

The moment-generating function of the log-normal distribution using the star probability measure

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Abstract The motivation of this study is to investigate new methods for the calculation of the moment-generating function of the lognormal distribution. Taylor expansion method on the moments of the lognormal suffers from divergence issues, saddle-point approximation is not exact, and integration methods can be complicated. In the present paper we introduce a new probability measure that we refer to as the star probability measure as an alternative approach to compute the moment-generating function of normal variate functionals such as the lognormal distribution.

Keywords moment-generating function · lognormal distribution · star probability measure

1 Introduction

In the present study, we investigate how we might use a stochastic approach to compute the moment-generating function of normal variate functionals. The moment-generating function of a random variable X is expressed as $M(\theta) = \mathbb{E}(e^{\theta X})$ by definition. By Taylor expansion of $e^{\theta X}$ centered in zero, we would get $\mathbb{E}(e^{\theta X}) = \mathbb{E}\left(1 + \frac{\theta X}{1!} + \frac{\theta^2 X^2}{2!} + \dots + \frac{\theta^n X^n}{n!}\right)$. The approach using Taylor expansion fails when applied to the moment-generating function of the lognormal distribution. Its expression $M(\theta) = \sum_{n=0}^{\infty} \frac{\theta^n}{n!} e^{n\mu + n^2\sigma^2/2}$ diverges for all θ - mainly because the lognormal distribution is skewed to the right. The skew increases the likelihood of having occurrences that depart from the central point of the Taylor expansion. These occurrences produce moments of higher order, which are not offset by the factorial of n in the denominator; therefore, the Taylor series diverges. Also [6] showed that the lognormal

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distribution is not uniquely determined by its moments, meaning there exists a different distribution which has the same moments. According to [9], the moment-generating function of the lognormal distribution does not exist when $\theta > 0$, which is inexact. Actually, there exists a small interval of $\theta > 0$ at the right of zero where the moment-generating function is defined.

Saddle points and their representations are worth considering, although they are not always accurate in all domains. An expression for the characteristic function using the classical saddle-point approximation as described in [4] was obtained by [7]. Several other representations of the characteristic function have been obtained using asymptotic series [2,8]. Integration methods to evaluate the characteristic function such as the Hermite-Gauss quadrature proposed by [5] appear to be a natural choice. It was noted in [3] that, due to the oscillatory integrand and the slow decay rate of the tail of the lognormal density function, numerical integration is difficult. Other numerical methods include the work of [10], which employs a contour integral passing through the saddle point at the steepest descent rate to overcome the oscillatory behavior of the integrand and the slow decay rate at the tail. A closed-form approximation of the Laplace transform of the lognormal distribution, which is fairly accurate over its domain of definition was derived by [1]. Although this equation is not a true benchmark, we can compare it with our results due to its simplicity and because the Laplace transform of the density function is the moment-generating function with a negative sign in front of the argument θ .

In the present approach we introduce the stochastic process associated to the problem, which is obtained by applying Itô's lemma with the function $f(x) = e^{\theta Y(x)}$ to the base process $dx_t = \mu dt + \sigma dW_t$, where μ and σ are respectively the drift and volatility parameters, W_t is a Wiener process, and $Y(x)$ is the random variable of the distribution expressed as a function of a normal random variable x . The purpose is to compute $\mathbb{E}(f_t) = \mathbb{E}(f(x_t))$ to evaluate the moment-generating function of the distribution. While the method leads to the known closed form for the normal distribution, numerical methods cannot be avoided for the calculation of the moment-generating function of the lognormal distribution. Because of the dependence between the real and imaginary parts of the associated stochastic process, the method fails when computing the characteristic function of the lognormal distribution due to loss of normality of the process. It appears that the stochastic approach provides some interesting information about the resolvability of the moment-generating function of normal variate functionals.

Section 2 introduces the theoretical background with the star probability measure and its use to compute the expected value of normal variate functionals. An illustration of the method is provided in section 3 for the calculation of the moment-generating function of the normal distribution. In section 4, we apply the method to the calculation of the moment-generating function of the lognormal distribution. This section covers the derivation of the stochastic differential equation, the differential equation of the first moment and of the

variance. Issues arising when extending the calculations to the characteristic function are also highlighted. Lastly, we perform a numerical calculation of the moment-generating function of the lognormal distribution, where we compare the present study's results against the closed-form approximation of the Laplace transform. In section 5, we offer our conclusion.

2 Theoretical background

2.1 The star probability measure

Let us assume that $f: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and X a random variable. Let us say F is the primitive of f . The star probability measure relative to $F(X)$ is defined such that:

$$\mathbb{E}_*(F(X)) = F(\mathbb{E}(X)), \quad (1)$$

where \mathbb{E}_* is the expectation in the star probability measure and \mathbb{E} is the expectation in the natural probability measure.

Note that the star probability measure as defined in (1) does not exist in all cases. For example, if $F(X) = X^2$ and X is normally distributed centered in zero, then the star probability measure does not exist. If F is an increasing function on its domain of definition, then the star probability measure exists. Conversely, if F is a decreasing function on its domain of definition, then the star probability measure exists.

