

Single-Valued Neutrosophic Planar Graphs

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Abstract. We apply the concept of single-valued neutrosophic sets to multigraphs, planar graphs and dual graphs. We introduce the notions of single-valued neutrosophic multigraphs, single-valued neutrosophic planar graphs, and single-valued neutrosophic dual graphs. We illustrate these concepts with examples. We also investigate some of their properties.

1. Introduction

Zadeh [27] introduced the concept of fuzzy set. Atanassov [11] introduced the intuitionistic fuzzy sets which is a generalization of fuzzy sets. Fuzzy set theory and intuitionistic fuzzy set theory are useful models for dealing with uncertainty and incomplete information. But they may not be sufficient in modeling of indeterminate and inconsistent information encountered in real world. In order to cope with this issue, neutrosophic set theory was proposed by Smarandache [20] as a generalization of fuzzy sets and intuitionistic fuzzy sets. However, since neutrosophic sets are identified by three functions called truth-membership (T), indeterminacy-membership (I) and falsity-membership (F) whose values are real standard or non-standard subset of unit interval $]0^-, 1^+[$.

There are some difficulties in modeling of some problems in engineering and sciences. To overcome these difficulties, in 2010, the concept of single-valued neutrosophic sets and its operations were defined by Wang et al. [22] as a generalization of intuitionistic fuzzy sets. Yang *et al* [23] introduced the concept of single-valued neutrosophic relation based on single-valued neutrosophic set. Ye [25] introduced a multicriteria decision making method using aggregation operators.

Rosenfeld [18] described some properties of fuzzy graphs. Later on, Bhattacharya [12] worked on fuzzy graphs. Mordeson and Nair [16] discussed some operations on fuzzy graphs. Abdul-jabbar *et al.* [1] introduced the concept of a fuzzy dual graph and discussed some of its interesting properties. Samanta and Pal [17, 19] introduced and investigated the concept of fuzzy planar graphs and studied several properties. On other hand, Alshehri and Akram [10] introduced the concept of intuitionistic fuzzy planar graphs. Akram et al. [5] discussed the concept of bipolar fuzzy planar graphs. Dhavaseelan et al. [14] defined strong neutrosophic graphs. Broumi et al. [13] proposed single-valued neutrosophic graphs. Akram and Shahzadi [6] introduced the notions of neutrosophic graphs and neutrosophic soft graphs. They also presented application of neutrosophic soft graphs. Representation of graphs using intuitionistic

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neutrosophic soft sets was discussed in [7]. Akram et al. [8] introduced the notion of single-valued neutrosophic hypergraphs. In this article, we apply the concept of single-valued neutrosophic sets to multigraphs, planar graphs and dual graphs. We introduce the notions of single-valued neutrosophic multigraphs, single-valued neutrosophic planar graphs, single-valued neutrosophic dual graphs. We present some of their interesting properties. This paper is a continuation of Alshehri and Akram's work [10].

2. Preliminaries

Let X be a nonempty set. A *fuzzy set* [27] A drawn from X is defined as $A = \{ \langle x : \mu_A(x) \rangle : x \in X \}$, where $\mu : X \rightarrow [0, 1]$ is the membership function of the fuzzy set A . A *fuzzy binary relation* [28] on X is a fuzzy subset μ on $X \times X$. By a fuzzy relation, we mean a fuzzy binary relation given by $\mu : X \times X \rightarrow [0, 1]$. A *fuzzy graph* [15] $G = (V, \sigma, \mu)$ is a non-empty set V together with a pair of functions $\sigma : V \rightarrow [0, 1]$ and $\mu : V \times V \rightarrow [0, 1]$ such that for all $x, y \in V$, $\mu(x, y) \leq \min(\sigma(x), \sigma(y))$, where $\sigma(x)$ and $\mu(x, y)$ represent the membership values of the vertex x and of the edge (x, y) in G , respectively. A loop at a vertex x in a fuzzy graph is represented by $\mu(x, x) \neq 0$. An edge is non-trivial if $\mu(x, y) \neq 0$. Let G be a fuzzy graph and for a certain geometric representation, the graph has only one crossing between two fuzzy edges $((w, x), \mu(w, x))$ and $((y, z), \mu(y, z))$. If $\mu(w, x) = 1$ and $\mu(y, z) = 0$, then we say that the fuzzy graph has no crossing. Similarly, if $\mu(w, x)$ has value near to 1 and $\mu(y, z)$ has value near to 0, the crossing will not be important for the planarity. If $\mu(y, z)$ has value near to 1 and $\mu(w, x)$ has value near to 1, then the crossing becomes very important for the planarity. Let X be a nonempty set. A *fuzzy multiset* [24] A drawn from X is characterized by a function, 'count membership' of A denoted by CM_A such that $CM_A : X \rightarrow Q$, where Q is the set of all crisp multisets drawn from the unit interval $[0, 1]$. Then for any $x \in X$, the value $CM_A(x)$ is a crisp multiset drawn from $[0, 1]$. For each $x \in X$, the membership sequence is defined as the decreasingly ordered sequence of elements in $CM_A(x)$. It is denoted by $(\mu_A^1(x), \mu_A^2(x), \mu_A^3(x), \dots, \mu_A^p(x))$ where $\mu_A^1(x) \geq \mu_A^2(x) \geq \mu_A^3(x) \geq \dots \geq \mu_A^p(x)$.

