Complex Continued Fractions , Numbers , Sequences , Number Pi

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Abstract

In this note we recall some formulas related with continued fractions , numbers , sequences and the constant pi.

Introduction: Two Examples

Example 1:

$$z = x + i\,y = \cfrac{i}{1+\cfrac{i}{1+\cfrac{i}{1+\dots}}} = \frac{1}{2}\left(\sqrt{\frac{\sqrt{17}+1}{2}}-1\right)+\frac{i}{2}\sqrt{\frac{\sqrt{17}-1}{2}} \quad , i=\sqrt{-1}$$

$$z_{n+1} = \frac{i}{1+z_n} \quad , z_1 = 0 \Rightarrow z_n \to z = x + i\,y$$

$$\pi = 2\sum_{n=1}^{\infty}\frac{1}{n}\left((-1)^{n-1}\operatorname{Im}\left(z^n\right)-\operatorname{Im}\left((1-z)^n\right)\right)$$

$$\pi = 2\sum_{n=1}^{\infty}\frac{1}{n}\left(2(-1)^{n-1}\operatorname{Im}\left(z^n\right)-\operatorname{Im}\left((1+z)^{-n}\right)\right)$$

Example 2:

$$z = u + iv = \cfrac{i}{i+\cfrac{i}{i+\cfrac{i}{i+\dots}}} = \frac{1}{2}\sqrt{\frac{\sqrt{17}-1}{2}}+\frac{i}{2}\left(\sqrt{\frac{\sqrt{17}+1}{2}}-1\right)$$

$$z_{n+1} = \frac{i}{i+z_n} \quad , z_1 = 0 \Rightarrow z_n \to z = u + iv$$

$$\pi = -4\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left((1-z)^n\left(1+(1+i)^{-n}\right)\right)$$

Formulas

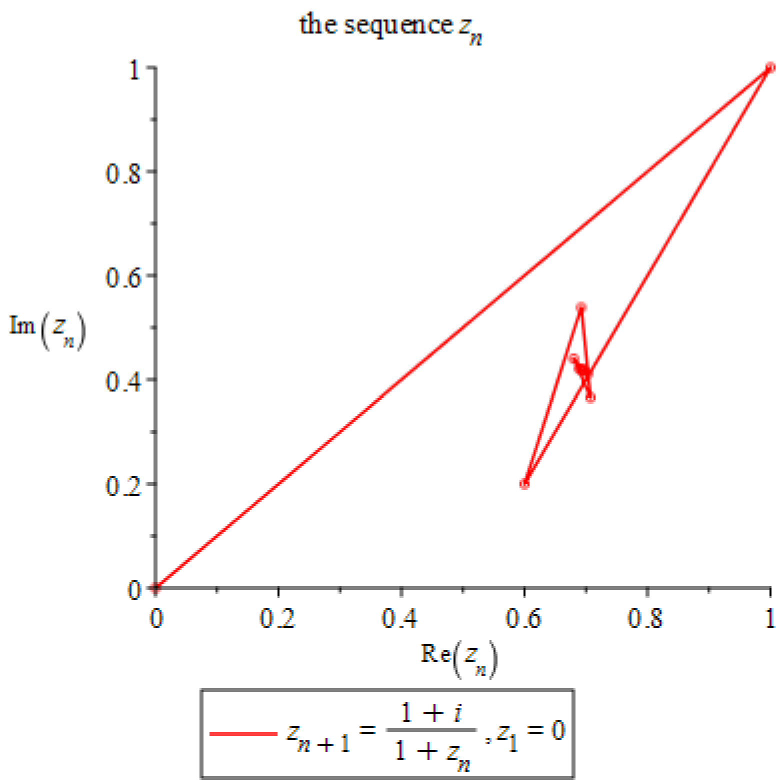
$$z = x + iy = \cfrac{1+i}{1+\cfrac{1+i}{1+\cfrac{1+i}{1+\ldots}}} \tag{1}$$

$$z = x + iy = \frac{1}{2}\left(\sqrt{\frac{\sqrt{41}+5}{2}} - 1\right) + \frac{i}{2}\sqrt{\frac{\sqrt{41}-5}{2}} \tag{2}$$

$$z_{n+1} = \frac{1+i}{1+z_n} \quad , z_1 = 0 \Rightarrow \lim_{n\to\infty} z_n = z \tag{3}$$

$$\pi = 4\sum_{n=1}^{\infty} \frac{1}{n}\left(1+2^{-n}\right) f_n \tag{4}$$

$$f_{n+2} = 2(1-x)\, f_{n+1} - \left((1-x)^2 + y^2\right) f_n \quad , f_1 = y, f_2 = 2(1-x)\, y \tag{5}$$



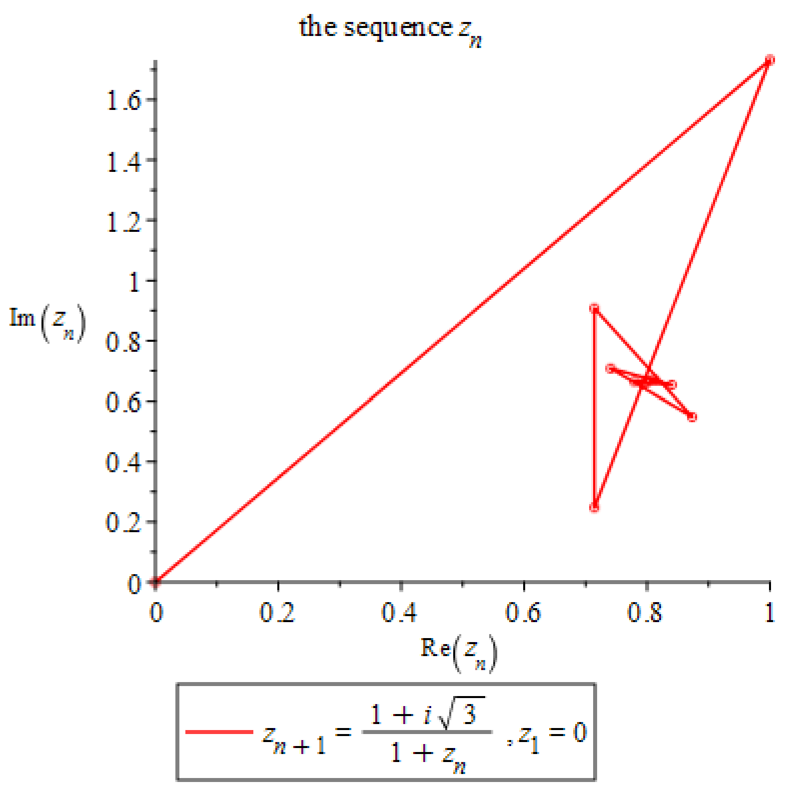
$$z = x + iy \frac{1+i\sqrt{3}}{1+\frac{1+i\sqrt{3}}{1+\frac{1+i\sqrt{3}}{1+\dots}}} \tag{6}$$