Because of (1) we can write:

$$\mathbb{E}_* \left(\int_{x_0}^X f(x) dx \right) = \int_{x_0}^{\mathbb{E}(X)} f(x) dx, \quad (2)$$

This expression is critical for calculating the expected value of a stochastic process; hence the introduction of the star probability measure.

One property of the expectation in the star probability measure is linearity. For any arbitrary star probability measure, we have:

$$\mathbb{E}_*(a + bX) = a + b\mathbb{E}_*(X), \quad (3)$$

where a and b are constants, and X a random variable. This property is straightforward as we assert there exists a density function of the random variable X in the star probability measure. The expectation in the star probability measure is the integral of the expression $a + bX$ times the density function of X in the star probability measure over the domain of definition of X . The calculation yields (3).

We enforce the below constraint:

$$\mathbb{E}_*(W_t) = 0, \quad (4)$$

where W_t is a Wiener process under the natural probability measure. The rationale for this constraint is explained below.

Given a base process $dx_t = \mu dt + \sigma dW_t$ where μ and σ are parameters and W_t a Wiener process, we define the Itô process $f_t = f(x_t)$ where $f: \mathbb{R} \rightarrow \mathbb{R}$ is a twice-differentiable continuous function which is invertible. The Itô process of f_t obtained by applying Itô's lemma to the base process x_t with the function $f(x)$ is as follows:

$$df_t = \left(\mu \frac{\partial f}{\partial x}(x_t) + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial x^2}(x_t) \right) dt + \sigma \frac{\partial f}{\partial x}(x_t) dW_t, \quad (5)$$

where W_t is a Wiener process.

When normalizing the Itô process of f_t by dividing both sides of (5) by $\frac{\partial f}{\partial x}(x_t)$, we obtain a stochastic differential equation of the form:

$$[f^{-1}]'(f_t) df_t = \left(\mu + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial x^2}(x_t) \Big/ \frac{\partial f}{\partial x}(x_t) \right) dt + \sigma dW_t. \quad (6)$$

Applying Itô's lemma to the Itô process of f_t given by (5) with the function $f^{-1}(x)$ yields:

$$d(f^{-1}(f_t)) = [f^{-1}]'(f_t) df_t - h(f_t) dt, \quad (7)$$

where $h(f_t)$ is expressed as follows:

$$h(f_t) = h(f(x_t)) = \frac{1}{2} \sigma^2 \frac{f''(x_t)}{f'(x_t)}. \quad (8)$$

The term $h(f_t)dt$ is the convexity adjustment due to the transform $f_t \rightarrow f^{-1}(f_t)$. Removing the convexity adjustment term in (7) when solving the normalized stochastic differential equation (6) causes f_t to be shifted to \tilde{f}_t . We say \tilde{f}_t is the Itô process of f_t in the probability measure Tilda. The constraint $\mathbb{E}_*(W_t) = 0$ cancels out the convexity adjustment of the reverse transform $f^{-1}(\tilde{f}_t) \rightarrow \tilde{f}_t$. Hence, the constraint $\mathbb{E}_*(W_t) = 0$ where W_t is the Wiener process associated with \tilde{f}_t causes \tilde{f}_t to shift to f_t through the star probability transformation (1). If we apply the star probability measure to $f^{-1}(\tilde{f}_t)$ with the constraint (4), we get $\mathbb{E}_*(f^{-1}(\tilde{f}_t)) = f^{-1}(\mathbb{E}(f_t))$.

Proposition 1 For any arbitrary star probability measure defined on a bijective function f of a random variable X where $f(X)$ is Gaussian, we have:

$$\mathbb{E}_*(f(X)) = \mathbb{E}(f(X)), \quad (9)$$

provided the constraint $\mathbb{E}_*(W_t) = 0$ is enforced and there is a bijective map between X and W_t .

Proof Let us say we have an arbitrary function f and a random variable X such that $f(X)$ is Gaussian, and a Wiener process W_t . If $f(X)$ is Gaussian and there is a bijective map between X and W_t , then there exists two constants a and b such that $f(X) = a + bW_t$. Because $\mathbb{E}_*(W_t) = 0$, we get $\mathbb{E}_*(f(X)) = a = \mathbb{E}(f(X))$.

2.2 The stochastic differential equation

Assume x_t is a stochastic process that satisfies the below stochastic differential equation:

$$dx_t = \mu_1 dt + \sigma_1 dW_t, \quad (10)$$

where W_t is a Wiener process, μ_1 a constant drift, and σ_1 a constant volatility parameter.

