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Neutrosophic subalgebras of BCK/BCI-algebras based on neutrosophic points

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Abstract. Properties on neutrosophic $\in\vee q$-subsets and neutrosophic q-subsets are investigated. Relations between an $(\in, \in\vee q)$-neutrosophic subalgebra and a $(q, \in\vee q)$-neutrosophic subalgebra are considered. Characterization of an $(\in, \in\vee q)$-neutrosophic subalgebra by using neutrosophic $\in$-subsets are discussed. Conditions for an $(\in, \in\vee q)$-neutrosophic subalgebra to be a $(q, \in\vee q)$-neutrosophic subalgebra are provided.

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1. Introduction

The concept of neutrosophic set (NS) developed by Smarandache [4, 5] is a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various part (refer to the site http://fs.gallup.unm.edu/neutrosophy.htm). Jun [2] introduced the notion of neutrosophic subalgebras in BCK/BCI-algebras with several types. He provided characterizations of an $(\in, \in)$-neutrosophic subalgebra and an $(\in, \in\vee q)$-neutrosophic subalgebra. Given special sets, so called neutrosophic $\in$-subsets, neutrosophic q-subsets and neutrosophic $\in\vee q$-subsets, he considered conditions for the neutrosophic $\in$-subsets, neutrosophic q-subsets and neutrosophic $\in\vee q$-subsets to be subalgebras. He discussed conditions for a neutrosophic set to be a $(q, \in\vee q)$-neutrosophic subalgebra.

In this paper, we give relations between an $(\in, \in\vee q)$-neutrosophic subalgebra and a $(q, \in\vee q)$-neutrosophic subalgebra. We discuss characterization of an $(\in, \in\vee q)$-neutrosophic subalgebra by using neutrosophic $\in$-subsets. We provide conditions for an $(\in, \in\vee q)$-neutrosophic subalgebra to be a $(q, \in\vee q)$-neutrosophic subalgebra. We investigate properties on neutrosophic q-subsets and neutrosophic $\in\vee q$-subsets.

2. Preliminaries

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms:

(a1) $((x * y) * (x * z)) * (z * y) = 0$,
(a2) $(x * (x * y)) * y = 0$,
(a3) $x * x = 0$,
(a4) $x * y = y * x = 0 \Rightarrow x = y$,

for all $x, y, z \in X$. If a BCI-algebra X satisfies the axiom

(a5) $0 * x = 0$ for all $x \in X$,

then we say that X is a *BCK-algebra*. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

We refer the reader to the books [1] and [3] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

If $\Lambda = \{1, 2\}$, we will also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee\{a_i \mid i \in \Lambda\}$ and $\bigwedge\{a_i \mid i \in \Lambda\}$, respectively.

Let X be a non-empty set. A neutrosophic set (NS) in X (see [4]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

3. Neutrosophic subalgebras of several types

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we consider the following sets:

$$\begin{aligned}
&T_\in(A;\alpha) := \{x \in X \mid A_T(x) \geq \alpha\},\\
&I_\in(A;\beta) := \{x \in X \mid A_I(x) \geq \beta\},\\
&F_\in(A;\gamma) := \{x \in X \mid A_F(x) \leq \gamma\},\\
&T_q(A;\alpha) := \{x \in X \mid A_T(x) + \alpha > 1\},\\
&I_q(A;\beta) := \{x \in X \mid A_I(x) + \beta > 1\},\\
&F_q(A;\gamma) := \{x \in X \mid A_F(x) + \gamma < 1\},\\
&T_{\in\vee\, q}(A;\alpha) := \{x \in X \mid A_T(x) \geq \alpha \text{ or } A_T(x) + \alpha > 1\},\\
&I_{\in\vee\, q}(A;\beta) := \{x \in X \mid A_I(x) \geq \beta \text{ or } A_I(x) + \beta > 1\},\\
&F_{\in\vee\, q}(A;\gamma) := \{x \in X \mid A_F(x) \leq \gamma \text{ or } A_F(x) + \gamma < 1\}.
\end{aligned}$$

We say $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are *neutrosophic $\in$-subsets*; $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are *neutrosophic q-subsets*; and $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are *neutrosophic $\in\vee\, q$-subsets*. For $\Phi \in \{\in, q, \in\vee\, q\}$, the element of $T_\Phi(A;\alpha)$ (resp., $I_\Phi(A;\beta)$ and $F_\Phi(A;\gamma)$) is called a *neutrosophic T_Φ-point* (resp., *neutrosophic I_Φ-point* and *neutrosophic F_Φ-point*) with value α (resp., β and γ) (see [2]).