$$z = x + iy = \frac{1}{2}\left(\sqrt{\frac{\sqrt{73}+5}{2}} - 1\right) + \frac{i}{2}\sqrt{\frac{\sqrt{73}-5}{2}} \tag{7}$$

$$z_{n+1} = \frac{1+i\sqrt{3}}{1+z_n} \quad , z_1 = 0 \Rightarrow \lim_{n\to\infty} z_n = z \tag{8}$$

$$\pi = 3\sum_{n=1}^{\infty} \frac{1}{n}\left(1+2^{-n}\right) f_n \tag{9}$$

$$f_{n+2} = 2(1-x) f_{n+1} - \left((1-x)^2 + y^2\right) f_n \quad , f_1 = y, f_2 = 2(1-x)y \tag{10}$$



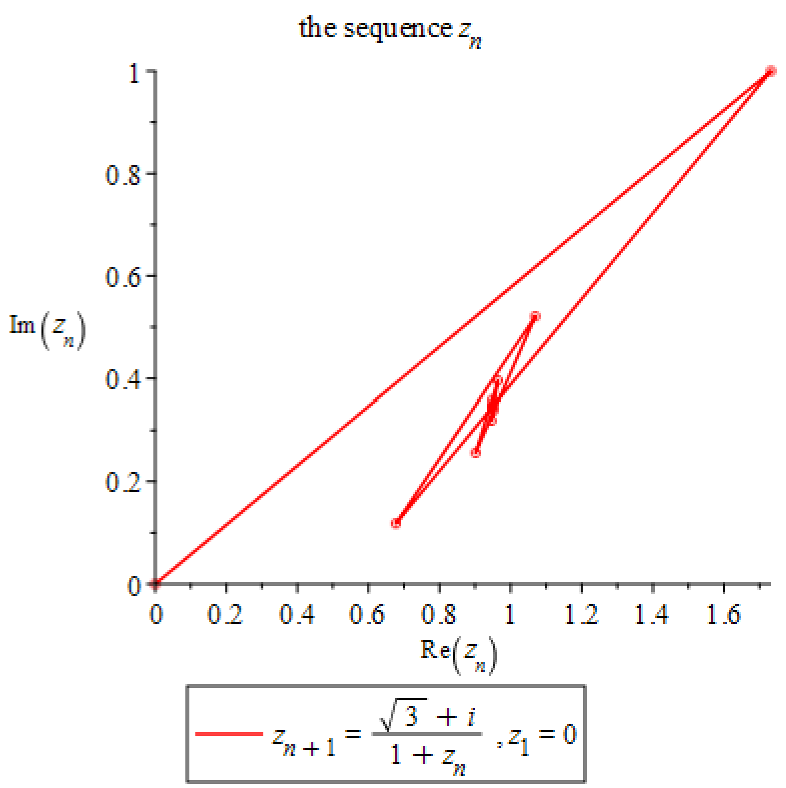
$$z = x + iy = \cfrac{\sqrt{3}+i}{1+\cfrac{\sqrt{3}+i}{1+\cfrac{\sqrt{3}+i}{1+\ldots}}} \tag{11}$$

$$z = x + iy = \frac{1}{2}\left(\sqrt{\frac{\sqrt{65+8\sqrt{3}}+4\sqrt{3}+1}{2}} - 1\right) + \frac{i}{2}\sqrt{\frac{\sqrt{65+8\sqrt{3}}-4\sqrt{3}-1}{2}} \tag{12}$$

$$z_{n+1} = \frac{\sqrt{3}+i}{1+z_n} \quad , z_1 = 0 \Rightarrow \lim_{n\to\infty} z_n = z \tag{13}$$

$$\pi = 6\sum_{n=1}^{\infty}\frac{1}{n}\left(1+2^{-n}\right) f_n \tag{14}$$

$$f_{n+2} = 2(1-x)\, f_{n+1} - \left((1-x)^2 + y^2\right) f_n \quad , f_1 = y, f_2 = 2(1-x)\, y \tag{15}$$