Let V be a non-empty set and $\sigma : V \rightarrow [0, 1]$ be a mapping and let $\mu = \{((x, y), \mu(x, y)_j), j = 1, 2, \dots, p_{xy} | (x, y) \in V \times V\}$ be a fuzzy multi-set of $V \times V$ such that $\mu(x, y)_j \leq \min\{\sigma(x), \sigma(y)\}$ for all $j = 1, 2, \dots, p_{xy}$, where $p_{xy} = \max\{j | \mu(x, y)_j \neq 0\}$. Then $G = (V, \sigma, \mu)$ is denoted as *fuzzy multigraph* [17] where $\sigma(x)$ and $\mu(x, y)_j$ represent the membership value of the vertex x and the membership value of the edge (x, y) in G , respectively.

Definition 2.1. [20] *Let X be a space of points (objects). A neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$. The functions $T_A(x)$, $I_A(x)$, and $F_A(x)$ are real standard or non-standard subsets of $]0^-, 1^+]$. That is, $T_A(x) : X \rightarrow]0^-, 1^+]$, $I_A(x) : X \rightarrow]0^-, 1^+]$ and $F_A(x) : X \rightarrow]0^-, 1^+]$ and $0^- \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$.*

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]0^-, 1^+]$. In real life applications in scientific and engineering problems, it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]0^-, 1^+]$. To apply neutrosophic sets in real-life problems more conveniently, Wang et al. [22] defined single-valued neutrosophic sets which takes the value from the subset of $[0, 1]$.

Definition 2.2. [26] *Let X be a nonempty set with generic elements in X denoted by x . A single valued neutrosophic multiset A drawn from X is characterized by the three functions: count truth-membership of CT_A , count indeterminacy-membership of CI_A , and count falsity-membership of CF_A such that $CT_A(x) : X \rightarrow R$, $CI_A(x) : X \rightarrow R$, $CF_A(x) : X \rightarrow R$ for $x \in X$, where R is the set of all real number multisets in the real unit interval $[0, 1]$. Then, a single valued neutrosophic multiset A is denoted by*

$$A = \{ \langle x, (T_A^1(x), T_A^2(x), \dots, T_A^q(x)), (I_A^1(x), I_A^2(x), \dots, I_A^q(x)), (F_A^1(x), F_A^2(x), \dots, F_A^q(x)) \rangle | x \in X \}$$

where the truth-membership sequence $(T_A^1(x), T_A^2(x), \dots, T_A^q(x))$, the indeterminacy-membership sequence $(I_A^1(x), I_A^2(x), \dots, I_A^q(x))$, and the falsity-membership sequence $(F_A^1(x), F_A^2(x), \dots, F_A^q(x))$ may be in decreasing or increasing order, and sum of $T_A^i(x), I_A^i(x), F_A^i(x) \in [0, 1]$ satisfies the condition $0 \leq \sup T_A^i(x) + \sup I_A^i(x) + \sup F_A^i(x) \leq 3$

for $x \in X$ and $i = 1, 2, \dots, q$. For convenience, a single valued neutrosophic multiset A can be denoted by the simplified form:

$$A = \{\langle x, T_A(x)_i, I_A(x)_i, F_A(x)_i \rangle | x \in X, i = 1, 2, \dots, q\}.$$

Definition 2.3. [26] Let $A = \{\langle x, T_A(x)_i, I_A(x)_i, F_A(x)_i \rangle | x \in X, i = 1, 2, \dots, q\}$ and $B = \{\langle x, T_B(x)_i, I_B(x)_i, F_B(x)_i \rangle | x \in X, i = 1, 2, \dots, q\}$ be two single-valued neutrosophic multisets in X . Then, there are the following operations:

- (1) $A \subseteq B$ if and only if $T_A(x)_i \leq T_B(x)_i, I_A(x)_i \geq I_B(x)_i, F_A(x)_i \geq F_B(x)_i$ for $i = 1, 2, 3, \dots, q$ and $x \in X$;
- (2) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$,
- (3) $A^c = \{\langle x, F_A(x)_i, 1 - I_A(x)_i, T_A(x)_i \rangle | x \in X, i = 1, 2, \dots, q\}$,
- (4) $A \cup B = \{\langle x, T_A(x)_i \vee T_B(x)_i, I_A(x)_i \wedge I_B(x)_i, F_A(x)_i \wedge F_B(x)_i \rangle | x \in X, i = 1, 2, \dots, q\}$,
- (5) $A \cap B = \{\langle x, T_A(x)_i \wedge T_B(x)_i, I_A(x)_i \vee I_B(x)_i, F_A(x)_i \vee F_B(x)_i \rangle | x \in X, i = 1, 2, \dots, q\}$.

Akram and Shahzadi [6] introduced notion of single-valued neutrosophic soft graphs. Akram and Shahzadi [9] highlighted some flaws in the definitions of Broumi et al. [13] and Shah-Hussain [21].

Definition 2.4. [6, 9] A single-valued neutrosophic graph is a pair $G = (A, B)$, where $A : V \rightarrow [0, 1]$ is single-valued neutrosophic set in V and $B : V \times V \rightarrow [0, 1]$ is single-valued neutrosophic relation on V such that

$$T_B(xy) \leq \min\{T_A(x), T_A(y)\},$$

$$I_B(xy) \leq \min\{I_A(x), I_A(y)\},$$

$$F_B(xy) \leq \max\{F_A(x), F_A(y)\}$$

for all $x, y \in V$.