It is clear that

$$T_{\in\vee\, q}(A;\alpha) = T_\in(A;\alpha) \cup T_q(A;\alpha), \tag{3.1}$$

$$I_{\in\vee\, q}(A;\beta) = I_\in(A;\beta) \cup I_q(A;\beta), \tag{3.2}$$

$$F_{\in\vee\, q}(A;\gamma) = F_\in(A;\gamma) \cup F_q(A;\gamma). \tag{3.3}$$

Definition 3.1 ([2])**.** Given $\Phi, \Psi \in \{\in, q, \in\vee\, q\}$, a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called a $(\Phi,\, \Psi)$-*neutrosophic subalgebra* of X if the following assertions are valid.

$$\begin{aligned}
&x \in T_\Phi(A;\alpha_x),\ y \in T_\Phi(A;\alpha_y) \ \Rightarrow\ x * y \in T_\Psi(A;\alpha_x \wedge \alpha_y),\\
&x \in I_\Phi(A;\beta_x),\ y \in I_\Phi(A;\beta_y) \ \Rightarrow\ x * y \in I_\Psi(A;\beta_x \wedge \beta_y),\\
&x \in F_\Phi(A;\gamma_x),\ y \in F_\Phi(A;\gamma_y) \ \Rightarrow\ x * y \in F_\Psi(A;\gamma_x \vee \gamma_y)
\end{aligned} \tag{3.4}$$

for all $x, y \in X$, α_x, α_y, $\beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

Lemma 3.2 ([2])**.** *A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is an $(\in,\, \in\vee\, q)$-neutrosophic subalgebra of X if and only if it satisfies:*

$$(\forall x, y \in X)\left(\begin{array}{l} A_T(x*y) \geq \bigwedge\{A_T(x), A_T(y), 0.5\}\\ A_I(x*y) \geq \bigwedge\{A_I(x), A_I(y), 0.5\}\\ A_F(x*y) \leq \bigvee\{A_F(x), A_F(y), 0.5\} \end{array}\right). \tag{3.5}$$

Theorem 3.3. *A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is an $(\in,\, \in\vee\, q)$-neutrosophic subalgebra of X if and only if the neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$.*

Proof. Assume that $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic subalgebra of X. For any $x, y \in X$, let $\alpha \in (0, 0.5]$ be such that $x, y \in T_\in(A;\alpha)$. Then $A_T(x) \geq \alpha$ and $A_T(y) \geq \alpha$. It follows from (3.5) that

$$A_T(x * y) \geq \bigwedge\{A_T(x), A_T(y), 0.5\} \geq \alpha \wedge 0.5 = \alpha$$

and so that $x * y \in T_\in(A;\alpha)$. Thus $T_\in(A;\alpha)$ is a subalgebra of X for all $\alpha \in (0, 0.5]$. Similarly, $I_\in(A;\beta)$ is a subalgebra of X for all $\beta \in (0, 0.5]$. Now, let $\gamma \in [0.5, 1)$ be such that $x, y \in F_\in(A;\gamma)$. Then $A_F(x) \leq \gamma$ and $A_F(y) \leq \gamma$. Hence

$$A_F(x * y) \leq \bigvee\{A_F x), A_F(y), 0.5\} \leq \gamma \vee 0.5 = \gamma$$

by (3.5), and so $x * y \in F_\in(A;\gamma)$. Thus $F_\in(A;\gamma)$ is a subalgebra of X for all $\gamma \in [0.5, 1)$.

Conversely, let $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$ be such that $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are subalgebras of X. If there exist $a, b \in X$ such that

$$A_I(a * b) < \bigwedge\{A_I(a), A_I(b), 0.5\},$$

then we can take $\beta \in (0, 1)$ such that

$$A_I(a * b) < \beta < \bigwedge\{A_I(a), A_I(b), 0.5\}. \tag{3.6}$$

Thus $a, b \in I_\in(A;\beta)$ and $\beta < 0.5$, and so $a * b \in I_\in(A;\beta)$. But, the left inequality in (3.6) induces $a * b \notin I_\in(A;\beta)$, a contradiction. Hence

$$A_I(x * y) \geq \bigwedge\{A_I(x), A_I(y), 0.5\}$$

for all $x, y \in X$. Similarly, we can show that

$$A_T(x * y) \geq \bigwedge\{A_T(x), A_T(y), 0.5\}$$

for all $x, y \in X$. Now suppose that

$$A_F(a * b) > \bigvee\{A_F(a), A_F(b), 0.5\}$$

for some $a, b \in X$. Then there exists $\gamma \in (0, 1)$ such that

$$A_F(a * b) > \gamma > \bigvee\{A_F(a), A_F(b), 0.5\}.$$

It follows that $\gamma \in (0.5, 1)$ and $a, b \in F_\in(A;\gamma)$. Since $F_\in(A;\gamma)$ is a subalgebra of X, we have $a * b \in F_\in(A;\gamma)$ and so $A_F(a * b) \leq \gamma$. This is a contradiction, and thus

$$A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), 0.5\}$$

for all $x, y \in X$. Using Lemma 3.2, $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic subalgebra of X. □

Using Theorem 3.3 and [2, Theorem 3.8], we have the following corollary.