Let us define a twice-differentiable continuous function $f(x)$ which is invertible such that:

$$f(x) = \int_{x_0}^x \sigma_2(s) ds, \quad (11)$$

Let us apply Itô's lemma to the Itô process (10) with the function defined in (11). We have

$$\frac{\partial f(x)}{\partial x} = \sigma_2(x), \quad (12)$$

and

$$\frac{\partial^2 f(x)}{\partial x^2} = \frac{\partial \sigma_2(x)}{\partial x} = \sigma_2'(x), \quad (13)$$

By applying Itô's lemma we get:

$$df = \left(\frac{\partial f}{\partial t} + \mu_1 \frac{\partial f}{\partial x} + \frac{1}{2} \sigma_1^2 \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma_1 \frac{\partial f}{\partial x} dW_t, \quad (14)$$

Therefore

$$df_t = \left(\mu_1 \sigma_2(f^{-1}(f_t)) + \frac{1}{2} \sigma_1^2 \sigma_2'(f^{-1}(f_t)) \right) dt + \sigma_1 \sigma_2(f^{-1}(f_t)) dW_t, \quad (15)$$

where f^{-1} is the inverse function of f , and f_t the Itô process defined by $f_t = f(x_t)$.

We normalize (15) by $\sigma_2(f^{-1}(f_t))$ to obtain:

$$\frac{df_t}{\sigma_2(f^{-1}(f_t))} = \left(\mu_1 + \frac{1}{2} \sigma_1^2 \frac{\sigma_2'(f^{-1}(f_t))}{\sigma_2(f^{-1}(f_t))} \right) dt + \sigma_1 dW_t, \quad (16)$$

Let us set the drift of the stochastic differential equation (15) to $\mu_2(f^{-1}(f_t)) = (\mu_1\sigma_2(f^{-1}(f_t)) + \frac{1}{2}\sigma_1^2\sigma_2'(f^{-1}(f_t)))$; therefore we get:

$$\frac{df_t}{\sigma_2(f^{-1}(f_t))} = \frac{\mu_2(f^{-1}(f_t))}{\sigma_2(f^{-1}(f_t))}dt + \sigma_1 dW_t. \quad (17)$$

We integrate (17) between the boundary conditions to obtain:

$$\int_{f_0}^{f_t} \frac{df}{\sigma_2(f^{-1}(f))} = \int_0^t \frac{\mu_2(f^{-1}(f_s))}{\sigma_2(f^{-1}(f_s))} ds + \int_0^t \sigma_1 dW_t. \quad (18)$$

The derivative of the inverse of a function $h(x)$ is expressed as follows:

$$[h^{-1}]'(x) = \frac{1}{h'(h^{-1}(x))}. \quad (19)$$

Hence:

$$f^{-1}(x) = \int_{f_0}^x \frac{ds}{\sigma_2(f^{-1}(s))}, \quad (20)$$

Because f_t is an Itô process, we should apply Itô's lemma to the process of f_t with the function $f^{-1}(x)$, and we get $f^{-1}(f_t) \neq \int_{f_0}^{f_t} \frac{df}{\sigma_2(f^{-1}(f))}$. We define the probability measure Tilda which excludes the convexity adjustment in the differential operator. Hence, we can replace f_t by \tilde{f}_t in (18) such that we can use (20) with \tilde{f}_t directly. We say \tilde{f}_t is the Itô process of f_t under the probability measure Tilda. Therefore, we get:

$$f^{-1}(\tilde{f}_t) = \int_0^t \frac{\mu_2(f^{-1}(\tilde{f}_s))}{\sigma_2(f^{-1}(\tilde{f}_s))} ds + \sigma_1 W_t, \quad (21)$$

Proposition 2 If f^{-1} spans \mathbb{R} on the image of f , then $f^{-1}(\tilde{f}_t)$ is Gaussian with distribution $\mathcal{N}(m_t, v_t)$ where m_t is the mean and v_t the variance of the process. As a counterexample, if the function $f(x) = \sqrt{x}$ in (11) where $x \in \mathbb{R}^+$, then $f^{-1}(\tilde{f}_t)$ is not Gaussian.

Proof Let us consider the stochastic process X_t defined such that $X_t = \int_0^t g(X_s) ds + \sigma W_t$ where W_t is a Wiener process, σ the volatility parameter and $g: \mathbb{R} \rightarrow \mathbb{R}$ is a function which is differentiable. Hence, we get: $dX_t = g(X_t)dt + \sigma dW_t$. To show that X_t is Gaussian, we need to show that X_t can be decomposed into an infinite sum of infinitesimal independent Gaussian random variables. Let us consider a discretization of the process into infinitesimal time intervals δt . We get: $X_{t+1} - X_t = g(X_t)\delta t + \sigma\sqrt{\delta t}Z_t$ where Z_t is a standard normal random variable. The first iteration is expressed as follows: $X_1 = X_0 + g(X_0)\delta t + \sigma\sqrt{\delta t}Z_0$. The corresponding increment is $\delta X_0 = g(X_0)\delta t + \sigma\sqrt{\delta t}Z_0$, which is a Gaussian infinitesimal increment. The second iteration is $X_2 = X_1 + g(X_0 + \delta X_0)\delta t + \sigma\sqrt{\delta t}Z_1$. The corresponding increment is $\delta X_1 = g(X_0 + \delta X_0)\delta t + \sigma\sqrt{\delta t}Z_1$. Because the increment δX_0 is infinitesimal, we approximate $g(X_0 + \delta X_0)$ by its first order Taylor expansion