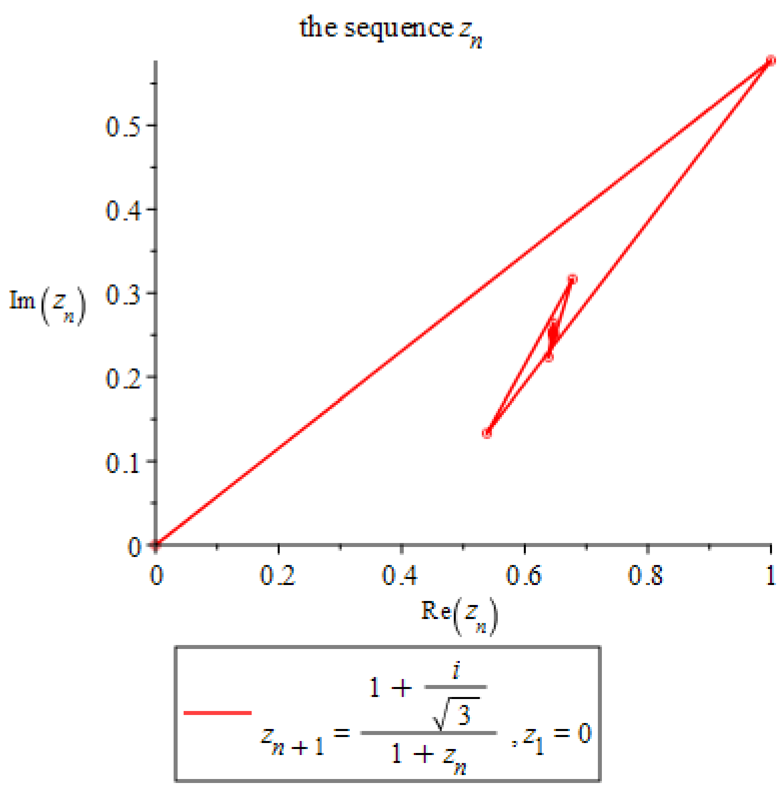
$$z = x + iy = \cfrac{\sqrt{3}+i}{\sqrt{3}+\cfrac{\sqrt{3}+i}{\sqrt{3}+\cfrac{\sqrt{3}+i}{\sqrt{3}+\ldots}}} \tag{16}$$

$$z = x + iy = \frac{1}{2}\left(\frac{1}{\sqrt[4]{3}}\sqrt{\frac{\sqrt{91}+5\sqrt{3}}{2}}-1\right)+\frac{i}{2\sqrt[4]{3}}\sqrt{\frac{\sqrt{91}-5\sqrt{3}}{2}} \tag{17}$$

$$z_{n+1} = \frac{\sqrt{3}+i}{\sqrt{3}\left(1+z_n\right)} \quad , z_1 = 0 \Rightarrow \lim_{n\to\infty} z_n = z \tag{18}$$

$$\pi = 6\sum_{n=1}^{\infty}\frac{1}{n}\left(1+2^{-n}\right) f_n \tag{19}$$

$$f_{n+2} = 2\left(1-x\right) f_{n+1} - \left(\left(1-x\right)^2 + y^2\right) f_n \quad , f_1 = y, \, f_2 = 2\left(1-x\right) y \tag{20}$$



$$z = x + iy = (1+i)e^{-(1+i)e^{-(1+i)e^{-(1+i)\cdots}}} \tag{21}$$

$$z_{n+1} = (1+i)e^{-z_n} \quad , z_1 = 0 \Rightarrow \lim_{n\to\infty} z_n = z \tag{22}$$

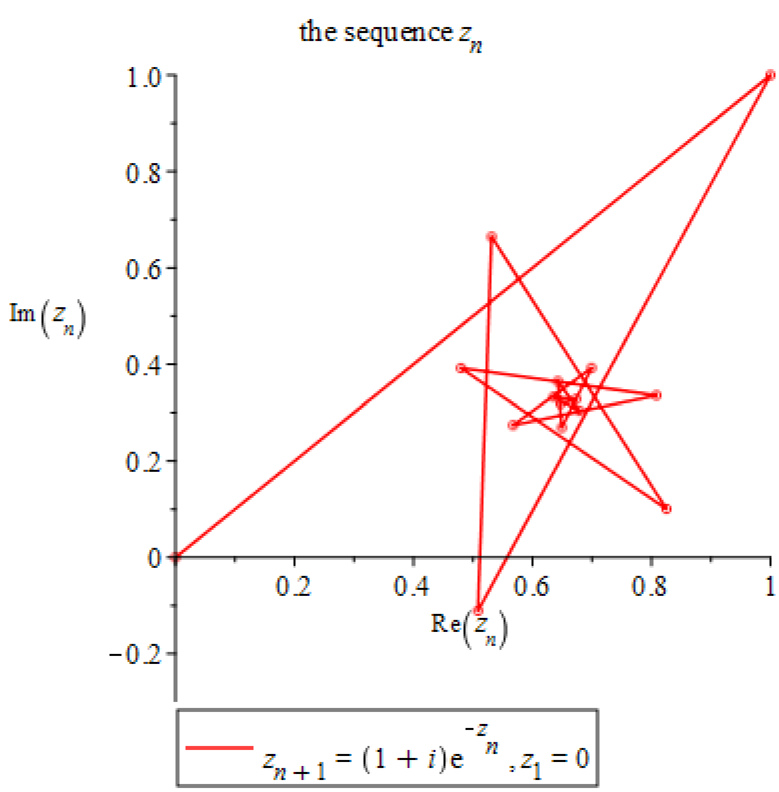
$$\begin{cases} x_{n+1} = e^{-x_n}(\cos y_n + \sin y_n) \\ y_{n+1} = e^{-x_n}(\cos y_n - \sin y_n) \end{cases} , x_1 = y_1 = 0 \Rightarrow \begin{cases} \lim\limits_{n\to\infty} x_n = x \\ \lim\limits_{n\to\infty} y_n = y \end{cases} \tag{23}$$

$$\pi = 4y + 4\tan^{-1}(y/x) \tag{24}$$

$$\pi = 4y - 4\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Im}\left((1-z)^n\right) \tag{25}$$

$$z = W(1+i) \quad , W(x) \text{ is the Lambert function} \tag{26}$$

$$z = 0.656966\ldots + i\,0.325450\ldots \tag{27}$$



$$z = x + iy = \left(1 + i\sqrt{3}\right) e^{-\left(1+i\sqrt{3}\right) e^{-\left(1+i\sqrt{3}\right)\cdots}} \tag{28}$$

$$z_{n+1} = \left(1 + i\sqrt{3}\right) e^{-z_n} \quad , z_1 = 0 \Rightarrow \lim_{n\to\infty} z_n = z \tag{29}$$