Corollary 3.4. *For a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, if the nonempty neutrosophic $\in\vee q$-subsets $T_{\in\vee q}(A;\alpha)$, $I_{\in\vee q}(A;\beta)$ and $F_{\in\vee q}(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, then the neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$.*

Theorem 3.5. *Given neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, the nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$ if and only if the following assertion is valid.*

$$(\forall x, y \in X) \left(\begin{array}{l} A_T(x * y) \vee 0.5 \geq A_T(x) \wedge A_T(y) \\ A_I(x * y) \vee 0.5 \geq A_I(x) \wedge A_I(y) \\ A_F(x * y) \wedge 0.5 \leq A_F(x) \vee A_F(y) \end{array} \right). \tag{3.7}$$

Proof. Assume that the nonempty neutrosophic $\in$-subsets $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$. Suppose that there are $a, b \in X$ such that $A_T(a * b) \vee 0.5 < A_T(a) \wedge A_T(b) := \alpha$. Then $\alpha \in (0.5, 1]$ and $a, b \in T_\in(A;\alpha)$. Since $T_\in(A;\alpha)$ is a subalgebra of X, it follows that $a * b \in T_\in(A;\alpha)$, that is, $A_T(a * b) \geq \alpha$ which is a contradiction. Thus

$$A_T(x * y) \vee 0.5 \geq A_T(x) \wedge A_T(y)$$

for all $x, y \in X$. Similarly, we know that $A_I(x * y) \vee 0.5 \geq A_I(x) \wedge A_I(y)$ for all $x, y \in X$. Now, if $A_F(x * y) \wedge 0.5 > A_F(x) \vee A_F(y)$ for some $x, y \in X$, then $x, y \in F_\in(A;\gamma)$ and $\gamma \in [0, 0.5)$ where $\gamma = A_F(x) \vee A_F(y)$. But, $x * y \notin F_\in(A;\gamma)$ which is a contradiction. Hence $A_F(x * y) \wedge 0.5 \leq A_F(x) \vee A_F(y)$ for all $x, y \in X$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X satisfying the condition (3.7). Let $x, y, a, b \in X$ and $\alpha, \beta \in (0.5, 1]$ be such that $x, y \in T_\in(A;\alpha)$ and $a, b \in I_\in(A;\beta)$. Then

$$\begin{aligned} A_T(x * y) \vee 0.5 &\geq A_T(x) \wedge A_T(y) \geq \alpha > 0.5, \\ A_I(a * b) \vee 0.5 &\geq A_I(a) \wedge A_I(b) \geq \beta > 0.5. \end{aligned}$$

It follows that $A_T(x * y) \geq \alpha$ and $A_I(a * b) \geq \beta$, that is, $x * y \in T_\in(A;\alpha)$ and $a * b \in I_\in(A;\beta)$. Now, let $x, y \in X$ and $\gamma \in [0, 0.5)$ be such that $x, y \in F_\in(A;\gamma)$. Then $A_F(x * y) \wedge 0.5 \leq A_F(x) \vee A_F(y) \leq \gamma < 0.5$ and so $A_F(x * y) \leq \gamma$, i.e., $x * y \in F_\in(A;\gamma)$. This completes the proof. □

We consider relations between a $(q, \in\vee q)$-neutrosophic subalgebra and an $(\in, \in\vee q)$-neutrosophic subalgebra.

Theorem 3.6. *In a BCK/BCI-algebra, every $(q, \in\vee q)$-neutrosophic subalgebra is an $(\in, \in\vee q)$-neutrosophic subalgebra.*