(higher order terms are negligible). We get $g(X_0 + \delta X_0) \doteq g(X_0) + g'(X_0)\delta X_0$. Hence $\delta X_1 = (g(X_0) + g'(X_0)\delta X_0)\delta t + \sigma\sqrt{\delta t}Z_1$. Therefore, when $\delta t \rightarrow 0$, δX_1 tends to a Gaussian infinitesimal increment. The third iteration is $X_3 = X_2 + g(X_0 + \delta X_0 + \delta X_1)\delta t + \sigma\sqrt{\delta t}Z_2$. The corresponding increment is $\delta X_2 = g(X_0 + \delta X_0 + \delta X_1)\delta t + \sigma\sqrt{\delta t}Z_2$. Because δX_0 and δX_1 are infinitesimal increments, we approximate $g(X_0 + \delta X_0 + \delta X_1)$ by the first order bivariate Taylor expansion of $\varphi(x, y) = g(X_0 + x + y)$ in $(x, y) = (\delta X_0, \delta X_1)$. We get $g(X_0 + \delta X_0 + \delta X_1) \doteq \varphi(0, 0) + \varphi_x(0, 0)\delta X_0 + \varphi_y(0, 0)\delta X_1 = g(X_0) + g'(X_0)(\delta X_0 + \delta X_1)$. Hence, $\delta X_2 = (g(X_0) + g'(X_0)(\delta X_0 + \delta X_1))\delta t + \sigma\sqrt{\delta t}Z_2$. Therefore, when $\delta t \rightarrow 0$, δX_2 tends to a Gaussian infinitesimal increment. By applying this process recursively, we get that for all integers $i = 0, \dots, n$, δX_i is a linear combination of $\delta X_0, \dots, \delta X_{i-1}$ and Z_i . By substitution of the expression of δX_0 into δX_1 and so on, we obtain that for all integers $i = 0, \dots, n$, δX_i can be expressed as a linear combination of Z_0, \dots, Z_i . Adding together all the δX_i , we obtain that X_t can be expressed as a linear combination of Z_0, \dots, Z_n which are independent standard normal random variables. Therefore, we can conclude that X_t is Gaussian. In (21), $X_t = f^{-1}(\tilde{f}_t)$. For X_t to be Gaussian, f^{-1} must span \mathbb{R} on the image of f as $f_t = f(x_t)$; otherwise X_t could not take any value in \mathbb{R} and would not be Gaussian. This concludes the proof of *proposition 2*.

If $f^{-1}(\tilde{f}_t)$ is Gaussian, we can use *proposition 1* and apply the star probability measure with respect to $f^{-1}(\tilde{f}_t)$. Hence, we get:

$$\mathbb{E}_*(f^{-1}(\tilde{f}_t)) = \mathbb{E}(f^{-1}(\tilde{f}_t)) = m_t, \quad (22)$$

where the constraint $\mathbb{E}_*(W_t) = 0$ is enforced.

Because the constraint $\mathbb{E}_*(W_t) = 0$ causes \tilde{f}_t to shift to f_t through the star probability transformation, we get:

$$f^{-1}(\mathbb{E}(f_t)) = m_t. \quad (23)$$

Finally

$$\mathbb{E}(f_t) = f(m_t), \quad (24)$$

where $\mathbb{E}(f_t) = \mathbb{E}(f(x_t))$.

3 An illustration of the star probability measure with the moment-generating function of the normal distribution

In the present section we show how to derive the moment-generating function of the normal distribution with parameter μ and σ using the star probability measure.

Let us consider the stochastic process x_t which satisfies the following stochastic differential equation:

$$dx_t = \mu dt + \sigma dW_t, \quad (25)$$

where W_t is a Wiener process, μ the drift, σ the volatility, and $x_0 = 0$. This process has distribution $x_t \sim \mathcal{N}(\mu t, \sigma^2 t)$.

Let us apply Itô's lemma to (25) using the function $f(x) = e^{\theta x}$; we get:

$$\frac{df_t}{f_t} = \left(\mu\theta + \frac{1}{2}\sigma^2\theta^2 \right) dt + \theta\sigma dW_t. \quad (26)$$

Note that due to Itô's lemma $d(\ln f_t) \neq \frac{df_t}{f_t}$. We get $d(\ln f_t) = \frac{df_t}{f_t} - \frac{1}{2}\sigma^2\theta^2 dt$.

Under the probability measure Tilda, f_t is shifted to \tilde{f}_t in (26) such that $\frac{d\tilde{f}_t}{\tilde{f}_t} = d(\ln \tilde{f}_t)$. Let us integrate this equation between the boundary conditions; we get:

$$\int_{f_0}^{\tilde{f}_t} \frac{df}{f} = \int_0^t \left(\mu\theta + \frac{1}{2}\sigma^2\theta^2 \right) dt + \theta\sigma W_t. \quad (27)$$

Hence

$$\ln(\tilde{f}_t) = \left(\mu\theta + \frac{1}{2}\sigma^2\theta^2 \right) t + \theta\sigma W_t. \quad (28)$$