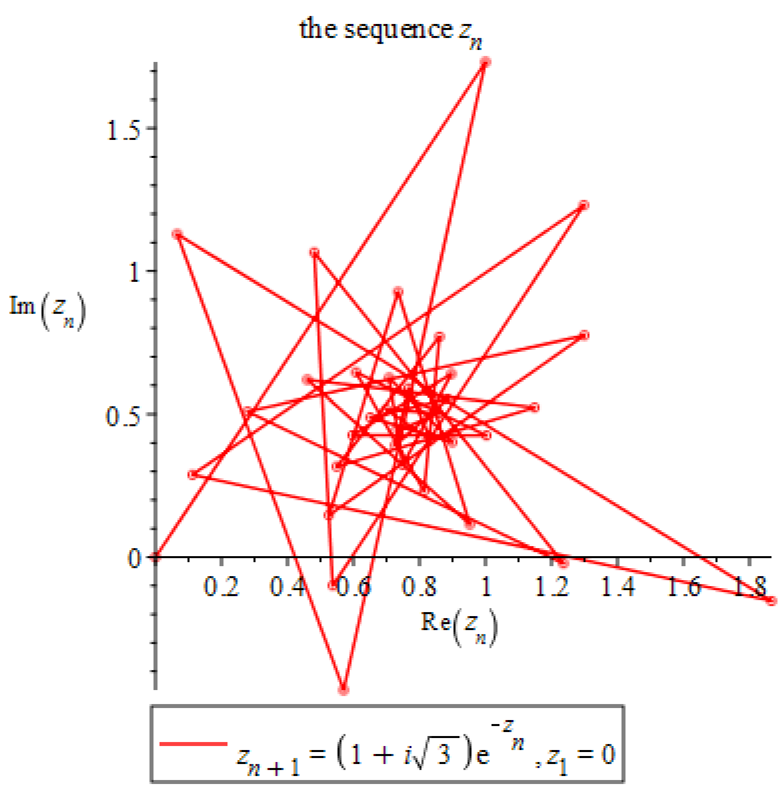
$$\begin{cases} x_{n+1} = e^{-x_n}\left(\cos y_n + \sqrt{3}\sin y_n\right) \\ y_{n+1} = e^{-x_n}\left(\sqrt{3}\cos y_n - \sin y_n\right) \end{cases} , x_1 = y_1 = 0 \Rightarrow \begin{cases} \lim_{n\to\infty} x_n = x \\ \lim_{n\to\infty} y_n = y \end{cases} \tag{30}$$

$$\pi = 3y + 3\tan^{-1}\left(y/x\right) \tag{31}$$

$$\pi = 3y - 3\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-z\right)^n\right) \tag{32}$$

$$z = W\left(1 + i\sqrt{3}\right) \quad , W\left(x\right) \textit{ is the Lambert function} \tag{33}$$

$$z = 0.778283\ldots + i\,0.487555\ldots \tag{34}$$



$$z = x + iy = \left(\sqrt{3}+i\right)e^{-\left(\sqrt{3}+i\right)e^{-\left(\sqrt{3}+i\right)\cdots}} \quad (35)$$

$$z_{n+1} = \left(\sqrt{3}+i\right)e^{-z_n} \quad , z_1 = 0 \Rightarrow \lim_{n\to\infty} z_n = z \quad (36)$$

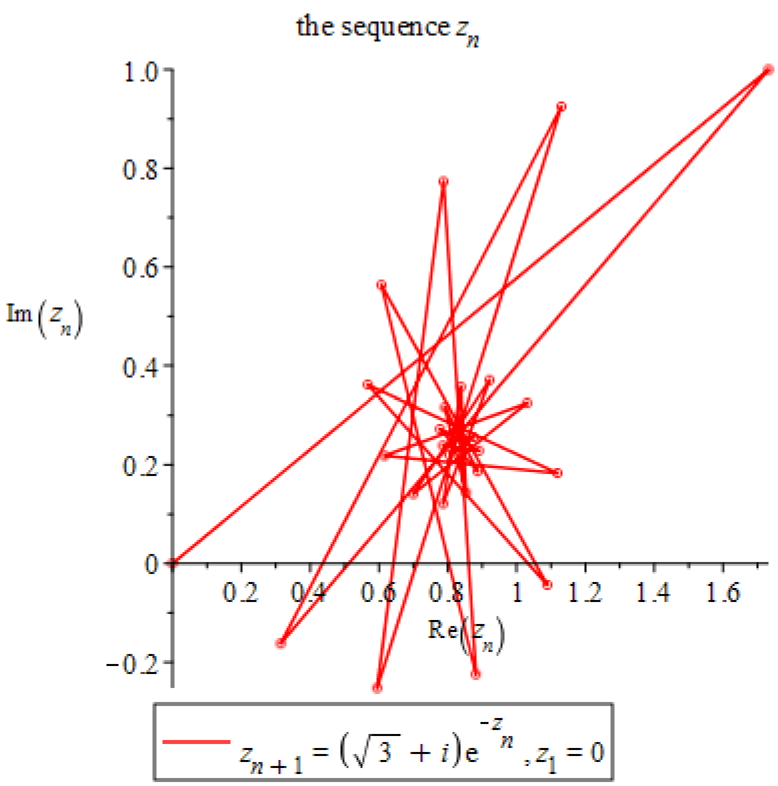
$$\begin{cases} x_{n+1} = e^{-x_n}\left(\sqrt{3}\cos y_n + \sin y_n\right) \\ y_{n+1} = e^{-x_n}\left(\cos y_n - \sqrt{3}\sin y_n\right) \end{cases} , x_1 = y_1 = 0 \Rightarrow \begin{cases} \lim_{n\to\infty} x_n = x \\ \lim_{n\to\infty} y_n = y \end{cases} \quad (37)$$

$$\pi = 6y + 6\tan^{-1}\left(y/x\right) \quad (38)$$

$$\pi = 6y - 6\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left(\left(1-z\right)^n\right) \quad (39)$$

$$z = W\left(\sqrt{3}+i\right) \quad , W\left(x\right) \textit{ is the Lambert function} \quad (40)$$

$$z = 0.834173\ldots + i\,0.241640\ldots \quad (41)$$



$$z = x + iy = \left(1 + \frac{i}{\sqrt{3}}\right) e^{-\left(1+\frac{i}{\sqrt{3}}\right) e^{-\left(1+\frac{i}{\sqrt{3}}\right)\cdots}} \tag{42}$$

$$z_{n+1} = \left(1 + \frac{i}{\sqrt{3}}\right) e^{-z_n} \quad , z_1 = 0 \Rightarrow \lim_{n\to\infty} z_n = z \tag{43}$$