Proof. Let $A = (A_T, A_I, A_F)$ be a $(q, \in\vee q)$-neutrosophic subalgebra of a BCK/BCI-algebra X and let $x, y \in X$. Let $\alpha_x, \alpha_y \in (0, 1]$ be such that $x \in T_\in(A;\alpha_x)$ and $y \in T_\in(A;\alpha_y)$. Then $A_T(x) \geq \alpha_x$ and $A_T(y) \geq \alpha_y$. Suppose $x * y \notin T_{\in\vee q}(A;\alpha_x \wedge \alpha_y)$. Then

$$A_T(x * y) < \alpha_x \wedge \alpha_y, \tag{3.8}$$

$$A_T(x * y) + (\alpha_x \wedge \alpha_y) \leq 1. \tag{3.9}$$

It follows that

$$A_T(x * y) < 0.5. \tag{3.10}$$

Combining (3.8) and (3.10), we have

$$A_T(x*y) < \bigwedge\{\alpha_x, \alpha_y, 0.5\}$$

and so

$$\begin{aligned} 1 - A_T(x*y) &> 1 - \bigwedge\{\alpha_x, \alpha_y, 0.5\} \\ &= \bigvee\{1-\alpha_x, 1-\alpha_y, 0.5\} \\ &\geq \bigvee\{1 - A_T(x), 1 - A_T(y), 0.5\}. \end{aligned}$$

Hence there exists $\alpha \in (0, 1]$ such that

$$1 - A_T(x*y) \geq \alpha > \bigvee\{1 - A_T(x), 1 - A_T(y), 0.5\}. \tag{3.11}$$

The right inequality in (3.11) induces $A_T(x) + \alpha > 1$ and $A_T(y) + \alpha > 1$, that is, $x, y \in T_q(A;\alpha)$. Since $A = (A_T, A_I, A_F)$ is a $(q, \in \vee\, q)$-neutrosophic subalgebra of X, we have $x*y \in T_{\in\vee\, q}(A;\alpha)$. But, the left inequality in (3.11) implies that $A_T(x*y)+\alpha \leq 1$, i.e., $x*y \notin T_q(A;\alpha)$, and $A_T(x*y) \leq 1-\alpha < 1-0.5 = 0.5 < \alpha$, i.e., $x*y \notin T_\in(A;\alpha)$. Hence $x*y \notin T_{\in\vee\, q}(A;\alpha)$, a contradiction. Thus $x*y \in T_{\in\vee\, q}(A;\alpha_x \wedge \alpha_y)$. Similarly, we can show that if $x \in I_\in(A;\beta_x)$ and $y \in I_\in(A;\beta_y)$ for $\beta_x, \beta_y \in (0, 1]$, then $x*y \in I_{\in\vee\, q}(A;\beta_x \wedge \beta_y)$. Now, let $\gamma_x, \gamma_y \in [0, 1)$ be such that $x \in F_\in(A;\gamma_x)$ and $y \in F_\in(A;\gamma_y)$. $A_F(x) \leq \gamma_x$ and $A_F(y) \leq \gamma_y$. If $x*y \notin F_{\in\vee\, q}(A;\gamma_x \vee \gamma_y)$, then

$$A_F(x*y) > \gamma_x \vee \gamma_y, \tag{3.12}$$

$$A_F(x*y) + (\gamma_x \vee \gamma_y) \geq 1. \tag{3.13}$$

It follows that

$$A_F(x*y) > \bigvee\{\gamma_x, \gamma_y, 0.5\}$$

and so that

$$\begin{aligned} 1 - A_F(x*y) &< 1 - \bigvee\{\gamma_x, \gamma_y, 0.5\} \\ &= \bigwedge\{1-\gamma_x, 1-\gamma_y, 0.5\} \\ &\leq \bigwedge\{1 - A_F(x), 1 - A_F(y), 0.5\}. \end{aligned}$$

Thus there exists $\gamma \in [0, 1)$ such that

$$1 - A_F(x*y) \leq \gamma < \bigwedge\{1 - A_F(x), 1 - A_F(y), 0.5\}. \tag{3.14}$$

It follows from the right inequality in (3.14) that $A_F(x)+\gamma < 1$ and $A_F(y)+\gamma < 1$, that is, $x, y \in F_q(A;\gamma)$, which implies that $x*y \in F_{\in\vee\, q}(A;\gamma)$. But, we have $x*y \notin F_{\in\vee\, q}(A;\gamma)$ by the left inequality in (3.14). This is a contradiction, and so $x*y \in F_{\in\vee\, q}(A;\gamma_x \vee \gamma_y)$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee\, q)$-neutrosophic subalgebra of X. □

The following example shows that the converse of Theorem 3.6 is not true.