3. Single-valued Neutrosophic planar graphs

We first introduce the notion of a single-valued neutrosophic multigraph using the concept of a single-valued neutrosophic multiset.

Definition 3.1. Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic set on V and let

$$B = \{\langle xy, T_B(xy)_i, I_B(xy)_i, F_B(xy)_i \rangle, i = 1, 2, \dots, m | xy \in V \times V\}$$

be a single-valued neutrosophic multiset of $V \times V$ such that

$$T_B(xy)_i \leq \min\{T_A(x), T_A(y)\},$$

$$I_B(xy)_i \leq \min\{I_A(x), I_A(y)\},$$

$$F_B(xy)_i \leq \max\{F_A(x), F_A(y)\}$$

for all $i = 1, 2, \dots, m$. Then $G = (A, B)$ is called a single-valued neutrosophic multigraph.

Note that there may be more than one edge between the vertices x and y . $T_B(xy)_i, I_B(xy)_i, F_B(xy)_i$ represent truth-membership value, indeterminacy-membership value and falsity-membership value of the edge xy in G , respectively. m denotes the number of edges between the vertices. In single-valued neutrosophic multigraph G , B is said to be single-valued neutrosophic multiedge set.

Example 3.2. Consider a multigraph $G^* = (V, E)$ such that $V = \{a, b, c, d\}$, $E = \{ab, ab, ab, bc, bd\}$. Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic set on V and $B = (T_B, I_B, F_B)$ be a single-valued neutrosophic multiedge set on $V \times V$ defined in Table 1 and Table 2.

Table 1: Single-valued neutrosophic set A

A	a	b	c	d
T_A	0.5	0.4	0.5	0.4
I_A	0.3	0.2	0.4	0.3
F_A	0.3	0.4	0.3	0.4

Table 2: single-valued neutrosophic multiedge set B

B	ab	ab	ab	bc	bd
T_B	0.2	0.1	0.2	0.3	0.1
I_B	0.2	0.1	0.2	0.1	0.2
F_B	0.2	0	0.2	0.3	0.2

By routine calculations, it is easy to see from Fig. 1 that it is a single-valued neutrosophic multigraph.

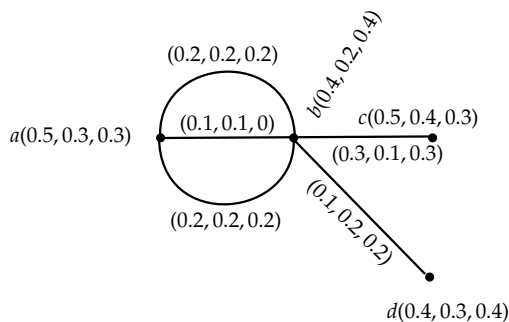


Figure 1: Single-valued neutrosophic multigraph.

Definition 3.3. Let $B = \{(xy, T_B(xy)_i, I_B(xy)_i, F_B(xy)_i), i = 1, 2, \dots, m | xy \in V \times V\}$ be a single-valued neutrosophic multiedge set in single-valued neutrosophic multigraph G . The degree of a vertex $x \in V$ is denoted by $\text{deg}(x)$ and is defined by $\text{deg}(x) = (\sum_{i=1}^m T_B(xy)_i, \sum_{i=1}^m I_B(xy)_i, \sum_{i=1}^m F_B(xy)_i)$ for all $y \in V$.

Example 3.4. In Example 3.2, the degree of vertices a, b, c, d are $\text{deg}(a) = (0.5, 0.5, 0.4)$, $\text{deg}(b) = (0.9, 0.8, 0.9)$, $\text{deg}(c) = (0.3, 0.1, 0.3)$ and $\text{deg}(d) = (0.1, 0.2, 0.2)$.

Definition 3.5. Let $B = \{(xy, T_B(xy)_i, I_B(xy)_i, F_B(xy)_i), i = 1, 2, \dots, m | xy \in V \times V\}$ be a single-valued neutrosophic multiedge set in single-valued neutrosophic multigraph G . A multiedge xy of G is strong if

$$\frac{1}{2} \min\{T_A(x), T_A(y)\} \leq T_B(xy)_i,$$

$$\frac{1}{2} \min\{I_A(x), I_A(y)\} \leq I_B(xy)_i,$$

$$\frac{1}{2} \max\{F_A(x), F_A(y)\} \geq F_B(xy)_i,$$

for all $i = 1, 2, \dots, m$.

Definition 3.6. Let $B = \{(xy, T_B(xy)_i, I_B(xy)_i, F_B(xy)_i), i = 1, 2, \dots, m | xy \in V \times V\}$ be a single-valued neutrosophic multiedge set in single-valued neutrosophic multigraph G . An single-valued neutrosophic multigraph G is complete if

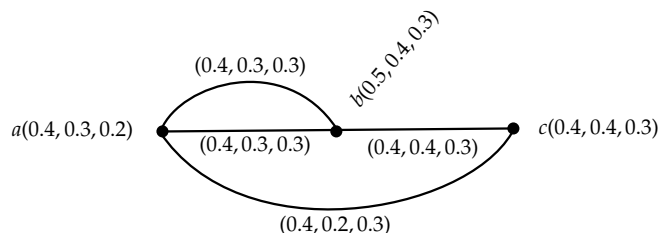
$$\min\{T_A(x), T_A(y)\} = T_B(xy)_i,$$

$$\min\{I_A(x), I_A(y)\} = I_B(xy)_i,$$

$$\max\{F_A(x), F_A(y)\} = F_B(xy)_i$$

for all $i = 1, 2, \dots, m$ and for all $x, y \in V$.