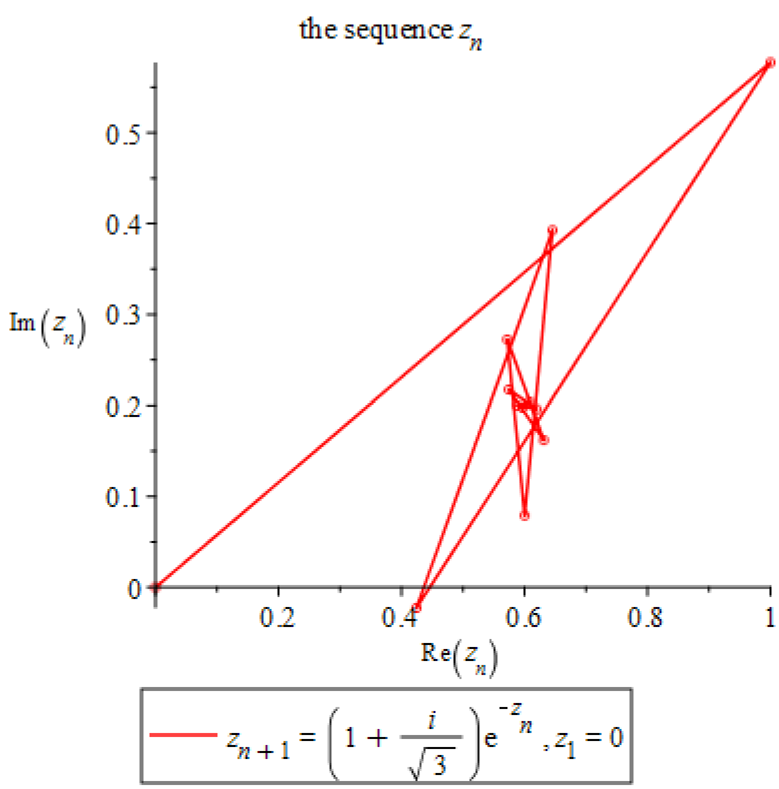
$$\begin{cases} x_{n+1} = e^{-x_n}\left(\cos y_n + \frac{1}{\sqrt{3}}\sin y_n\right) \\ y_{n+1} = e^{-x_n}\left(\frac{1}{\sqrt{3}}\cos y_n - \sin y_n\right) \end{cases} , x_1 = y_1 = 0 \Rightarrow \begin{cases} \lim_{n\to\infty} x_n = x \\ \lim_{n\to\infty} y_n = y \end{cases} \tag{44}$$

$$\pi = 6y + 6\tan^{-1}(y/x) \tag{45}$$

$$\pi = 6y - 6\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{Im}\left((1-z)^n\right) \tag{46}$$

$$z = W\left(1 + \frac{i}{\sqrt{3}}\right) \quad , W(x) \text{ is the Lambert function} \tag{47}$$

$$z = 0.600621\ldots + i\,0.200864\ldots \tag{48}$$



$$z = x + iy = \left(1 + i\left(\sqrt{2} - 1\right)\right) e^{-\left(1+i\left(\sqrt{2}-1\right)\right) e^{-\left(1+i\left(\sqrt{2}-1\right)\right)\cdots}} \tag{49}$$

$$z_{n+1} = \left(1 + i\left(\sqrt{2} - 1\right)\right) e^{-z_n} \quad , z_1 = 0 \Rightarrow \lim_{n\to\infty} z_n = z \tag{50}$$

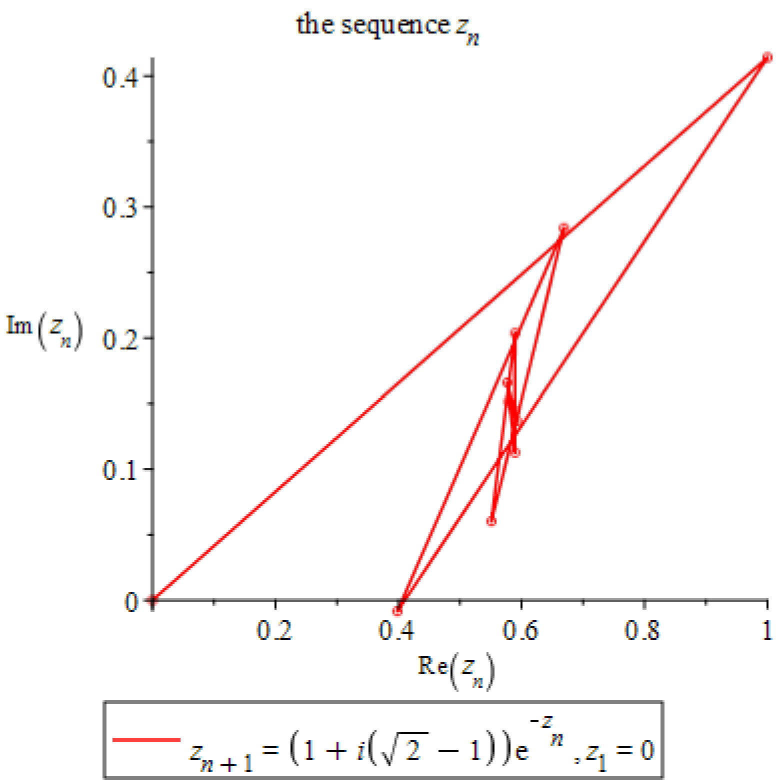
$$\begin{cases} x_{n+1} = e^{-x_n}\left(\cos y_n + \left(\sqrt{2}-1\right)\sin y_n\right) \\ y_{n+1} = e^{-x_n}\left(\left(\sqrt{2}-1\right)\cos y_n - \sin y_n\right) \end{cases} , x_1 = y_1 = 0 \Rightarrow \begin{cases} \lim_{n\to\infty} x_n = x \\ \lim_{n\to\infty} y_n = y \end{cases} \tag{51}$$