Table 1. Cayley table of the operation $*$

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	1	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.6	0.8	0.3
1	0.2	0.3	0.6
2	0.2	0.3	0.6
3	0.7	0.1	0.7
4	0.4	0.4	0.9

Example 3.7. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.
Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X defined by
Then

$$T_{\in}(A;\alpha) = \begin{cases} \{0,3\} & \text{if } \alpha \in (0.4, 0.5], \\ \{0,3,4\} & \text{if } \alpha \in (0.2, 0.4], \\ X & \text{if } \alpha \in (0, 0.2], \end{cases}$$

$$I_{\in}(A;\beta) = \begin{cases} \{0\} & \text{if } \beta \in (0.4, 0.5], \\ \{0,4\} & \text{if } \beta \in (0.3, 0.4], \\ \{0,1,2,4\} & \text{if } \beta \in (0.1, 0.3], \\ X & \text{if } \beta \in (0, 0.1], \end{cases}$$

$$F_{\in}(A;\gamma) = \begin{cases} X & \text{if } \gamma \in (0.9, 1), \\ \{0,1,2,3\} & \text{if } \gamma \in [0.7, 0.9), \\ \{0,1,2\} & \text{if } \gamma \in [0.6, 0.7), \\ \{0\} & \text{if } \gamma \in [0.5, 0.6), \end{cases}$$

which are subalgebras of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$. Using Theorem 3.3, $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic subalgebra of X. But it is not a $(q, \in\vee q)$-neutrosophic subalgebra of X since $2 \in T_q(A; 0.83)$ and $3 \in T_q(A; 0.4)$, but $2 * 3 = 2 \notin T_{\in\vee q}(A; 0.4)$.

We provide conditions for an $(\in, \in\vee q)$-neutrosophic subalgebra to be a $(q, \in\vee q)$-neutrosophic subalgebra.

Theorem 3.8. *Assume that any neutrosophic T_Φ-point and neutrosophic I_Φ-point has the value α and β in $(0, 0.5]$, respectively, and any neutrosophic F_Φ-point has the value γ in $[0.5, 1)$ for $\Phi \in \{\in, q, \in\vee q\}$. Then every $(\in, \in\vee q)$-neutrosophic subalgebra is a $(q, \in\vee q)$-neutrosophic subalgebra.*

Proof. Let X be a BCK/BCI-algebra and let $A = (A_T, A_I, A_F)$ be an $(\in, \in\vee q)$-neutrosophic subalgebra of X. For $x, y, a, b \in X$, let $\alpha_x, \alpha_y, \beta_a, \beta_b \in (0, 0.5]$ be such that $x \in T_q(A;\alpha_x)$, $y \in T_q(A;\alpha_y)$, $a \in I_q(A;\beta_a)$ and $b \in T_q(A;\beta_b)$. Then $A_T(x) + \alpha_x > 1$, $A_T(y) + \alpha_y > 1$, $A_I(a) + \beta_a > 1$ and $A_I(b) + \beta_b > 1$. Since $\alpha_x, \alpha_y, \beta_a, \beta_b \in (0, 0.5]$, it follows that $A_T(x) > 1 - \alpha_x \geq \alpha_x$, $A_T(y) > 1 - \alpha_y \geq \alpha_y$, $A_I(a) > 1 - \beta_a \geq \beta_a$ and $A_I(b) > 1 - \beta_b \geq \beta_b$, that is, $x \in T_\in(A;\alpha_x)$, $y \in T_\in(A;\alpha_y)$, $a \in I_\in(A;\beta_a)$ and $b \in I_\in(A;\beta_b)$. Also, let $x \in F_q(A;\gamma_x)$ and $y \in F_q(A;\gamma_y)$ for $x, y \in X$ and $\gamma_x, \gamma_y \in [0.5, 1)$. Then $A_F(x) + \gamma_x < 1$ and $A_F(y) + \gamma_y < 1$, and so $A_F(x) < 1 - \gamma_x \leq \gamma_x$ and $A_F(y) < 1 - \gamma_y \leq \gamma_y$ since $\gamma_x, \gamma_y \in [0.5, 1)$. This shows that $x \in F_\in(A;\gamma_x)$ and $y \in F_\in(A;\gamma_y)$. It follows from (3.4) that $x * y \in T_{\in\vee q}(A;\alpha_x \wedge \alpha_y)$, $a * b \in I_{\in\vee q}(A;\beta_a \wedge \beta_b)$, and $x * y \in F_{\in\vee q}(A;\gamma_x \vee \gamma_y)$. Consequently, $A = (A_T, A_I, A_F)$ is a $(q, \in\vee q)$-neutrosophic subalgebra of X. □

Theorem 3.9. *Both $(\in, \in)$-neutrosophic subalgebra and $(\in\vee q, \in\vee q)$-neutrosophic subalgebra are an $(\in, \in\vee q)$-neutrosophic subalgebra.*