Example 3.7. Consider a single-valued neutrosophic multigraph G as shown in Fig. ?? . By routine calculations, it is easy to see that Fig. ?? is a single-valued neutrosophic complete multigraph.



Suppose that geometric insight for single-valued neutrosophic graphs has only one crossing between single valued neutrosophic edges $(ab, T_B(ab)_i, I_B(ab)_i, F_B(ab)_i)$ and $(cd, T_B(cd)_i, I_B(cd)_i, F_B(cd)_i)$. We note that:

- If $(ab, T_B(ab)_i, I_B(ab)_i, F_B(ab)_i) = (1, 1, 1)$ and $(cd, T_B(cd)_i, I_B(cd)_i, F_B(cd)_i) = (0, 0, 0)$ or $(ab, T_B(ab)_i, I_B(ab)_i, F_B(ab)_i) = (0, 0, 0)$, $(cd, T_B(cd)_i, I_B(cd)_i, F_B(cd)_i) = (1, 1, 1)$, then single-valued neutrosophic graph has no crossing,
- If $(ab, T_B(ab)_i, I_B(ab)_i, F_B(ab)_i) = (1, 1, 1)$ and $(cd, T_B(cd)_i, I_B(cd)_i, F_B(cd)_i) = (1, 1, 1)$, then there exists a crossing for the representation of the graph.

Definition 3.8. The strength of the single-valued neutrosophic edge ab can be measured by the value

$$S_{ab} = ((S_T)_{ab}, (S_I)_{ab}, (S_F)_{ab}) = \left(\frac{T_B(ab)_i}{\min(T_A(a), T_A(b))}, \frac{I_B(ab)_i}{\min(I_A(a), I_A(b))}, \frac{F_B(ab)_i}{\max(F_A(a), F_A(b))} \right).$$

Definition 3.9. Let G be a single-valued neutrosophic multigraph. An edge ab is said to be a single-valued neutrosophic strong if $(S_T)_{ab} \geq 0.5$, $(S_I)_{ab} \geq 0.5$, $(S_F)_{ab} \geq 0.5$, otherwise, we call weak edge.

Definition 3.10. Let $G = (A, B)$ be a single-valued neutrosophic multigraph such that B contains two edges $(ab, T_B(ab)_i, I_B(ab)_i, F_B(ab)_i)$ and $(cd, T_B(cd)_j, I_B(cd)_j, F_B(cd)_j)$, intersected at a point P , where i and j are fixed integers. We define the intersecting value at the point P by

$$S_P = ((S_T)_P, (S_I)_P, (S_F)_P) = \left(\frac{(S_T)_{ab} + (S_T)_{cd}}{2}, \frac{(S_I)_{ab} + (S_I)_{cd}}{2}, \frac{(S_F)_{ab} + (S_F)_{cd}}{2} \right).$$

If the number of point of intersections in a single-valued neutrosophic multigraph increases, planarity decreases. Thus for single-valued neutrosophic multigraph, S_P is inversely proportional to the planarity. We now introduce the concept of a single-valued neutrosophic planar graph.

Definition 3.11. Let G be a single-valued neutrosophic multigraph and P_1, P_2, \dots, P_z be the points of intersection between the edges for a certain geometrical representation, G is said to be a single-valued neutrosophic planar graph with single-valued neutrosophic planarity value $f = (f_T, f_I, f_F)$, where

$$f = (f_T, f_I, f_F) = \left(\frac{1}{1 + \{(S_T)_{P_1} + (S_T)_{P_2} + \dots + (S_T)_{P_z}\}}, \frac{1}{1 + \{(S_I)_{P_1} + (S_I)_{P_2} + \dots + (S_I)_{P_z}\}}, \frac{1}{1 + \{(S_F)_{P_1} + (S_F)_{P_2} + \dots + (S_F)_{P_z}\}} \right).$$

Clearly, $f = (f_T, f_I, f_F)$ is bounded and $0 < f_T \leq 1, 0 < f_I \leq 1, 0 < f_F \leq 1$.

If there is no point of intersection for a certain geometrical representation of a single-valued neutrosophic planar graph, then its single-valued neutrosophic planarity value is $(1, 1, 1)$. In this case, the underlying crisp graph of this single-valued neutrosophic graph is the crisp planar graph. If f_T and f_I decrease and f_F increases, then the number of points of intersection between the edges increases and decreases, respectively, and the nature of planarity decreases and increases, respectively. We conclude that every single-valued neutrosophic graph is a single-valued neutrosophic planar graph with certain single-valued neutrosophic planarity value.

Example 3.12. Consider a multigraph $G^* = (V, E)$ such that $V = \{a, b, c, d, e\}$,

$$E = \{ab, ac, ad, ad, bc, bd, cd, ce, ae, de, be\}.$$

Let $A = (T_A, I_A, F_A)$ be a single-valued neutrosophic set of V and let $B = (T_B, I_B, F_B)$ be a single-valued neutrosophic multiedge set of $V \times V$ defined in Table 3 and Table 4.

