Yet another proof that $\zeta(2) = \frac{\pi^2}{6}$

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Abstract

Here is presented a proof that $\displaystyle\int_0^1 \frac{\ln(1-x)}{x}\,dx = \frac{\pi^2}{6}$.
I don't know if this proof has been already published.

1 Value of J

Let,

$$J := \int_0^1 \frac{\ln(1+x)}{x} dx = \frac{\pi^2}{12}$$

Let f, be a function such that, for $s \in [0;1]$,

$$f(s) = \int_0^{\frac{\pi}{2}} \arctan\left(\frac{\sin t}{\cos t + s}\right) dt$$

Observe that,

$$\begin{aligned} f(0) &= \int_0^{\frac{\pi}{2}} \arctan\left(\frac{\sin t}{\cos t}\right) dt \\ &= \int_0^{\frac{\pi}{2}} t\,dt \\ &= \left[\frac{t^2}{2}\right]_0^{\frac{\pi}{2}} \\ &= \frac{\pi^2}{8} \end{aligned}$$

For t in $\left[0, \frac{\pi}{2}\right]$,

$$\begin{aligned}
\frac{\sin t}{\cos t + 1} &= \frac{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}{\cos^2\left(\frac{t}{2}\right) - \sin^2\left(\frac{t}{2}\right) + 1} \\
&= \frac{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}{2\cos^2\left(\frac{t}{2}\right)} \\
&= \tan\left(\frac{t}{2}\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
f(1) &= \int_0^{\frac{\pi}{2}} \arctan\left(\frac{\sin t}{\cos t + 1}\right) dt \\
&= \int_0^{\frac{\pi}{2}} \arctan\left(\tan\left(\frac{t}{2}\right)\right) dt \\
&= \int_0^{\frac{\pi}{2}} \frac{t}{2}\, dt \\
&= \left[\frac{t^2}{4}\right]_0^{\frac{\pi}{2}} \\
&= \frac{\pi^2}{16}
\end{aligned}$$

For s in $[0,1]$,

$$\begin{aligned} f'(s) &= -\int_0^{\frac{\pi}{2}} \frac{\sin t}{1+2s\cos t+s^2}\,dt \\ &= \left[\frac{\ln(1+2s\cos t+s^2)}{2s}\right]_{t=0}^{t=\frac{\pi}{2}} \\ &= \frac{1}{2}\frac{\ln\left(1+s^2\right)}{s}-\frac{\ln\left(1+s\right)}{s} \end{aligned}$$

Therefore,

$$\begin{aligned} f(1)-f(0) &= \int_0^1 f'(s)ds \\ &= \frac{1}{2}\int_0^1 \frac{\ln\left(1+s^2\right)}{s}\,ds-\int_0^1 \frac{\ln\left(1+s\right)}{s}\,ds \end{aligned}$$

In the first integral perform the change of variable $y=s^2$, therefore,

$$f(1)-f(0) = -\frac{3}{4}J$$

But,

$$\begin{aligned} f(1)-f(0) &= \frac{\pi^2}{16}-\frac{\pi^2}{8} \\ &= -\frac{\pi^2}{16} \end{aligned}$$

Therefore,

$$\boxed{J=\frac{\pi^2}{12}}$$

2 The relation of J and $\zeta(2)$

To obtain the value of J knowing that,

$$\zeta(2) = -\int_0^1 \frac{\ln(1-x)}{x}dx$$

$$\int_0^1 \frac{\ln(1+t)}{t}\,dt + \int_0^1 \frac{\ln(1-t)}{t}\,dt = \int_0^1 \frac{\ln(1-t^2)}{t}\,dt$$

Perform the change of variable $y = t^2$ in the integral of the RHS,

$$\int_0^1 \frac{\ln(1+t)}{t}\,dt + \int_0^1 \frac{\ln(1-t)}{t}\,dt = \frac{1}{2}\int_0^1 \frac{\ln(1-t)}{t}\,dt$$

Therefore,

$$\int_0^1 \frac{\ln(1+t)}{t}\,dt = -\frac{1}{2}\int_0^1 \frac{\ln(1-t)}{t}\,dt$$

That is,

$$\boxed{J = \frac{1}{2}\zeta(2)}$$

Therefore,

$$\boxed{\zeta(2) = \frac{\pi^2}{6}}$$