Proof. It is clear that $(\in, \in)$-neutrosophic subalgebra is an $(\in, \in\vee q)$-neutrosophic subalgebra. Let $A = (A_T, A_I, A_F)$ be an $(\in\vee q, \in\vee q)$-neutrosophic subalgebra of X. For any $x, y, a, b \in X$, let α_x, α_y, β_a, $\beta_b \in (0, 1]$ be such that $x \in T_\in(A;\alpha_x)$, $y \in T_\in(A;\alpha_y)$, $a \in I_\in(A;\beta_a)$ and $b \in I_\in(A;\beta_b)$. Then $x \in T_{\in\vee q}(A;\alpha_x)$, $y \in T_{\in\vee q}(A;\alpha_y)$, $a \in I_{\in\vee q}(A;\beta_a)$ and $b \in I_{\in\vee q}(A;\beta_b)$ by (3.1) and (3.2). It follows that $x * y \in T_{\in\vee q}(A;\alpha_x \wedge \alpha_y)$ and $a * b \in I_{\in\vee q}(A;\beta_a \wedge \beta_b)$. Now, let $x, y \in X$ and $\gamma_x, \gamma_y \in [0, 1)$ be such that $x \in F_\in(A;\gamma_x)$ and $y \in F_\in(A;\gamma_y)$. Then $x \in F_{\in\vee q}(A;\gamma_x)$ and $y \in F_{\in\vee q}(A;\gamma_y)$ by (3.3). Hence $x * y \in F_{\in\vee q}(A;\gamma_x \vee \gamma_y)$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic subalgebra of X. □

The converse of Theorem 3.9 is not true in general. In fact, the $(\in, \in\vee q)$-neutrosophic subalgebra $A = (A_T, A_I, A_F)$ in Example 3.7 is neither an $(\in, \in)$-neutrosophic subalgebra nor an $(\in\vee q, \in\vee q)$-neutrosophic subalgebra.

Theorem 3.10. *For a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, if the nonempty neutrosophic q-subsets $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in (0, 0.5)$, then*

$$\begin{aligned} & x \in T_\in(A;\alpha_x),\ y \in T_\in(A;\alpha_y) \ \Rightarrow\ x * y \in T_q(A;\alpha_x \vee \alpha_y), \\ & x \in I_\in(A;\beta_x),\ y \in I_\in(A;\beta_y) \ \Rightarrow\ x * y \in I_q(A;\beta_x \vee \beta_y), \\ & x \in F_\in(A;\gamma_x),\ y \in F_\in(A;\gamma_y) \ \Rightarrow\ x * y \in F_q(A;\gamma_x \wedge \gamma_y) \end{aligned} \tag{3.15}$$

for all $x, y \in X$, α_x, α_y, $\beta_x, \beta_y \in (0.5, 1]$ and $\gamma_x, \gamma_y \in (0, 0.5)$.

Proof. Let $x, y, a, b, u, v \in X$ and α_x, α_y, β_a, $\beta_b \in (0.5, 1]$ and $\gamma_u, \gamma_v \in (0, 0.5)$ be such that $x \in T_\in(A;\alpha_x)$, $y \in T_\in(A;\alpha_y)$, $a \in I_\in(A;\beta_a)$, $b \in I_\in(A;\beta_b)$, $u \in F_\in(A;\gamma_u)$ and $v \in F_\in(A;\gamma_v)$. Then $A_T(x) \geq \alpha_x > 1 - \alpha_x$, $A_T(y) \geq \alpha_y > 1 - \alpha_y$, $A_I(a) \geq \beta_a > 1 - \beta_a$, $A_I(b) \geq \beta_b > 1 - \beta_b$, $A_F(u) \leq \gamma_u < 1 - \gamma_u$ and $A_F(v) \leq \gamma_v < 1 - \gamma_v$. It follows that $x, y \in T_q(A;\alpha_x \vee \alpha_y)$, $a, b \in I_q(A;\beta_a \vee \beta_b)$ and $u, v \in F_q(A;\gamma_u \wedge \gamma_v)$. Since $\alpha_x \vee \alpha_y, \beta_a \vee \beta_b \in (0.5, 1]$ and $\gamma_u \wedge \gamma_v \in (0, 0.5)$, we have $x * y \in T_q(A;\alpha_x \vee \alpha_y)$, $a * b \in I_q(A;\beta_a \vee \beta_b)$ and $u * v \in F_q(A;\gamma_u \wedge \gamma_v)$ by hypothesis. This completes the proof. □

The following corollary is by Theorem 3.10 and [2, Theorem 3.7].