Table 3: single-valued neutrosophic set A

A	a	b	c	d	e
T_A	0.5	0.4	0.3	0.6	0.6
I_A	0.5	0.4	0.3	0.6	0.6
F_A	0.2	0.1	0.1	0.2	0.1

Table 4: single-valued neutrosophic multiedge set B

B	ab	ac	ad	ad	bc	bd	cd	ae	ce	de	be
T_B	0.2	0.2	0.2	0.3	0.2	0.2	0.2	0.2	0.2	0.2	0.2
I_B	0.2	0.2	0.2	0.3	0.2	0.2	0.2	0.2	0.2	0.2	0.2
F_B	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1

The single-valued neutrosophic multigraph as shown in Fig. 2 has two point of intersections P_1 and P_2 . P_1 is a point between the edges $(ad, 0.2, 0.2, 0.1)$ and $(bc, 0.2, 0.2, 0.1)$ and P_2 is between $(ad, 0.3, 0.3, 0.1)$ and $(bc, 0.2, 0.2, 0.1)$. For the edge $(ad, 0.2, 0.2, 0.1)$, $S_{ad} = (0.4, 0.4, 0.5)$, For the edge $(ad, 0.3, 0.3, 0.1)$, $S_{ad} = (0.6, 0.6, 0.5)$ and for the edge $(bc, 0.2, 0.2, 0.1)$, $S_{bc} = (0.6667, 0.6667, 1)$.

For the first point of intersection P_1 , intersecting value S_{P_1} is $(0.5334, 0.5334, 0.75)$ and that for the second point of intersection P_2 , $S_{P_2} = (0.63335, 0.63335, 0.75)$. Therefore, the single-valued neutrosophic planarity value for the single-valued neutrosophic multigraph shown in Fig. 2 is $(0.461, 0.461, 0.4)$.

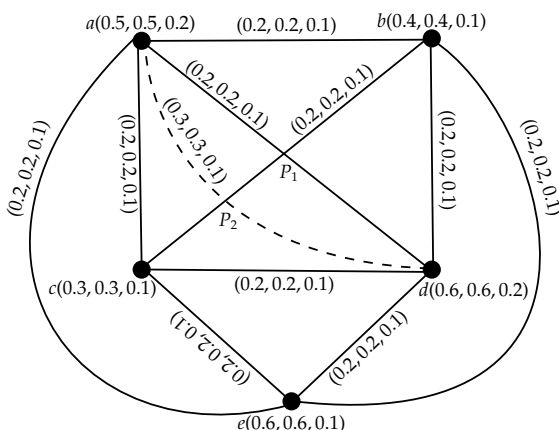


Figure 2: Single-valued neutrosophic planar graph

Theorem 3.13. Let G be a single-valued neutrosophic complete multigraph. The planarity value, $f = (f_T, f_I, f_F)$ of G is given by $f_T = \frac{1}{1+n_p}$, $f_I = \frac{1}{1+n_p}$ and $f_F = \frac{1}{1+n_p}$ such that $f_T + f_I + f_F \leq 3$, where n_p is the number of point of intersections between the edges in G

Definition 3.14. An single-valued neutrosophic planar graph G is called strong single-valued neutrosophic planar graph if the single-valued neutrosophic planarity value $f = (f_T, f_I, f_F)$ of the graph is $f_T \geq 0.5$, $f_I \geq 0.5$, $f_F \leq 0.5$.

Theorem 3.15. Let G be a strong single-valued neutrosophic planar graph. The number of point of intersections between strong edges in G is at most one.

Proof. Let G be a strong single-valued neutrosophic planar graph. Assume that G has at least two point of intersections P_1 and P_2 between two strong edges in G . For any strong edge $(ab, T_B(ab)_i, I_B(ab)_i, F_B(ab)_i)$,

$$T_B(ab)_i \geq \frac{1}{2} \min\{T_A(a), T_A(b)\}, I_B(ab)_i \geq \frac{1}{2} \min\{I_A(a), I_A(b)\}, F_B(ab)_i \leq \frac{1}{2} \max\{F_A(a), F_A(b)\}.$$

This shows that $(S_T)_{ab} \geq 0.5, (S_I)_{ab} \geq 0.5, (S_F)_{ab} \leq 0.5$. Thus for two intersecting strong edges $(ab, T_B(ab)_i, I_B(ab)_i, F_B(ab)_i)$ and $(cd, T_B(cd)_j, I_B(cd)_j, F_B(cd)_j)$,

$$\frac{(S_T)_{ab} + (S_T)_{cd}}{2} \geq 0.5, \frac{(S_I)_{ab} + (S_I)_{cd}}{2} \geq 0.5, \frac{(S_F)_{ab} + (S_F)_{cd}}{2} \leq 0.5,$$

that is, $(S_T)_{P_1} \geq 0.5, (S_I)_{P_1} \geq 0.5, (S_F)_{P_1} \leq 0.5$. Similarly, $(S_T)_{P_2} \geq 0.5, (S_I)_{P_2} \geq 0.5, (S_F)_{P_2} \leq 0.5$. This implies that $1 + (S_T)_{P_1} + (S_T)_{P_2} \geq 2, 1 + (S_I)_{P_1} + (S_I)_{P_2} \geq 2, 1 + (S_F)_{P_1} + (S_F)_{P_2} \leq 2$. Therefore, $f_T = \frac{1}{1+(S_T)_{P_1}+(S_T)_{P_2}} \leq 0.5, f_I = \frac{1}{1+(S_I)_{P_1}+(S_I)_{P_2}} \leq 0.5, f_F = \frac{1}{1+(S_F)_{P_1}+(S_F)_{P_2}} \geq 0.5$. It contradicts the fact that the single-valued neutrosophic graph is a strong single-valued neutrosophic planar graph. Thus number of point of intersections between strong edges can not be two. Obviously, if the number of point of intersections of strong single-valued neutrosophic edges increases, the single-valued neutrosophic planarity value decreases. Similarly, if the number of point of intersection of strong edges is one, then the single-valued neutrosophic planarity value $f_T > 0.5, f_I > 0.5, f_F > 0.5$. Any single-valued neutrosophic planar graph without any crossing between edges is a strong single-valued neutrosophic planar graph. Thus, we conclude that the maximum number of point of intersections between the strong edges in G is one. \square