$$\pi = 8y + 8\tan^{-1}(y/x) \tag{52}$$

$$\pi = 8y - 8\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Im}\left((1-z)^n\right) \tag{53}$$

$$z = W\left(1 + i\left(\sqrt{2} - 1\right)\right) \quad , W(x) \text{ is the Lambert function} \tag{54}$$

$$z = 0.584916\ldots + i\, 0.146800\ldots \tag{55}$$



$$z = x + iy = \left(1 + i\left(2 - \sqrt{3}\right)\right) e^{-\left(1+i\left(2-\sqrt{3}\right)\right) e^{-\left(1+i\left(2-\sqrt{3}\right)\right)\cdots}} \tag{56}$$

$$z_{n+1} = \left(1 + i\left(2 - \sqrt{3}\right)\right) e^{-z_n} \quad , z_1 = 0 \Rightarrow \lim_{n\to\infty} z_n = z \tag{57}$$

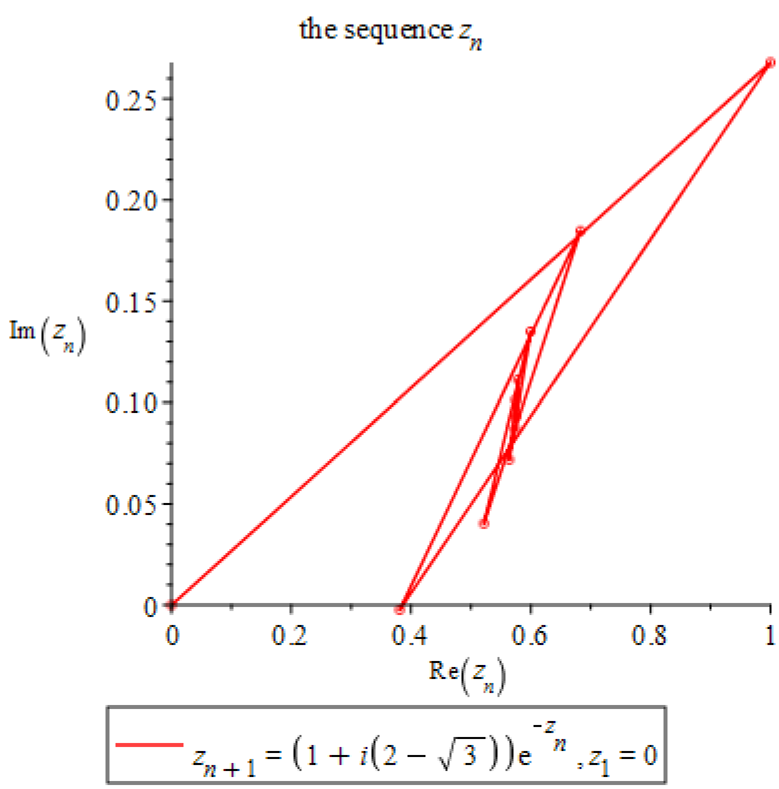
$$\begin{cases} x_{n+1} = e^{-x_n}\left(\cos y_n + \left(2-\sqrt{3}\right)\sin y_n\right) \\ y_{n+1} = e^{-x_n}\left(\left(2-\sqrt{3}\right)\cos y_n - \sin y_n\right) \end{cases} , x_1 = y_1 = 0 \Rightarrow \begin{cases} \lim_{n\to\infty} x_n = x \\ \lim_{n\to\infty} y_n = y \end{cases} \tag{58}$$

$$\pi = 12y + 12\tan^{-1}\left(y/x\right) \tag{59}$$

$$\pi = 12y - 12\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Im}\left(\left(1-z\right)^n\right) \tag{60}$$

$$z = W\left(1 + i\left(2-\sqrt{3}\right)\right) \quad , W\left(x\right) \textit{ is the Lambert function} \tag{61}$$

$$z = 0.574731\ldots + i\,0.096108\ldots \tag{62}$$



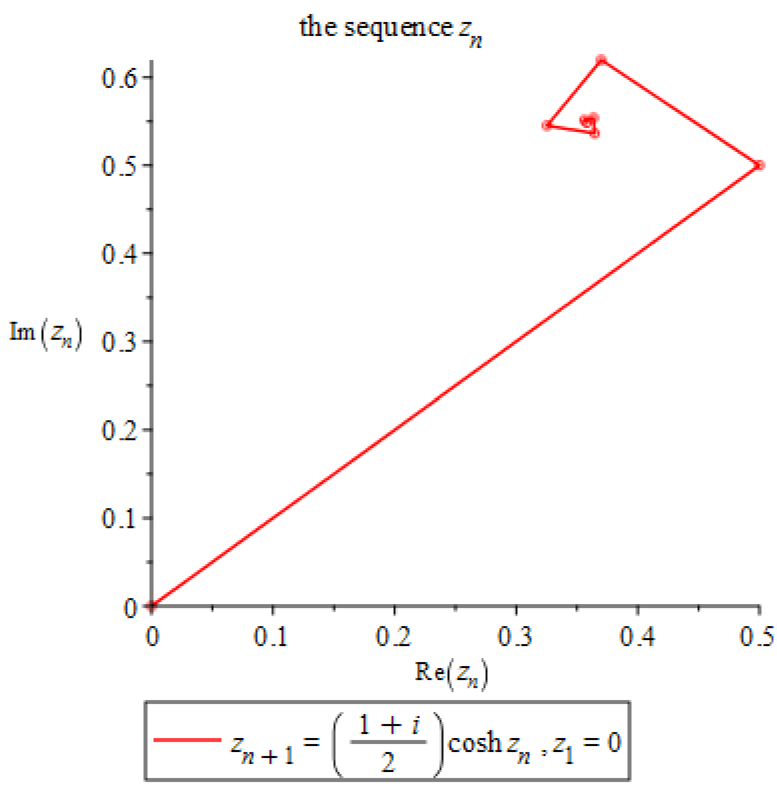
$$z = x + iy = \left(\frac{1+i}{2}\right)\cosh\left(\left(\frac{1+i}{2}\right)\cosh\left(\left(\frac{1+i}{2}\right)\cosh\left(\left(\frac{1+i}{2}\right)\ldots\right)\right)\right) \tag{63}$$

