after their introduction in Muktibodh [30] and [31]. In Jaíyéọlá [23], the universality of some Smarandache loops of Bol-Moufang types was studied and neccessary and sufficient conditions for their universality were established.

In this short note, the question 'Under what type of S-isotopism will a pair of S-quasigroups(S-loops) form any variety?' is answered by presenting a pair of specially S-isotopic S-quasigroups(loops) that both belong to the same variety of S-quasigroups(S-loops). This is important because pairs of specially S-isotopic S-quasigroups(e.g Smarandache cross inverse property quasigroups) that are of the same variety are useful for applications(e.g cryptography).

2 Definitions and Notations

Definition 2.1 *Let L be a non-empty set. Define a binary operation $(\cdot)$ on L : If $x\cdot y\in L\ \forall\ x,y\in L$, $(L,\cdot)$ is called a groupoid. If the system of equations ; $a\cdot x=b$ and $y\cdot a=b$ have unique solutions for x and y respectively, then $(L,\cdot)$ is called a quasigroup. Furthermore, if there exists a unique element $e\in L$ called the identity element such that $\forall\ x\in L$, $x\cdot e=e\cdot x=x$, $(L,\cdot)$ is called a loop.*

If there exists at least a non-empty and non-trivial subset M of a groupoid(quasigroup or semigroup or loop) L such that $(M,\cdot)$ is a non-trivial subsemigroup(subgroup or subgroup or subgroup) of $(L,\cdot)$, then L is called a Smarandache: groupoid(S-groupoid)$\Big($quasigroup(S-quasigroup) or semigroup(S-semigroup) or loop(S-loop)$\Big)$ with Smarandache: subsemigroup(S-subsemigroup)$\Big($subgroup(S-subgroup) or subgroup(S-subgroup) or subgroup(S-subgroup)$\Big)$ M.

A quasigroup(loop) is called a Smarandache "certain" quasigroup(loop) if it has at least *a non-trivial subquasigroup(subloop) with the "certain" property and the latter is referred to as the Smarandache "certain" subquasigroup(subloop). For example, a loop is called a Smarandache Bol-loop if it has at least a non-trivial subloop that is a Bol-loop and the latter is referred to as the Smarandache Bol-subloop. By an "initial S-quasigroup" L with an "initial S-subquasigroup"* L', *we mean L and* L' *are pure quasigroups, i.e. they do not obey a "certain" property(not of any variety).*

Let $(G, \cdot)$ *be a quasigroup(loop). The bijection* $L_x : G \to G$ *defined as* $yL_x = x \cdot y \;\forall\; x, y \in G$ *is called a left translation(multiplication) of G while the bijection* $R_x : G \to G$ *defined as* $yR_x = y \cdot x \;\forall\; x, y \in G$ *is called a right translation(multiplication) of* G.

The set $SYM(L, \cdot) = SYM(L)$ *of all bijections in a groupoid* $(L, \cdot)$ *forms a group called the permutation(symmetric) group of the groupoid* $(L, \cdot)$. *If L is a S-groupoid with a S-subsemigroup H, then the set* $SSYM(L, \cdot) = SSYM(L)$ *of all bijections A in L such that* $A : H \to H$ *forms a group called the Smarandache permutation(symmetric) group of the S-groupoid. In fact,* $SSYM(L) \leq SYM(L)$.

Definition 2.2 *If* $(L, \cdot)$ *and* $(G, \circ)$ *are two distinct groupoids, then the triple* $(U, V, W) : (L, \cdot) \to (G, \circ)$ *such that* $U, V, W : L \to G$ *are bijections is called an isotopism if and only if*

$$xU \circ yV = (x \cdot y)W \;\forall\; x, y \in L.$$

So we call L and G groupoid isotopes.

If $U = V = W$, *then U is called an isomorphism, hence we write* $(L, \cdot) \cong (G, \circ)$.

Now, if $(L, \cdot)$ *and* $(G, \circ)$ *are S-groupoids with S-subsemigroups* L' *and* G' *respectively such that* $A : L' \to G'$, *where* $A \in \{U, V, W\}$, *then the isotopism* $(U, V, W) : (L, \cdot) \to (G, \circ)$ *is called a Smarandache isotopism(S-isotopism).*

Thus, if $U = V = W$, *then U is called a Smarandache isomorphism, hence we write* $(L, \cdot) \succsim (G, \circ)$.