Face of a single-valued neutrosophic planar graph is an important parameter. Face of a single-valued neutrosophic graph is a region bounded by single-valued neutrosophic edges. Every single-valued neutrosophic face is characterized by single-valued neutrosophic edges in its boundary. If all the edges in the boundary of a single-valued neutrosophic face have T, I, F values $(1, 1, 1)$ and $(0, 0, 0)$, respectively, it becomes crisp face. If one of such edges is removed or has T, I and F values $(0, 0, 0)$ and $(1, 1, 1)$, respectively, the single-valued neutrosophic face does not exist. So the existence of a single-valued neutrosophic face depends on the minimum value of strength of single-valued neutrosophic edges in its boundary. A single-valued neutrosophic face and its T, I and F values of a single-valued neutrosophic graph are defined below.

Definition 3.16. Let G be a single-valued neutrosophic planar graph and $B = \{(xy, T_B(xy)_i, I_B(xy)_i, F_B(xy)_i), i = 1, 2, \dots, m | xy \in V \times V\}$. A single-valued neutrosophic face of G is a region, bounded by the set of single-valued neutrosophic edges $E' \subset E$, of a geometric representation of G . The membership and nonmembership values of the single-valued neutrosophic face are:

$$\begin{aligned} & \min \left\{ \frac{T_B(xy)_i}{\min\{T_A(x), T_A(y)\}}, i = 1, 2, \dots, m | xy \in E' \right\}, \\ & \min \left\{ \frac{I_B(xy)_i}{\min\{I_A(x), I_A(y)\}}, i = 1, 2, \dots, m | xy \in E' \right\}, \\ & \max \left\{ \frac{F_B(xy)_i}{\max\{F_A(x), F_A(y)\}}, i = 1, 2, \dots, m | xy \in E' \right\}. \end{aligned}$$

Definition 3.17. An single-valued neutrosophic face is called strong single-valued neutrosophic face if its true-membership value is greater than 0.5, indeterminacy value is greater than 0.5, false-membership value is lesser than 0.5, and weak face otherwise. Every single-valued neutrosophic planar graph has an infinite region which is called outer single-valued neutrosophic face. Other faces are called inner single-valued neutrosophic faces.

Example 3.18. Consider a single-valued neutrosophic planar graph as shown in Fig. 3. The single-valued neutrosophic planar graph has the following faces:

- single-valued neutrosophic face F_1 is bounded by the edges

$$(v_1v_2, 0.5, 0.5, 0.1), (v_2v_3, 0.6, 0.6, 0.1), (v_1v_3, 0.5, 0.5, 0.1).$$

- outer single-valued neutrosophic face F_2 surrounded by edges

$$(v_1v_3, 0.5, 0.5, 0.1), (v_1v_4, 0.5, 0.5, 0.1), (v_2v_4, 0.6, 0.6, 0.1), (v_2v_3, 0.6, 0.6, 0.1),$$

- single-valued neutrosophic face F_3 is bounded by the edges

$$(v_1v_2, 0.5, 0.5, 0.1), (v_2v_4, 0.6, 0.6, 0.1), (v_1v_4, 0.5, 0.5, 0.1).$$

Clearly, the truth-membership value, indeterminacy-membership value and falsity-membership value of a single-valued neutrosophic face F_1 are 0.833, 0.833 and 0.333, respectively. The truth-membership value, indeterminacy-membership value and falsity-membership value of a single-valued neutrosophic face F_3 are also 0.833, 0.833 and 0.333, respectively. Thus F_1 and F_3 are strong single-valued neutrosophic faces.

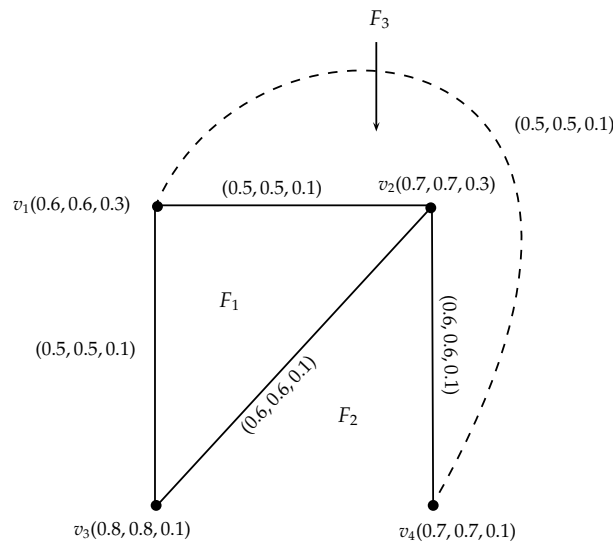


Figure 3: Faces in single-valued neutrosophic planar graph

We now introduce dual of single-valued neutrosophic planar graph. In single-valued neutrosophic dual graph, vertices are corresponding to the strong single-valued neutrosophic faces of the single-valued neutrosophic planar graph and each single-valued neutrosophic edge between two vertices is corresponding to each edge in the boundary between two faces of single-valued neutrosophic planar graph. The formal definition is given below.