$$z_{n+1} = \left(\frac{1+i}{2}\right)\cosh z_n \quad , z_1 = 0 \Rightarrow \lim_{n\to\infty} z_n = z \tag{64}$$

$$\begin{cases} x_{n+1} = \frac{1}{2}\left(\cosh x_n \cos y_n - \sinh x_n \sin y_n\right) \\ y_{n+1} = \frac{1}{2}\left(\cosh x_n \cos y_n + \sinh x_n \sin y_n\right) \end{cases}, x_1 = y_1 = 0 \Rightarrow \begin{cases} \lim_{n\to\infty} x_n = x \\ \lim_{n\to\infty} y_n = y \end{cases} \tag{65}$$

$$\pi = 4y + 4\tan^{-1}\left(\frac{x}{y}\right) - 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}e^{-2nx}\sin(2ny) \tag{66}$$

$$z = 0.358398\ldots + i\,0.549658\ldots \tag{67}$$



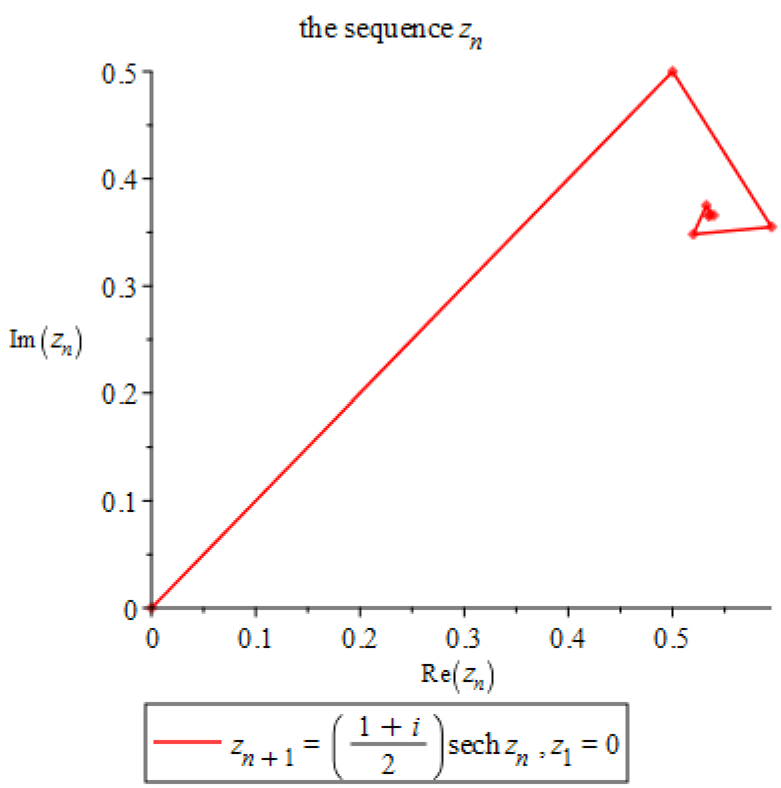
$$z = x + i\,y = \left(\frac{1+i}{2}\right)\operatorname{sech}\left(\left(\frac{1+i}{2}\right)\operatorname{sech}\left(\left(\frac{1+i}{2}\right)\operatorname{sech}\left(\left(\frac{1+i}{2}\right)\ldots\right)\right)\right) \tag{68}$$

$$z_{n+1} = \left(\frac{1+i}{2}\right)\operatorname{sech} z_n \quad , z_1 = 0 \Rightarrow \lim_{n\to\infty} z_n = z \tag{69}$$

$$\begin{cases} x_{n+1} = \dfrac{\cosh x_n \cos y_n + \sinh x_n \sin y_n}{2\left(\left(\cosh x_n\right)^2 - \left(\sin y_n\right)^2\right)} \\ y_{n+1} = \dfrac{\cosh x_n \cos y_n - \sinh x_n \sin y_n}{2\left(\left(\cosh x_n\right)^2 - \left(\sin y_n\right)^2\right)} \end{cases} , x_1 = y_1 = 0 \Rightarrow \begin{cases} \lim\limits_{n\to\infty} x_n = x \\ \lim\limits_{n\to\infty} y_n = y \end{cases} \tag{70}$$

$$\pi = 4y + 4\tan^{-1}\left(\frac{y}{x}\right) - 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} e^{-2nx}\sin(2ny) \tag{71}$$