Let us apply the star probability measure with respect to $\ln(\tilde{f}_t)$ to (28), we get:

$$\mathbb{E}_* \left(\ln(\tilde{f}_t) \right) = \mathbb{E}_* \left(\left(\mu\theta + \frac{1}{2}\sigma^2\theta^2 \right) t \right) + \mathbb{E}_*(\theta\sigma W_t). \quad (29)$$

Using (1), (3) and (4), we get:

$$\ln(\mathbb{E}(f_t)) = \left(\mu\theta + \frac{1}{2}\sigma^2\theta^2 \right) t. \quad (30)$$

Therefore

$$\mathbb{E}(f_t) = e^{(\mu\theta + \frac{1}{2}\sigma^2\theta^2)t}. \quad (31)$$

The moment-generating function of the normal distribution is $M(\theta) = \mathbb{E}(f_1)$, which is:

$$M(\theta) = e^{(\mu\theta + \frac{1}{2}\sigma^2\theta^2)}. \quad (32)$$

where $\theta \in \mathbb{R}$. The characteristic function of the normal distribution is expressed as $\varphi(\psi) = M(\psi i)$.

For a pair of random variables defined by their respective stochastic processes, the realization of both processes is pairwise correlated; although, both processes can be pathwise independent. The stochastic processes are said to be pathwise independent if their Brownian motions are independent and their

respective variances do not share a common variable. The real and imaginary parts of (26) are pathwise independent when θ is in the complex plane. Another example of distribution where the real and imaginary parts of the stochastic differential equation of the problem are pathwise independent is the uniform distribution. Given a Gaussian random variable $X \sim \mathcal{N}(\mu, \sigma^2)$, the uniform distribution $Y \sim U(0, 1)$ is constructed with the transformation $Y = \phi(X)$, where ϕ is the cumulative density function of X . We then apply Itô's lemma with the function $f(x) = \phi(x)$ to the base process.

4 Application to the moment-generating function of the lognormal distribution

In this section we apply the previous method to the moment-generating function of the lognormal distribution. The moment-generating function of the lognormal distribution of parameter μ and σ is expressed as $M(\theta) = \mathbb{E}(e^{\theta e^x})$ where $x \sim \mathcal{N}(\mu, \sigma^2)$ and $\theta \in \mathbb{R}$.

4.1 The stochastic differential equation

Let us consider the stochastic process x_t which satisfies the following equation:

$$dx_t = \mu dt + \sigma dW_t, \quad (33)$$

where W_t is a Wiener process, μ the drift, σ the volatility, and $x_0 = 0$. This process has distribution $x_t \sim \mathcal{N}(\mu t, \sigma^2 t)$.

Let us apply Itô's lemma to (33) using the function $f(x) = e^{\theta e^x}$ and apply the normalisation in (16); we get:

$$\frac{df_t}{f_t \ln f_t} = \left(\mu + \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2 \ln f_t \right) dt + \sigma dW_t, \quad (34)$$

Because of Itô's lemma $d(\ln \ln f_t) \neq \frac{df_t}{f_t \ln f_t}$. We get $d(\ln \ln f_t) = \frac{df_t}{f_t \ln f_t} - \frac{1}{2}\sigma^2(1 + \ln f_t)dt$. Under the probability measure Tilda, f_t is shifted to \tilde{f}_t such that $d(\ln \ln \tilde{f}_t) = \frac{d\tilde{f}_t}{\tilde{f}_t \ln \tilde{f}_t}$. Let us consider (34) under the measure Tilda and integrate this equation between the boundary conditions; we get:

$$\int_{f_0}^{\tilde{f}_t} \frac{df}{f \ln f} = \int_0^t \left(\mu + \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2 \ln \tilde{f}_s \right) ds + \sigma W_t. \quad (35)$$

Whenever $\theta > 0$:

$$\ln \ln(\tilde{f}_t) = \ln \theta + \left(\mu + \frac{1}{2}\sigma^2 \right) t + \frac{1}{2}\sigma^2 \int_0^t \ln(\tilde{f}_s) ds + \sigma W_t. \quad (36)$$

Whenever $\theta < 0$:

$$\ln(-\ln(\tilde{f}_t)) = \ln(-\theta) + \left(\mu + \frac{1}{2}\sigma^2\right)t + \frac{1}{2}\sigma^2 \int_0^t \ln(\tilde{f}_s) ds + \sigma W_t. \quad (37)$$

Whenever θ is real and positive, $\ln \ln(\tilde{f}_t)$ is Gaussian with distribution $\mathcal{N}(m_t, v_t)$, where m_t is the first moment and v_t the variance. The same can be said about $\ln(-\ln(\tilde{f}_t))$ when θ is negative. In the present study we only consider the case when $\theta > 0$. The case when $\theta < 0$ is left to the reader.