Definition 3.19. Let G be a single-valued neutrosophic planar graph and let

$$B = \{(xy, T_B(xy)_i, I_B(xy)_i, F_B(xy)_i), i = 1, 2, \dots, m | xy \in V \times V\}.$$

Let F_1, F_2, \dots, F_k be the strong single-valued neutrosophic faces of G . The single-valued neutrosophic dual graph of G is a single-valued neutrosophic planar graph $G' = (V', A', B')$, where $V' = \{x_i, i = 1, 2, \dots, k\}$, and the vertex x_i of G' is considered for the face F_i of G . The true-truth- membership, indeterminacy and false-truth- membership values of vertices are given by the mapping $A' = (T_{A'}, I_{A'}, F_{A'}) : V' \rightarrow [0, 1] \times [0, 1] \times [0, 1]$ such that $T_{A'}(x_i) = \max\{T_{B'}(uv)_i, i = 1, 2, \dots, p | uv \text{ is an edge of the boundary of the strong single-valued neutrosophic face } F_i\}$, $I_{A'}(x_i) = \max\{I_{B'}(uv)_i, i = 1, 2, \dots, p | uv \text{ is an edge of the boundary of the strong single-valued neutrosophic face } F_i\}$, $F_{A'}(x_i) = \min\{F_{B'}(uv)_i, i = 1, 2, \dots, p | uv \text{ is an edge of the boundary of the strong single-valued neutrosophic face } F_i\}$.

There may exist more than one common edges between two faces F_i and F_j of G . Thus there may be more than one edges between two vertices x_i and x_j in single-valued neutrosophic dual graph G' . Let $T_B^l(x_i x_j)$ denote the truth- membership value of the l -th edge between x_i and x_j , and $F_B^l(x_i x_j)$ denote the falsity-membership value of the l -th edge between x_i and x_j .

The truth- membership, indeterminacy-membership and falsity-membership values of the single-valued neutrosophic edges of the single-valued neutrosophic dual graph are given by $T_{B'}(x_i x_j)_l = T_B^l(uv)_j, I_{B'}(x_i x_j)_l = I_B^l(uv)_j, F_{B'}(x_i x_j)_l = F_B^l(uv)_j$ where $(uv)_i$ is an edge in the boundary between two strong single-valued neutrosophic faces F_i and F_j and $l = 1, 2, \dots, s$, where s is the number of common edges in the boundary between F_i and F_j or the number of edges between x_i and x_j .

If there be any strong pendant edge in the single-valued neutrosophic planar graph, then there will be a self loop in G' corresponding to this pendant edge. The edge truth- membership, indeterminacy-membership and falsity-membership value of the self loop is equal to the truth- membership, indeterminacy-membership and falsity-membership value of the pendant edge.

Single-valued neutrosophic dual graph of single-valued neutrosophic planar graph does not contain point of intersection of edges for a certain representation, so it is single-valued neutrosophic planar graph with planarity value $(1, 1, 1)$. Thus the single-valued neutrosophic face of single-valued neutrosophic dual graph can be similarly described as in single-valued neutrosophic planar graphs.

Example 3.20. Consider a single-valued neutrosophic planar graph $G = (V, A, B)$ as shown in Fig. 4 such that $V = \{a, b, c, d\}$, $A = (a, 0.6, 0.6, 0.2), (b, 0.7, 0.7, 0.2), (c, 0.8, 0.8, 0.2), (d, 0.9, 0.9, 0.1)$, and

$B = \{(ab, 0.5, 0.5, 0.01), (ac, 0.4, 0.4, 0.01), (ad, 0.55, 0.55, 0.01), (bc, 0.45, 0.45, 0.01), (bc, 0.6, 0.6, 0.01), (cd, 0.7, 0.7, 0.01)\}$.

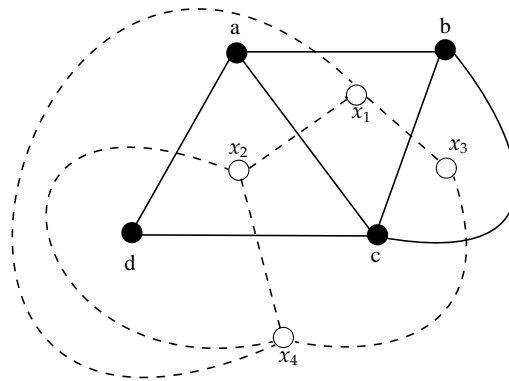


Figure 4: Single-valued neutrosophic dual graph

The single-valued neutrosophic planar graph has the following faces:

- single-valued neutrosophic face F_1 is bounded by $(ab, 0.5, 0.5, 0.01), (ac, 0.4, 0.4, 0.01), (bc, 0.45, 0.45, 0.01)$,
- single-valued neutrosophic face F_2 is bounded by $(ad, 0.55, 0.55, 0.01), (cd, 0.7, 0.7, 0.01), (ac, 0.4, 0.4, 0.01)$,
- single-valued neutrosophic face F_3 is bounded by $(bc, 0.45, 0.45, 0.01), (bc, 0.6, 0.6, 0.01)$ and
- outer single-valued neutrosophic face F_4 is surrounded by
 $(ab, 0.5, 0.5, 0.01), (bc, 0.6, 0.6, 0.01), (cd, 0.7, 0.7, 0.01), (ad, 0.55, 0.55, 0.01)$.