If $(L, \cdot) = (G, \circ)$, *then the triple* $\alpha = (U, V, W)$ *of bijections on* $(L, \cdot)$ *is called an autotopism of the groupoid(quasigroup, loop)* $(L, \cdot)$. *Such triples form a group* $AUT(L, \cdot)$ *called the autotopism group of* $(L, \cdot)$. *Furthermore, if* $U = V = W$, *then U is called an automorphism of the groupoid(quasigroup, loop)* $(L, \cdot)$. *Such bijections form a group* $AUM(L, \cdot)$ *called the automorphism group of* $(L, \cdot)$.

Similarly, if $(L, \cdot)$ *is an S-groupoid with S-subsemigroup* L' *such that* $A \in \{U, V, W\}$ *is a Smarandache permutation, then the autotopism* (U, V, W) *is called a Smarandache autotopism (S-autotopism) and they form a group* $SAUT(L, \cdot)$ *which will be called the Smarandache autotopism group of* $(L, \cdot)$. *Observe that* $SAUT(L, \cdot) \leq AUT(L, \cdot)$.

Discussions To be more precise about the notion of S-isotopism in Definition 2.2, the following explanations are given. For a given S-groupoid, the S-subsemigroup is arbitrary. But in the proofs, we make use of one arbitrary S-subsemigroup for an S-groupoid at a time for our arguments. Now, if $(L, \cdot)$ and $(G, \circ)$ are S-isotopic S-groupoids with arbitrary S-subsemigroups L' and G' respectively under the triple (U, V, W). In case the S-subsemigroup L' of the S-groupoid L is replaced with another S-groupoid L'' of L(i.e a situation where by L has at least two S-subsemigroups), then under the same S-isotopism (U,V,W), the S-groupoid isotope G has a second S-subsemigroups G''. Hence, when studying the S-isotopism (U,V,W), it will be for the system

$$\{(L,\cdot),(L',\cdot)\}\to\{(G,\circ),(G',\circ)\} \text{ or } \{(L,\cdot),(L'',\cdot)\}\to\{(G,\circ),(G'',\circ)\}$$

and not

$$\{(L,\cdot),(L',\cdot)\}\to\{(G,\circ),(G'',\circ)\} \text{ or } \{(L,\cdot),(L'',\cdot)\}\to\{(G,\circ),(G',\circ)\}.$$

This is because $|L'|=|G'|$ and $|L''|=|G''|$ since $(L')A=G'$ and $(L'')A=G''$ for all $A\in\{U,V,W\}$ while it is not compulsory that $|L'|=|G''|$ and $|L''|=|G'|$. It is very easy to see from the definition that the component transformations U,V,W of isotopy after restricting them to the S-subsemigroup or S-subgroup L' are bijections. Let $x_1,x_2\in L'$, then $x_1A=x_2A$ implies that $x_1=x_2$ because $x_1,x_2\in L'$ implies $x_1,x_2\in L$, hence $x_1A=x_2A$ in L implies $x_1=x_2$. The mappings $A\ :L'\to G'$ and $A\ :L-L'\to G-G'$ are bijections because $A\ :L\to G$ is a bijection. Our explanations above are illustrated with the following examples.

Example 2.1 *The systems $(L,\cdot)$ and $(L,*)$, $L=\{0,1,2,3,4\}$ with the multiplication tables below are S-quasigroups with S-subgroups $(L',\cdot)$ and $(L'',*)$ respectively, $L'=\{0,1\}$ and $L''=\{1,2\}$. $(L,\cdot)$ is taken from Example 2.2 of [31]. The triple (U,V,W) such that*

$$U=\begin{pmatrix}0&1&2&3&4\\1&2&3&4&0\end{pmatrix},\ V=\begin{pmatrix}0&1&2&3&4\\1&2&4&0&3\end{pmatrix}\ \textit{and}\ W=\begin{pmatrix}0&1&2&3&4\\1&2&0&4&3\end{pmatrix}$$

are permutations on L, is an S-isotopism of $(L,\cdot)$ onto $(L,*)$. Notice that $A(L')=L''$ for all $A\in\{U,V,W\}$ and $U,V,W:L'\to L''$ are all bijcetions.