4.2 The dependence structure between the real and imaginary parts

If we set $y_t = \ln \ln \tilde{f}_t$, we can rewrite (36) as follows:

$$y_t = \ln \theta + \left(\mu + \frac{1}{2}\sigma^2\right)t + \frac{1}{2}\sigma^2 \int_0^t e^{y_s} ds + \sigma W_t. \quad (38)$$

As we move into the complex plane we need to introduce $y_t = y_{R,t} + i y_{I,t}$, where $y_{R,t}$ is the real part and $y_{I,t}$ the imaginary part. In a number of settings, the bivariate normal assumption relies on the law of large numbers, a prior holistic to multivariate distributions. Given (38), the dependence structure between the marginals in our problem is as follows:

$$y_{R,t} = \Re(\ln \theta) + \left(\mu + \frac{1}{2}\sigma^2\right)t + \frac{1}{2}\sigma^2 \int_0^t \cos(y_{I,s}) e^{y_{R,s}} ds + \sigma W_t, \quad (39)$$

and

$$y_{I,t} = \Im(\ln \theta) + \frac{1}{2}\sigma^2 \int_0^t \sin(y_{I,s}) e^{y_{R,s}} ds. \quad (40)$$

We note that the real part $y_{R,t}$ diffuses according to a Gaussian process, whereas $y_{I,t}$ is correlated to its real counterpart. The joint distribution of the variables $y_{R,t}$ and $y_{I,t}$ deviates from the multivariate normal distribution due to the structure resulting from the application of Euler's formula to $e^{i y_{I,t}}$. In the present case, we can say that the real part is normally distributed given (38) and that $y_{I,t}$ is a function of $y_{R,t}$ over a full filtration \mathcal{F}_t of W_t . Nonetheless, the distribution of the imaginary part is not well defined in the proper way of a parametric distribution. We can say that $y_{I,t}$ is not Gaussian because $y_{I,t}$ is not linear in $y_{R,t}$ which is Gaussian. Note that the complex logarithm in (39) and (40) is defined from the inverse of the complex exponential function.

The parametrization of a bivariate normal distribution yields :

$$Z = \rho_t Z_1 + \sqrt{1 - \rho_t^2} Z_2, \quad (41)$$

where ρ_t is the correlation between Z and Z_1 at time t , and Z_1 and Z_2 are two independent standard normal random variables.

Assuming the real and imaginary parts of y_t have bivariate normal distribution, we would get:

$$y_t = a + bt + \sigma_1 Z_1 + i \sigma_2 \left(\rho_t Z_1 + \sqrt{1 - \rho_t^2} Z_2 \right), \quad (42)$$

where a and b are complex numbers, σ_1 and σ_2 are respectively the standard deviations of the real and imaginary parts, and Z_1 and Z_2 are two independent standard normal random variables.

The bivariate normal example is an ideal scenario provided for illustration purpose. Another representation of the dependence between the real and imaginary parts of the process is using the Sklar's theorem. In this setting, the multivariate cumulative distribution of the random vector $(y_{R,t}, y_{I,t})$ may be expressed in terms of its marginals and a Copula which are time dependent. In the following sections, to avoid complexity arising from the dependence between the real and imaginary parts of the process, we restrict θ to be real. This approach allows the calculation of the moment-generating function of the lognormal distribution, but its characteristic function remains intractable.

4.3 Differential equation of the first moment

Let us rewrite (38):

$$y_t = \ln \theta + \left(\mu + \frac{1}{2} \sigma^2 \right) t + \frac{1}{2} \sigma^2 \int_0^t e^{y_s} ds + \sigma W_t, \quad (43)$$

where $y_t = \ln \ln \tilde{f}_t$.

Whenever θ is real, we have $y_t \sim \mathcal{N}(m_t, v_t)$.

Let us take the expectation of the stochastic differential equation (43). In addition, we use the fact that $\mathbb{E}(\int_0^t h(s) ds) = \int_0^t \mathbb{E}(h(s)) ds$. Hence, we get:

$$\mathbb{E}(y_t) = \ln \theta + \left(\mu + \frac{1}{2} \sigma^2 \right) t + \frac{1}{2} \sigma^2 \int_0^t \mathbb{E}(e^{y_s}) ds. \quad (44)$$

Because the expected value of a lognormal variable $z = e^x$, where $x \sim \mathcal{N}(\mu, \sigma)$ is $\mathbb{E}(z) = e^{\mu + \frac{1}{2} \sigma^2}$, we get:

$$\mathbb{E}(e^{y_t}) = e^{(m_t + \frac{1}{2} v_t)}. \quad (45)$$

Note that (45) holds when θ is real and positive, which is the case for the calculation of the moment-generating function. However, this equation is no longer valid when θ is a complex number because $y_t = y_{R,t} + i y_{I,t}$, and only the real part $y_{R,t}$ is Gaussian, whereas the imaginary part $y_{I,t}$ as shown in the previous section, is not normally distributed. The linear combination of a Gaussian random variable with a non-Gaussian random variable is not Gaussian, hence we can no longer use the Gaussianity of y_t to compute the expected value $\mathbb{E}(e^{y_t})$.