$$z = 0.535711\ldots + i\;0.366271\ldots \tag{72}$$



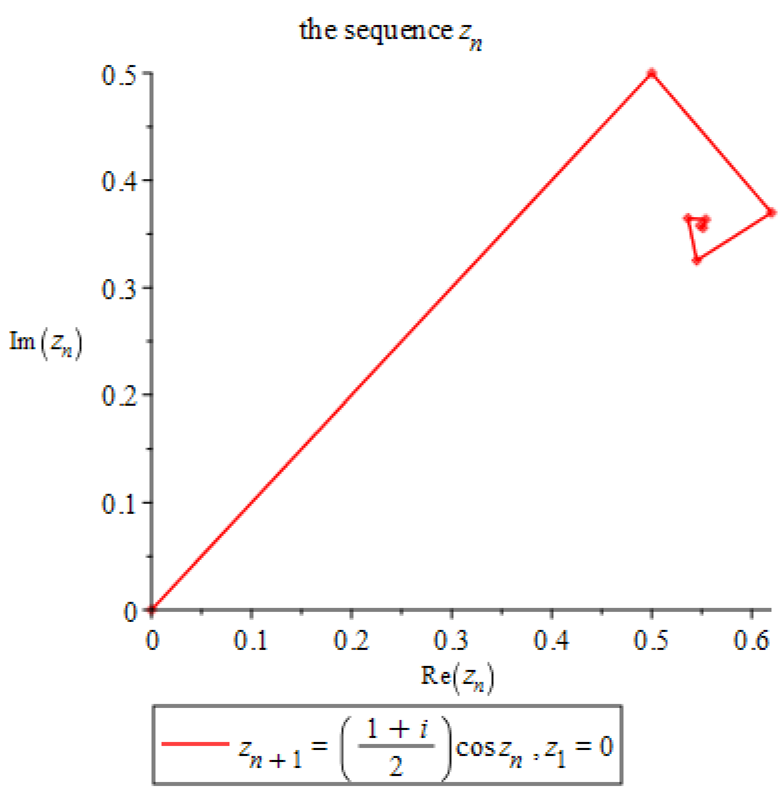
$$z = x + iy = \left(\frac{1+i}{2}\right)\cos\left(\left(\frac{1+i}{2}\right)\cos\left(\left(\frac{1+i}{2}\right)\cos\left(\left(\frac{1+i}{2}\right)\ldots\right)\right)\right) \tag{73}$$

$$z_{n+1} = \left(\frac{1+i}{2}\right)\cos z_n \quad , z_1 = 0 \Rightarrow \lim_{n\to\infty} z_n = z \tag{74}$$

$$\begin{cases} x_{n+1} = \frac{1}{2}\left(\cos x_n \cosh y_n + \sin x_n \sinh y_n\right) \\ y_{n+1} = \frac{1}{2}\left(\cos x_n \cosh y_n - \sin x_n \sinh y_n\right) \end{cases} , x_1 = y_1 = 0 \Rightarrow \begin{cases} \lim_{n\to\infty} x_n = x \\ \lim_{n\to\infty} y_n = y \end{cases} \tag{75}$$

$$\pi = 4\tan^{-1}\left(\frac{y}{x}\right) + 4\tan^{-1}\left(\frac{\sin x \sinh y}{\cos x \cosh y}\right) \tag{76}$$

$$z = 0.549658\ldots + i\,0.358398\ldots \tag{77}$$



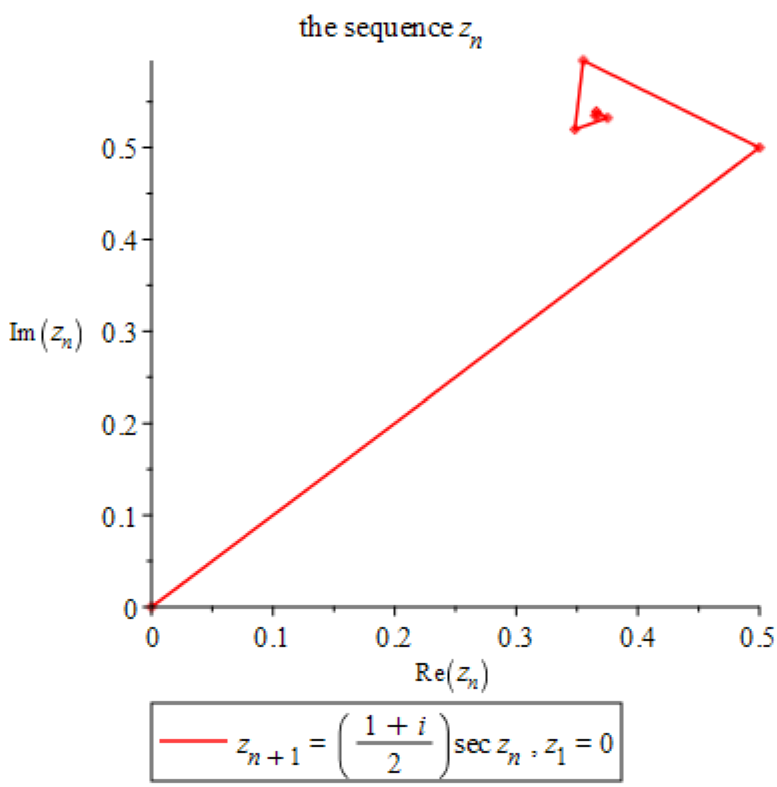
$$z = x + iy = \left(\frac{1+i}{2}\right)\sec\left(\left(\frac{1+i}{2}\right)\sec\left(\left(\frac{1+i}{2}\right)\sec\left(\left(\frac{1+i}{2}\right)\ldots\right)\right)\right) \quad (78)$$

$$z_{n+1} = \left(\frac{1+i}{2}\right)\sec z_n \quad , z_1 = 0 \Rightarrow \lim_{n\to\infty} z_n = z \quad (79)$$

$$\begin{cases} x_{n+1} = \dfrac{\cos x_n \cosh y_n - \sin x_n \sinh y_n}{2\left((\cosh y_n)^2 - (\sin x_n)^2\right)} \\ y_{n+1} = \dfrac{\cos x_n \cosh y_n + \sin x_n \sinh y_n}{2\left((\cosh y_n)^2 - (\sin x_n)^2\right)} \end{cases} , x_1 = y_1 = 0 \Rightarrow \begin{cases} \lim_{n\to\infty} x_n = x \\ \lim_{n\to\infty} y_n = y \end{cases} \quad (80)$$

$$\pi = 4\tan^{-1}\left(\frac{x}{y}\right) + 4\tan^{-1}\left(\frac{\sin x \sinh y}{\cos x \cosh y}\right) \quad (81)$$

$$z = 0.366271\ldots + i\,0.535711\ldots \quad (82)$$



References

1. M. Abramowitz and I.A. Stegun. Handbook of Mathematical Functions. Dover Publications, New York , 1970.
2. D.H. Bailey and J.M. Borwein. Experimental Mathematics: examples , methods and implications. Notices Amer. Math. Soc. , 52(5):502-514,2005.