$\cdot$	0	1	2	3	4
0	0	1	3	4	2
1	1	0	2	3	4
2	3	4	1	2	0
3	4	2	0	1	3
4	2	3	4	0	1

$*$	0	1	2	3	4
0	1	0	4	2	3
1	3	1	2	0	4
2	4	2	1	3	0
3	0	4	3	1	2
4	2	3	0	4	1

Example 2.2 *According Example 4.2.2 of [43], the system $(\mathbb{Z}_6,\times_6)$ i.e the set $L=\mathbb{Z}_6$ under multiplication modulo 6 is an S-semigroup with S-subgroups $(L',\times_6)$ and $(L'',\times_6)$, $L'=\{2,4\}$ and $L''=\{1,5\}$. This can be deduced from its multiplication table, below. The triple (U,V,W) such that*

$$U=\begin{pmatrix}0&1&2&3&4&5\\4&3&5&1&2&0\end{pmatrix},\ V=\begin{pmatrix}0&1&2&3&4&5\\1&3&2&4&5&0\end{pmatrix}\ \textit{and}\ W=\begin{pmatrix}0&1&2&3&4&5\\1&0&5&4&2&3\end{pmatrix}$$

are permutations on L, is an S-isotopism of $(\mathbb{Z}_6,\times_6)$ unto an S-semigroup $(\mathbb{Z}_6,*)$ with S-subgroups $(L''',*)$ and $(L'''',*)$, $L'''=\{2,5\}$ and $L''''=\{0,3\}$ as shown in the second *table below. Notice that $A(L') = L'''$ and $A(L'') = L''''$ for all $A \in \{U,V,W\}$ and $U,V,W : L' \to L'''$ and $U,V,W : L'' \to L''''$ are all bijcetions.*

$\times_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

$*$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	4	1	1	4	4	1
2	5	1	5	2	1	2
3	3	1	5	0	4	2
4	1	1	1	1	1	1
5	2	1	2	5	1	5

From Example 2.1 and Example 2.2, it is very clear that the study of of S-isotopy of two S-groupoids or S-quasigroups or S-semigroups or S-loops is independent of the S-subsemigroup or S-subgroup that is in consideration. All results in this paper are true for any given S-subsemigroups or S-subgroups of two S-isotopic S-groupoids or S-quasigroups or S-semigroups or S-loops. More examples of S-isotopic S-groupoids can be constructed using S-groupoids in [42].

Remark 2.1 *Taking careful look at Definition 2.2 and comparing it with [Definition 4.4.1,[40]], it will be observed that the author did not allow the component bijections U,V and W in (U,V,W) to act on the whole S-loop L but only on the S-subloop(S-subgroup) L'. We feel this is necessary to adjust here so that the set $L - L'$ is not out of the study. Apart from this, our adjustment here will allow the study of Smarandache isotopy to be explorable. Therefore, the S-isotopism and S-isomorphism here are clearly special types of relations(isotopism and isomorphism) on the whole domain into the whole co-domain but those of Vasantha Kandasamy [40] only take care of the structure of the elements in the S-subloop and not the S-loop. Nevertheless, we do not fault her study for we think she defined them to apply them to some life problems as an applied algebraist.*

To every loop $(L,\cdot)$ with automorphism group $AUM(L,\cdot)$, there corresponds another loop. Let the set $H = (L,\cdot) \times AUM(L,\cdot)$. If we define '$\circ$' on H such that $(\alpha,x)\circ(\beta,y) = (\alpha\beta, x\beta\cdot y)$ for all $(\alpha,x),(\beta,y) \in H$, then $H(L,\cdot) = (H,\circ)$ is a loop as shown in Bruck [9] and is called the Holomorph of $(L,\cdot)$. Let $(L,\cdot)$ be an S-quasigroup(S-loop) with S-subgroup $(L',\cdot)$. Define the Smarandache automorphism of L to be the set $SAUM(L) = SAUM(L,\cdot) = \{\alpha \in AUM(L) : \alpha : L' \to L'\}$. It is easy to see that $SAUM(L) \leq AUM(L)$. So, $SAUM(L)$ will be called the Smarandache automorphism group(SAG) of L. Now, set $H_S = (L,\cdot) \times SAUM(L,\cdot)$. If we define '$\circ$' on H_S such that $(\alpha,x)\circ(\beta,y) = (\alpha\beta, x\beta\cdot y)$ for all $(\alpha,x),(\beta,y) \in H_S$, then $H_S(L,\cdot) = (H_S,\circ)$ is a S-quasigroup(S-loop) with S-subgroup $(H',\circ)$ where $H' = L' \times SAUM(L)$ and thus will be called the Smarandache Holomorph(SH) of $(L,\cdot)$.