We can rewrite (44) as follows:

$$m_t = \ln \theta + \left(\mu + \frac{1}{2} \sigma^2 \right) t + \frac{1}{2} \sigma^2 \int_0^t e^{(m_s + \frac{1}{2} v_s)} ds. \quad (46)$$

By deriving (46) with respect to time, we get:

$$\frac{\partial m_t}{\partial t} = \mu + \frac{1}{2} \sigma^2 + \frac{1}{2} \sigma^2 e^{(m_t + \frac{1}{2} v_t)}, \quad (47)$$

with initial condition $m_0 = \ln(\theta)$, which is the differential equation of the first moment. Note this equation is only valid when θ is real and positive but not in the complex plane.

4.4 Differential equation of the variance

Whenever θ is real and positive, the process (43) is Gaussian with $y_t \sim \mathcal{N}(m_t, v_t)$. Hence, we can write:

$$\tilde{y}_t = m_t + \sqrt{v_t} Z, \quad (48)$$

where $Z \sim \mathcal{N}(0, 1)$. We use the notation \tilde{y}_t to denote y_t in its Gaussian representation. Before we can extract the differential equation of the variance, we first need to follow a few steps.

Let us derive (48) with respect to time. Hence, we get:

$$\frac{\partial \tilde{y}_t}{\partial t} = m'_t + \frac{1}{2} \frac{v'_t}{\sqrt{v_t}} Z, \quad (49)$$

The variance of (49) is as follows:

$$\text{Var} \left(\frac{\partial \tilde{y}_t}{\partial t} \right) = \frac{1}{4} \frac{v'_t{}^2}{v_t}. \quad (50)$$

Let us set $y_t = \ln \ln \tilde{f}_t$. Hence, (34) can be expressed as follows:

$$dy = \left(\mu + \frac{1}{2} \sigma^2 + \frac{1}{2} \sigma^2 e^{y_t} \right) dt + \sigma dW_t, \quad (51)$$

Let us integrate (51), therefore we get:

$$y_t - y_0 = \int_0^t \left(\mu + \frac{1}{2} \sigma^2 + \frac{1}{2} \sigma^2 e^{y_t} \right) dt + \sigma W_t, \quad (52)$$

Now let us derive (52) with respect to time introducing $W_t = \sqrt{t} Z$, where $Z \sim \mathcal{N}(0, 1)$. We get:

$$\frac{\partial y_t}{\partial t} = \mu + \frac{1}{2} \sigma^2 + \frac{1}{2} \sigma^2 e^{y_t} + \frac{1}{2} \sigma t^{-1/2} Z, \quad (53)$$

with $y_t = m_t + \sqrt{v_t} Z$.

The variance of (53) is as follows:

$$\text{Var}\left(\frac{\partial y_t}{\partial t}\right) = \frac{1}{4}\sigma^4 \text{Var}\left(e^{m_t+\sqrt{v_t}Z}\right) + \frac{1}{4}\frac{\sigma^2}{t} + \frac{1}{2}\frac{\sigma^3}{t^{1/2}} \text{Cov}\left(e^{m_t+\sqrt{v_t}Z}, Z\right). \quad (54)$$

Note that to obtain (54) we used $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$. For a lognormal random variable $u = e^x$ where $x \sim \mathcal{N}(\mu, \sigma^2)$, we have $\text{Var}(u) = (e^{\sigma^2} - 1)e^{2\mu+\sigma^2}$. Therefore:

$$\text{Var}\left(e^{m_t+\sqrt{v_t}Z}\right) = (e^{v_t} - 1)e^{2m_t+v_t}. \quad (55)$$

To evaluate the covariance term we need to use the formula $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$. Because $\mathbb{E}(Z) = 0$, the covariance term is equal to $\mathcal{I} = \mathbb{E}\left(Ze^{m_t+\sqrt{v_t}Z}\right)$.

We have

$$\mathcal{I} = \int_{-\infty}^{\infty} x e^{m_t+\sqrt{v_t}x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx. \quad (56)$$

We can write

$$\mathcal{I} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{m_t+\sqrt{v_t}x-\frac{1}{2}x^2} dx, \quad (57)$$

which is equivalent to

$$\mathcal{I} = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sqrt{v_t} - x) e^{m_t+\sqrt{v_t}x-\frac{1}{2}x^2} dx + \sqrt{\frac{v_t}{2\pi}} \int_{-\infty}^{\infty} e^{m_t+\sqrt{v_t}x-\frac{1}{2}x^2} dx. \quad (58)$$

The left hand side integral is equal to zero as both branches converge asymptotically to zero, hence:

$$\mathcal{I} = \sqrt{\frac{v_t}{2\pi}} \int_{-\infty}^{\infty} e^{m_t+\sqrt{v_t}x-\frac{1}{2}x^2} dx. \quad (59)$$

Because we have $m_t + \sqrt{v_t}x - \frac{1}{2}x^2 = -\frac{1}{2}(x - \sqrt{v_t})^2 + m_t + \frac{1}{2}v_t$, we can write:

$$\mathcal{I} = \sqrt{v_t} e^{(m_t+\frac{1}{2}v_t)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\sqrt{v_t})^2} dx. \quad (60)$$

