CONFIRMATION OF RIEMANN HYPOTHESIS ALLOWS THE CALCULATION OF THE 19TH AND 21ST MILLS' PRIMES

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Abstract. Using the η function, we show that the real part of the non-trivial zeros of the Riemann zeta fuction is $\frac{1}{2}$. Then, we calculate two big primes using the Riemann hypothesis as true. These two big primes have respectively more than one hundred millions digits and more than one billion digits.

Introduction

The Riemann zeta function is the function of the complex variable s, defined in the half-plane $\Re(s) > 1$ by the absolutely convergent series $\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}$ and in the whole complex plane C (except at s=1) by analytic continuation. The zeros of this function are the solutions $\rho \in \mathbb{C}$ of the equation $\zeta(\rho) = 0$. The attention of the present research on the Riemann zeta function deals with the zeros with real part in $[0; 1]$. These last zeros are called non-trivial zeros for certain reasons you could find in the official problem description. The Riemann hypothesis that has to be proved is that all the non-trivial zeros have real part equal to $\frac{1}{2}$.

Demonstration

One analytic continuation of $\zeta(s)$ is defined for $\Re(s) > 0$ by $\zeta(s) = \frac{\eta(s)}{1-2^{1-s}} = \frac{1}{1-2^{1-s}} \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^s}$ (except for $s = 1 + i\frac{2k\pi}{\ln(2)}$ for $k \in \mathbb{Z}$, $s = 1 + i\frac{2k\pi}{\ln(2)}$ being the zeros of $1-2^{1-s}$). Hence, $\zeta(s) = 0 \Rightarrow \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^s} = 0$. We know that the function ζ does not vanish at $s = 1 + ib$ for $b \in \mathbb{R}$, so the non-trivial zeros won't be found there. Indeed, the prime number theorem is equivalent to the fact that there are no zeros of the zeta function on the Re(s) = 1 line [1].

Let's have $s = a + ib$ where $a \in]0; 1[$ and $b \in \mathbb{R}$. Then,

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^s} = 0 \Rightarrow \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^{a+ib}} = 0 \Rightarrow \sum_{n=1}^{+\infty} (-1)^{n+1} n^{-a-ib} = 0.$$

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If the Riemann hypothesis is true, then the real part a equals $\frac{1}{2}$ and is unique in $]0;1[$. So, if we suppose that the real part is not unique and if we show that this supposition leads to an absurdity, we should have proved that the Riemann hypothesis is true.

So let's suppose that the real part a is not unique in $]0;1[$ by saying that it exists $\epsilon \neq 0$ such that $\zeta(a+\epsilon+ib)=0$. Then, we have:

$$\sum_{n=1}^{+\infty}(-1)^{n+1}n^{-a}n^{-\epsilon}n^{-ib}=0$$

Let's choose this ϵ very small. Then we have, with the Taylor expansion: $n^{-\epsilon} = 1-\epsilon\ln(n)+o(\epsilon)$, and $\zeta(s)=0$:

$$\sum_{n=1}^{+\infty}(-1)^{n+1}n^{-a}(-1)^{1}\ln(n)n^{-ib}=0$$

And we have also, with the Taylor expansion to the second degree : $n^{\epsilon} = 1-\epsilon\ln(n)+(-1)^2\dfrac{(\epsilon\ln(n))^2}{2!}+o(\epsilon^2)$, and $\zeta(s)=0$:

$$\sum_{n=1}^{+\infty}(-1)^{n+1}n^{-a}(-1)^{2}\frac{(\ln(n))^2}{2!}n^{-ib}=0$$

And so on, expanding to one more degree each time leads to:

$$\forall k\in\mathbb{N}\ \sum_{n=1}^{+\infty}(-1)^{n+1}n^{-a}(-1)^{k}\frac{(\ln(n))^k}{k!}n^{-ib}=0$$

So we can sum from $k=0$ to $+\infty$ and we have:

$$\sum_{n=1}^{+\infty}(-1)^{n+1}n^{-a}\sum_{k=0}^{+\infty}(-1)^{k}\frac{(\ln(n))^k}{k!}n^{-ib}$$

which leads to: $\displaystyle\sum_{n=1}^{+\infty}(-1)^{n+1}n^{-a-1}n^{-ib}=0$

which gives, for $0<\Re(s)<1$:

$$\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n^{s+1}}=0$$

$$\Rightarrow \frac{1}{1-2^{-s}}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n^{s+1}}=0$$

which gives: $\zeta(s+1)=0$.

But there is no zero for $1<\Re(s)<2$. Indeed, for $a>1$ and $b\in\mathbb{R}$, $\dfrac{\zeta(2a)}{\zeta(a)} \leq |\zeta(a+ib)|$.

So we were led to an absurdity. So we can conclude that the real part of the non-trivial zero is unique. Hardy showed that there was an infinity of non-trivial zeros with $\frac{1}{2}$ as real part [2]. So we can say that the real part of non-trivial zeros of ζ is $\frac{1}{2}$.

The calculation of two big primes

Aware of the truth about the Riemann hypothesis, there is an "easy" way to calculate big primes: by using the Mills' constant. Mills proved in 1947 that if the Riemann hypothesis is true, then it exists several real numbers A such that the integer part of A^{3^n} is a prime number [3], n being an integer. So, I have established two C++ programs, one to calculate big primes and another one to calculate digits of the smallest Mills' constant, because, in order to calculate big primes, you need a very high precision of the Mills' constant. So, wanting to be simple, I have been calculating the smallest Mills' constant by calculating Mills' primes and then the 3^nth root of primes to add digits to the smallest Mills' constant. I have been calculating 3^nth roots of primes with the Newton algorithm [4]. But I had one question to answer: how to be sure of the results?

First, you can find the primes, the biggest Mills constant I have calculated and the two C++ programs on this website: http://www.brouardthomas.org/

For the Mills' constant, I was sure of my results by comparing the last result of the Newton algorithm and the previous result (the one that the program was finding before the last step of the loop), to be sure to find two lists of digits that are equal. I began to do this with the hand then upgraded the program to make it automatic by removing the digits that were not matching the two numbers.

For the primes, I have been using each time the biggest Mills' constant I could use to have the highest possible precision and so, to get more easily the prime I was looking for. I calculated the primes and the Mills' constant until I was certain that the integer part that I obtained was not changing while increasing the number of digits of the Mills' constant. And indeed, from a certain number of digits of the Mills' constant, the integer part was not changing while rising the precision of the Mills' constant. The obtained number had not its integer part and its six zeros present after the comma changed while increasing the number of digits of the Mills' constant, which means the number was changing only from a certain rank in the decimal part, confirming then that the prime was found, as the integer part.

The 19th Mills' prime has more than one hundred millions digits. And the 21st Mills' prime has more than one billion digits. I was sure of these standing numbers of digits by trying to calculate the powered numbers on wolframalpha.com which was, at the time, giving a tenth power giving thus an approximate number of digits.

References

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3. William H. Mills (1947), "A prime-representing function", Bulletin of the American Mathematical Society, p. 604 et 1196

4. Newton nth root algorithm