3 Main Results

Theorem 3.1 *Let $U = (L, \oplus)$ and $V = (L, \otimes)$ be initial S-quasigroups such that $SAUM(U)$ and $SAUM(V)$ are conjugates in $SSYM(L)$ i.e there exists a $\psi \in SSYM(L)$ such that for any $\gamma \in SAUM(V)$, $\gamma = \psi^{-1}\alpha\psi$ where $\alpha \in SAUM(U)$. Then, $H_S(U) \succsim H_S(V)$ if and only if $x\delta \otimes y\gamma = (x\beta \oplus y)\delta\ \forall\ x, y \in L,\ \beta \in SAUM(U)$ and some $\delta, \gamma \in SAUM(V)$. Hence:*

1. *$\gamma \in SAUM(U)$ if and only if $(I, \gamma, \delta) \in SAUT(V)$.*
2. *if U is a initial S-loop, then;*
   (a) *$\mathcal{L}_{e\delta} \in SAUM(V)$.*
   (b) *$\beta \in SAUM(V)$ if and only if $\mathcal{R}_{e\gamma} \in SAUM(V)$.*

   where e is the identity element in U and $\mathcal{L}_x$, $\mathcal{R}_x$ are respectively the left and right translations mappings of $x \in V$.
3. *if $\delta = I$, then $|SAUM(U)| = |SAUM(V)| = 3$ and so $SAUM(U)$ and $SAUM(V)$ are boolean groups.*
4. *if $\gamma = I$, then $|SAUM(U)| = |SAUM(V)| = 1$.*

Proof
Let $H_S(L, \oplus) = (H_S, \circ)$ and $H_S(L, \otimes) = (H_S, \odot)$. $H_S(U) \succsim H_S(V)$ if and only if there exists a bijection $\phi\ :\ H_S(U) \to H_S(V)$ such that $[(\alpha, x)\circ(\beta, y)]\phi = (\alpha, x)\phi\odot(\beta, y)\phi$ and $(H', \oplus) \overset{\phi}{\cong} (H'', \otimes)$ where $H' = L'\times SAUM(U)$ and $H'' = L''\times SAUM(V)$, $(L', \oplus)$ and $(L'', \otimes)$ been the initial S-subquasigroups of U and V. Define $(\alpha, x)\phi = (\psi^{-1}\alpha\psi, x\psi^{-1}\alpha\psi)\ \forall\ (\alpha, x) \in (H_S, \circ)$ where $\psi \in SSYM(L)$.

$H_S(U) \cong H_S(V) \Leftrightarrow (\alpha\beta, x\beta \oplus y)\phi = (\psi^{-1}\alpha\psi, x\psi^{-1}\alpha\psi) \odot (\psi^{-1}\beta\psi, y\psi^{-1}\beta\psi) \Leftrightarrow (\psi^{-1}\alpha\beta\psi, (x\beta \oplus y)\psi^{-1}\alpha\beta\psi) = (\psi^{-1}\alpha\beta\psi, x\psi^{-1}\alpha\beta\psi \otimes y\psi^{-1}\beta\psi) \Leftrightarrow (x\beta \oplus y)\psi^{-1}\alpha\beta\psi = x\psi^{-1}\alpha\beta\psi \otimes y\psi^{-1}\beta\psi \Leftrightarrow x\delta \otimes y\gamma = (x\beta \oplus y)\delta$ where $\delta = \psi^{-1}\alpha\beta\psi$, $\gamma = \psi^{-1}\beta\psi$.

Note that, $\gamma\mathcal{L}_{x\delta} = L_{x\beta}\delta$ and $\delta\mathcal{R}_{y\gamma} = \beta R_y\delta\ \forall\ x, y \in L$. So, when U is an S-loop, $\gamma\mathcal{L}_{e\delta} = \delta$ and $\delta\mathcal{R}_{e\gamma} = \beta\delta$. These can easily be used to prove the remaining part of the theorem.

Theorem 3.2 *Let $\mathfrak{F}$ be any class of variety of S-quasigroups(loops). Let $U = (L, \oplus)$ and $V = (L, \otimes)$ be initial S-quasigroups(S-loops) that are S-isotopic under the triple of the form $(\delta^{-1}\beta, \gamma^{-1}, \delta^{-1})$ for all $\beta \in SAUM(U)$ and some $\delta, \gamma \in SAUM(V)$ such that their SAGs are non-trivial and are conjugates in $SSYM(L)$ i.e there exists a $\psi \in SSYM(L)$ such that for any $\gamma \in SAUM(V)$, $\gamma = \psi^{-1}\alpha\psi$ where $\alpha \in SAUM(U)$. Then, $U \in \mathfrak{F}$ if and only if $V \in \mathfrak{F}$.*

Proof
By Theorem 3.1, $H_S(U) \cong H_S(V)$. Let $U \in \mathfrak{F}$, then since $H(U)$ has an initial S-subquasigroup(S-subloop) that is isomorphic to U and that initial S-subquasigroup(S-subloop) is isomorphic to an S-subquasigroup(S-subloop) of $H(V)$ which is isomorphic to V, $V \in \mathfrak{F}$. The proof for the converse is similar.

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