The term inside the integral is the density function of a normal random variable with mean $\sqrt{v_t}$ and unit variance. Its integral over the domain is equal to one. Therefore, we get:

$$\mathcal{I} = \sqrt{v_t} e^{(m_t+\frac{1}{2}v_t)}. \quad (61)$$

Therefore, we have

$$\text{Var} \left(\frac{\partial y_t}{\partial t} \right) = \frac{1}{4} \sigma^4 (e^{v_t} - 1) e^{2m_t + v_t} + \frac{1}{4} \frac{\sigma^2}{t} + \frac{1}{2} \frac{\sigma^3}{t^{1/2}} \sqrt{v_t} e^{(m_t + \frac{1}{2} v_t)}. \quad (62)$$

By coupling (50) with (62) under $y_t = \tilde{y}_t$, we get:

$$\frac{\partial v_t}{\partial t} = \sqrt{\frac{v_t \sigma^2}{t} + v_t \sigma^4 (e^{v_t} - 1) e^{2m_t + v_t} + 2 \frac{\sigma^3}{t^{1/2}} v_t^{3/2} e^{(m_t + \frac{1}{2} v_t)}}, \quad (63)$$

with initial conditions $v_0 = 0$ and $v'_t|_{t=0} = \sigma^2$. This last condition is due to the convergence property of the variance: as t converges towards 0, the variance of the process converges towards $\sigma^2 t$.

Eq. (63) is the differential equation for the variance that we need to compute the moment-generating function of the lognormal distribution. As for the equation of the first moment, the differential equation of the variance is only valid when θ is real and positive but not in the complex plane.

4.5 Numerical application

The moment-generating function of the lognormal distribution with parameter μ and σ and with θ positive is $M(\theta) = \mathbb{E}(f_1)$ where \tilde{f}_t is given by the stochastic differential equation (36). The parameters m_1 and v_1 are evaluated by integration of the differential equation of the first moment (47) and the variance (63) over a unit time interval $\Delta T = [0, 1]$.

We solve these integrals by discretisation over the domain of integration ΔT , introducing small time increments δt . We start the calculation from time $t_0 = 0$ and iteratively compute $\frac{\partial m_t}{\partial t}$ and $\frac{\partial v_t}{\partial t}$ at each time step. Using piecewise linear segments, we compute m_t and v_t at the next time step until we reach $t_1 = 1$. The numerical scheme consists of:

$$m_{i+1} = m_i + \left. \frac{\partial m_t}{\partial t} \right|_i \delta t, \quad (64)$$

and

$$v_{i+1} = v_i + \left. \frac{\partial v_t}{\partial t} \right|_i \delta t, \quad (65)$$

for each iteration.

The initial conditions are given by $m_0 = \ln \theta$, $v_0 = 0$ and $v'_t|_{t=0} = \sigma^2$. Once we get the endpoint m_1 we can compute the moment-generating function of the lognormal distribution using (24). We get $M(\theta) = e^{e^{m_1}}$. This is the algorithm for the calculation of the moment-generating function of the lognormal distribution using the stochastic approach.

As a reference for this study let us use the approximation of the moment-generating function of the lognormal distribution given by the Laplace transform of the lognormal distribution [1]. This equation, derived via an asymptotic method, is as follows:

$$M(\theta) \approx \frac{\exp\left(-\frac{W^2(-\theta\sigma^2 e^\mu) + 2W(-\theta\sigma^2 e^\mu)}{2\sigma^2}\right)}{\sqrt{1 + W(-\theta\sigma^2 e^\mu)}}, \quad (66)$$

where W is the Lambert-W function.

Table 1 Calculations of the moment-generating function of the lognormal distribution

θ	0.1	0.3	0.5	1
Algorithm of the present study	1.105780	1.352504	1.654947	2.745844
Asmussen, Jensen and Rojas-Nandayapa approximation	1.105780	1.352504	1.654957	2.745950

The calculations of the moment-generating function of the lognormal distribution are summarized in table 1 for a set of θ values. We use the parameters $\sigma = 0.1$ and $\mu = 0$. For the discretization algorithm of the stochastic approach we used 2,000 equidistant time steps, and an accuracy of 1.0×10^{-6} for the Lambert function.

5 Conclusion

In the present study we introduced the star probability measure and illustrated some of its application to the calculation of the moment function of the normal and lognormal distributions. While for the normal distribution there exists a closed-form solution; the lognormal distribution does not have a known closed form. At this stage, the star probability method has its own limitations. It can be used to compute the moment-generating function of the lognormal distribution; however, the method is no longer applicable to the calculation of its characteristic function. This is due to the dependence between the real and imaginary parts of the process and the non-Gaussianity of the latter. It seems tangible that if the moment-generating function of the lognormal distribution $M(\theta)$ we obtained was analytical, meaning it can be expressed in terms of elementary functions, the characteristic function would be expressed as $\varphi(\psi) = M(\psi i)$. A sufficient condition for an analytical solution to exist is that the real and imaginary parts of the stochastic differential equation are pathwise independent; however, if this is not the case, the star probability method does not yield a closed-form solution. Although, this method is applicable to any functional of a normal random variable where the function on which we apply Itô's lemma is bijective and its inverse spans \mathbb{R} , the characteristic function of the lognormal distribution remains a topic of research today.

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