Corollary 3.11. *Every $(\in, \in\vee q)$-neutrosophic subalgebra $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X satisfies the condition (3.15).*

Corollary 3.12. *Every $(q, \in\vee q)$-neutrosophic subalgebra $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X satisfies the condition (3.15).*

Proof. It is by Theorem 3.6 and Corollary 3.11. □

Theorem 3.13. *For a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, if the nonempty neutrosophic q-subsets $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in (0.5, 1)$, then*

$$\begin{aligned} &x \in T_q(A;\alpha_x),\ y \in T_q(A;\alpha_y) \ \Rightarrow\ x * y \in T_\in(A;\alpha_x \vee \alpha_y),\\ &x \in I_q(A;\beta_x),\ y \in I_q(A;\beta_y) \ \Rightarrow\ x * y \in I_\in(A;\beta_x \vee \beta_y),\\ &x \in F_q(A;\gamma_x),\ y \in F_q(A;\gamma_y) \ \Rightarrow\ x * y \in F_\in(A;\gamma_x \wedge \gamma_y)\end{aligned} \tag{3.16}$$

for all $x, y \in X$, α_x, α_y, $\beta_x, \beta_y \in (0, 0.5]$ and $\gamma_x, \gamma_y \in (0.5, 1)$.

Proof. Let $x, y, a, b, u, v \in X$ and α_x, α_y, β_a, $\beta_b \in (0, 0.5]$ and $\gamma_u, \gamma_v \in (0.5, 1)$ be such that $x \in T_q(A;\alpha_x)$, $y \in T_q(A;\alpha_y)$, $a \in I_q(A;\beta_a)$, $b \in I_q(A;\beta_b)$, $u \in F_q(A;\gamma_u)$ and $v \in F_q(A;\gamma_v)$. Then $x, y \in T_q(A;\alpha_x \vee \alpha_y)$, $a, b \in I_q(A;\beta_a \vee \beta_b)$ and $u, v \in F_q(A;\gamma_u \wedge \gamma_v)$. Since $\alpha_x \vee \alpha_y, \beta_a \vee \beta_b \in (0, 0.5]$ and $\gamma_u \wedge \gamma_v \in (0.5, 1)$, it follows from the hypothesis that $x * y \in T_q(A;\alpha_x \vee \alpha_y)$, $a * b \in I_q(A;\beta_a \vee \beta_b)$ and $u * v \in F_q(A;\gamma_u \wedge \gamma_v)$. Hence

$$\begin{aligned} &A_T(x * y) > 1 - (\alpha_x \vee \alpha_y) \geq \alpha_x \vee \alpha_y, \text{ that is, } x * y \in T_\in(A;\alpha_x \vee \alpha_y),\\ &A_I(a * b) > 1 - (\beta_a \vee \beta_b) \geq \beta_a \vee \beta_b, \text{ that is, } a * b \in I_\in(A;\beta_a \vee \beta_b),\\ &A_F(u * v) < 1 - (\gamma_u \wedge \gamma_v) \leq \gamma_u \wedge \gamma_v, \text{ that is, } u * v \in F_\in(A;\gamma_u \wedge \gamma_v).\end{aligned}$$

Consequently, the condition (3.16) is valid for all $x, y \in X$, α_x, α_y, $\beta_x, \beta_y \in (0, 0.5]$ and $\gamma_x, \gamma_y \in (0.5, 1)$. □

Theorem 3.14. *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, if the nonempty neutrosophic $\in\vee q$-subsets $T_{\in\vee q}(A;\alpha)$, $I_{\in\vee q}(A;\beta)$ and $F_{\in\vee q}(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$, then the following assertions are valid.*

$$\begin{aligned} &x \in T_q(A;\alpha_x),\ y \in T_q(A;\alpha_y) \ \Rightarrow\ x * y \in T_{\in\vee q}(A;\alpha_x \vee \alpha_y),\\ &x \in I_q(A;\beta_x),\ y \in I_q(A;\beta_y) \ \Rightarrow\ x * y \in I_{\in\vee q}(A;\beta_x \vee \beta_y),\\ &x \in F_q(A;\gamma_x),\ y \in F_q(A;\gamma_y) \ \Rightarrow\ x * y \in F_{\in\vee q}(A;\gamma_x \wedge \gamma_y)\end{aligned} \tag{3.17}$$

for all $x, y \in X$, α_x, α_y, $\beta_x, \beta_y \in (0, 0.5]$ and $\gamma_x, \gamma_y \in [0.5, 1)$.