Routine calculations show that all faces are strong single-valued neutrosophic faces. For each strong single-valued neutrosophic face, we consider a vertex for the single-valued neutrosophic dual graph. So the vertex set $V' = \{x_1, x_2, x_3, x_4\}$, where the vertex x_i is taken corresponding to the strong single-valued neutrosophic face $F_i, i = 1, 2, 3, 4$. Thus

$$\begin{aligned} T_{A'}(x_1) &= \max\{0.5, 0.4, 0.45\} = 0.5, T_{A'}(x_2) = \max\{0.55, 0.7, 0.4\} = 0.7, \\ I_{A'}(x_1) &= \max\{0.5, 0.4, 0.45\} = 0.5, I_{A'}(x_2) = \max\{0.55, 0.7, 0.4\} = 0.7, \\ F_{A'}(x_1) &= \min\{0.01, 0.01, 0.01\} = 0.01, F_{A'}(x_2) = \min\{0.01, 0.01, 0.01\} = 0.01, \\ T_{A'}(x_3) &= \max\{0.45, 0.6\} = 0.6, T_{A'}(x_4) = \max\{0.5, 0.6, 0.7, 0.55\} = 0.7, \\ I_{A'}(x_3) &= \max\{0.45, 0.6\} = 0.6, I_{A'}(x_4) = \max\{0.5, 0.6, 0.7, 0.55\} = 0.7, \\ F_{A'}(x_3) &= \min\{0.01, 0.01\} = 0.01, F_{A'}(x_4) = \min\{0.01, 0.01, 0.01, 0.01\} = 0.01. \end{aligned}$$

There are two common edges ad and cd between the faces F_2 and F_4 in G . Hence between the vertices x_2 and x_4 , there exist two edges in the single-valued neutrosophic dual graph of G . Truth-membership, indeterminacy-membership and falsity-membership values of these edges are given by

$$\begin{aligned} T_{B'}(x_2x_4) &= T_B(cd) = 0.7, T_{B'}(x_2x_4) = T_B(ad) = 0.55, I_{B'}(x_2x_4) = I_B(cd) = 0.7, I_{B'}(x_2x_4) = I_B(ad) = 0.55, \\ F_{B'}(x_2x_4) &= F_B(cd) = 0.01, F_{B'}(x_2x_4) = F_B(ad) = 0.01. \end{aligned}$$

The truth-membership, indeterminacy-membership and falsity-membership values of other edges of the single-valued neutrosophic dual graph are calculated as

$$\begin{aligned} T_{B'}(x_1x_3) &= T_B(bc) = 0.45, T_{B'}(x_1x_2) = T_B(ac) = 0.4, T_{B'}(x_1x_4) = T_B(ab) = 0.5, T_{B'}(x_3x_4) = T_B'(bc) = 0.6, \\ I_{B'}(x_1x_3) &= I_B(bc) = 0.45, I_{B'}(x_1x_2) = I_B(ac) = 0.4, I_{B'}(x_1x_4) = I_B(ab) = 0.5, I_{B'}(x_3x_4) = I_B'(bc) = 0.6, \\ F_{B'}(x_1x_3) &= F_B(bc) = 0.01, F_{B'}(x_1x_2) = F_B(ac) = 0.01, F_{B'}(x_1x_4) = F_B(ab) = 0.01, F_{B'}(x_3x_4) = F_B(bc) = 0.01. \end{aligned}$$

Thus the edge set of single-valued neutrosophic dual graph is

$$\begin{aligned} B' &= \{(x_1x_3, 0.45, 0.45, 0.01), (x_1x_2, 0.4, 0.4, 0.01), (x_1x_4, 0.5, 0.5, 0.01), \\ &(x_3x_4, 0.6, 0.6, 0.01), (x_2x_4, 0.7, 0.7, 0.01), (x_2x_4, 0.55, 0.55, 0.01)\}. \end{aligned}$$

In Fig. 4, the single-valued neutrosophic dual graph $G' = (V', A', B')$ of G is drawn by dotted line.

Weak edges in planar graphs are not considered for any calculation in single-valued neutrosophic dual graphs. We state the following Theorems without their proofs.

Theorem 3.21. Let G be a single-valued neutrosophic planar graph whose number of vertices, number of single-valued neutrosophic edges and number of strong faces are denoted by n, p, m , respectively. Let G' be the single-valued neutrosophic dual graph of G . Then:

- the number of vertices of G' is equal to m ,
- number of edges of G' is equal to p ,
- number of single-valued neutrosophic faces of G' is equal to n .

Theorem 3.22. Let $G = (V, A, B)$ be a single-valued neutrosophic planar graph without weak edges and the single-valued neutrosophic dual graph of G be $G' = (V', A', B')$. The truth-membership indeterminacy-membership and falsity-membership values of single-valued neutrosophic edges of G' are equal to truth-membership indeterminacy-membership and falsity-membership values of the single-valued neutrosophic edges of G .

4. Conclusion

A single-valued neutrosophic graph is a generalization of intuitionistic fuzzy graph that is very useful to solve real life problems. In this research article, we have presented single-valued neutrosophic planar graphs. We are extending our work to (1) Bipolar neutrosophic planar graphs, (2) Intuitionistic neutrosophic planar graphs, (3) Interval-valued neutrosophic graphs, and (4) single-valued neutrosophic soft planar graphs.

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