Proof. Let $x, y, a, b, u, v \in X$ and α_x, α_y, β_a, $\beta_b \in (0, 0.5]$ and $\gamma_u, \gamma_v \in [0.5, 1)$ be such that $x \in T_q(A;\alpha_x)$, $y \in T_q(A;\alpha_y)$, $a \in I_q(A;\beta_a)$, $b \in I_q(A;\beta_b)$, $u \in F_q(A;\gamma_u)$ and $v \in F_q(A;\gamma_v)$. Then $x \in T_{\in\vee q}(A;\alpha_x)$, $y \in T_{\in\vee q}(A;\alpha_y)$, $a \in I_{\in\vee q}(A;\beta_a)$, $b \in I_{\in\vee q}(A;\beta_b)$, $u \in F_{\in\vee q}(A;\gamma_u)$ and $v \in F_{\in\vee q}(A;\gamma_v)$. It follows that $x, y \in T_{\in\vee q}(A;\alpha_x \vee \alpha_y)$, $a, b \in I_{\in\vee q}(A;\beta_a \vee \beta_b)$ and $u, v \in F_{\in\vee q}(A;\gamma_u \wedge \gamma_v)$ which imply from the hypothesis that $x * y \in T_{\in\vee q}(A;\alpha_x \vee \alpha_y)$, $a * b \in I_{\in\vee q}(A;\beta_a \vee \beta_b)$ and $u * v \in F_{\in\vee q}(A;\gamma_u \wedge \gamma_v)$. This completes the proof. □

Corollary 3.15. *Every $(\in, \in\vee q)$-neutrosophic subalgebra $A = (A_T, A_I, A_F)$ of a BCK/BCI-algebra X satisfies the condition (3.17).*

Proof. It is by Theorem 3.14 and [2, Theorem 3.9]. □

Theorem 3.16. *Given a neutrosophic set* $A = (A_T, A_I, A_F)$ *in a* BCK/BCI*-algebra* X*, if the nonempty neutrosophic* $\in\vee\, q$*-subsets* $T_{\in\vee\, q}(A;\alpha)$*,* $I_{\in\vee\, q}(A;\beta)$ *and* $F_{\in\vee\, q}(A;\gamma)$ *are subalgebras of* X *for all* $\alpha, \beta \in (0.5, 1]$ *and* $\gamma \in [0, 0.5)$*, then the following assertions are valid.*

$$\begin{aligned} &x \in T_q(A;\alpha_x),\ y \in T_q(A;\alpha_y) \ \Rightarrow\ x*y \in T_{\in\vee\, q}(A;\alpha_x \vee \alpha_y), \\ &x \in I_q(A;\beta_x),\ y \in I_q(A;\beta_y) \ \Rightarrow\ x*y \in I_{\in\vee\, q}(A;\beta_x \vee \beta_y), \\ &x \in F_q(A;\gamma_x),\ y \in F_q(A;\gamma_y) \ \Rightarrow\ x*y \in F_{\in\vee\, q}(A;\gamma_x \wedge \gamma_y) \end{aligned} \tag{3.18}$$

for all $x, y \in X$*,* α_x, α_y*,* $\beta_x, \beta_y \in (0.5, 1]$ *and* $\gamma_x, \gamma_y \in [0, 0.5)$*.*

Proof. It is similar to the proof Theorem 3.14. □

Corollary 3.17. *Every* $(q, \in\vee\, q)$*-neutrosophic subalgebra* $A = (A_T, A_I, A_F)$ *of a* BCK/BCI*-algebra* X *satisfies the condition* (3.18)*.*

Proof. It is by Theorem 3.16 and [2, Theorem 3.10]. □

Combining Theorems 3.14 and 3.16, we have the following corollary.

Corollary 3.18. *Given a neutrosophic set* $A = (A_T, A_I, A_F)$ *in a* BCK/BCI*-algebra* X*, if the nonempty neutrosophic* $\in\vee\, q$*-subsets* $T_{\in\vee\, q}(A;\alpha)$*,* $I_{\in\vee\, q}(A;\beta)$ *and* $F_{\in\vee\, q}(A;\gamma)$ *are subalgebras of* X *for all* $\alpha, \beta \in (0, 1]$ *and* $\gamma \in [0, 1)$*, then the following assertions are valid.*

$$\begin{aligned} &x \in T_q(A;\alpha_x),\ y \in T_q(A;\alpha_y) \ \Rightarrow\ x*y \in T_{\in\vee\, q}(A;\alpha_x \vee \alpha_y), \\ &x \in I_q(A;\beta_x),\ y \in I_q(A;\beta_y) \ \Rightarrow\ x*y \in I_{\in\vee\, q}(A;\beta_x \vee \beta_y), \\ &x \in F_q(A;\gamma_x),\ y \in F_q(A;\gamma_y) \ \Rightarrow\ x*y \in F_{\in\vee\, q}(A;\gamma_x \wedge \gamma_y) \end{aligned}$$

for all $x, y \in X$*,* α_x, α_y*,* $\beta_x, \beta_y \in (0, 1]$ *and* $\gamma_x, \gamma_y \in [0, 1)$*.*

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