Real roots of the equation:

$$x^6-3x^4-2x^3+9x^2+3x-26=0$$

Edgar Valdebenito

abstract

This note presents the real roots (in radicals) of the equation:

$$x^6-3x^4-2x^3+9x^2+3x-26=0$$

Keywords: sextic equation , galois group , solvable by radicals.

1. Introduction

- The equation:

$$f(x)=x^6-3x^4-2x^3+9x^2+3x-26=0 \tag{1}$$

- Galois criterion: A polynomial $f\in\mathbb{Q}[x]$ is solvable by radicals if and only if its Galois group $Gal(f)$ is soluble.
- We have (Maple):

$$Gal(f)= \text{"6T11"},\{\text{"2S\_4(6)"},\text{"2 wr S(3)"},\text{"[2^3]S(3)"}\},\text{"-"},48,\{\text{"(3 6)"},\text{"(1 5)(2 4)"},\text{"(2 4 6)(1 3 5)"}\} \tag{2}$$

- $Gal(f)$ is soluble (Maple).

2. Roots

$$f(x)=0\Rightarrow x=\begin{cases} x_1=r_1\in\mathbb{R} \\ x_2=r_2\in\mathbb{R} \\ x_3=z\in\mathbb{C},x_4=\overline{z} \\ x_5=w\in\mathbb{C},x_6=\overline{w}\end{cases} \tag{3}$$

- Graphics:

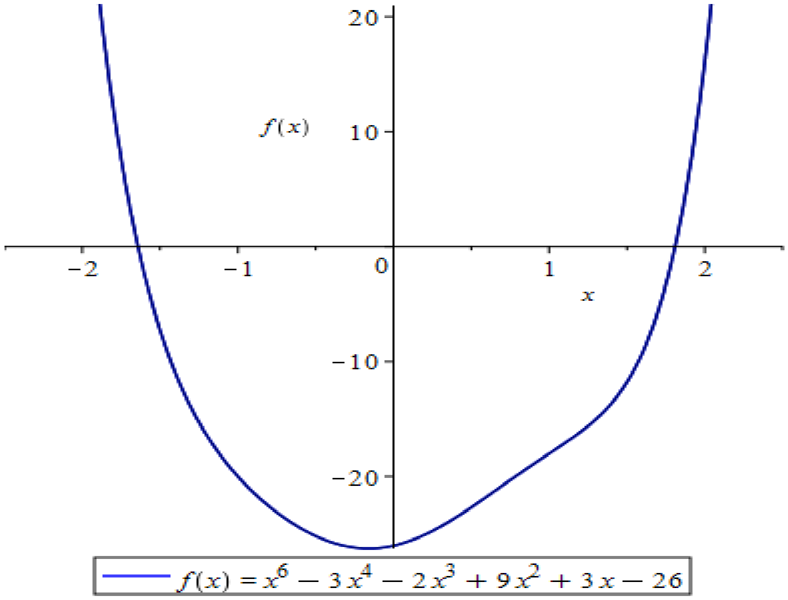

Fig. 1.

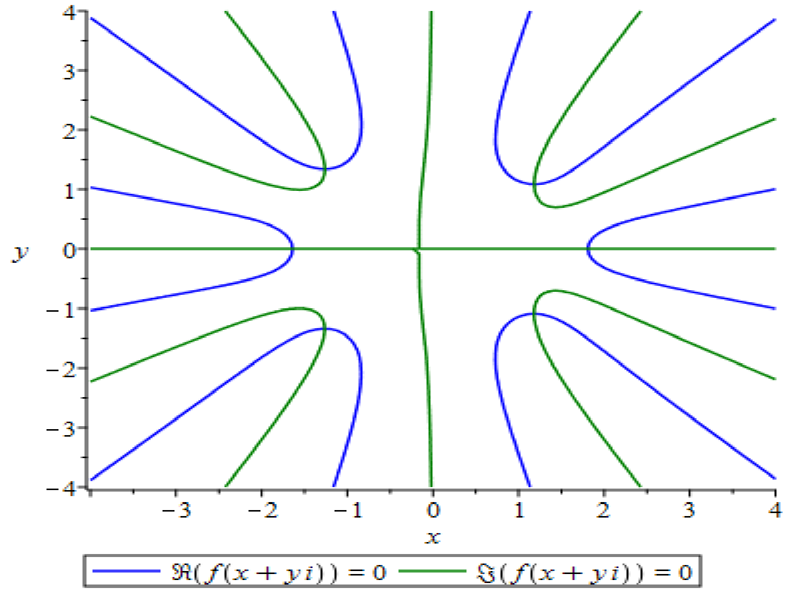

Fig. 2.

❖ Values:

$$r_1 = 1.809694055393... \tag{4}$$

$$r_2 = -1.642244864284... \tag{5}$$

$$z = -1.26445897... + i \times 1.34036171... \tag{6}$$

$$\bar{z} = -1.26445897... - i \times 1.34036171... \tag{7}$$

$$w = 1.18073438... + i \times 1.08738266... \tag{8}$$

$$\bar{w} = 1.18073438... - i \times 1.08738266... \tag{9}$$

3. Real roots in radicals

❖ The number $\alpha \in \mathbb{R}$:

$$\alpha = \sqrt[3]{1 + 6\sqrt[3]{1 + 6\sqrt[3]{1 + ...}}} \tag{10}$$

$$\alpha = \sqrt{6 + \frac{1}{\sqrt{6 + \frac{1}{\sqrt{6 + ...}}}}} \tag{11}$$

$$\alpha = 2\sqrt{2}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{1}{4\sqrt{2}}\right)\right) \tag{12}$$

$$\alpha = \frac{1}{2}\left(4 + 4i\sqrt{31}\right)^{1/3} + 4\left(4 + 4i\sqrt{31}\right)^{-1/3} \tag{13}$$

$$\alpha^3 - 6\alpha - 1 = 0 \tag{14}$$

❖ The number β :

$$\beta = \frac{1}{2}\sqrt[3]{4\sqrt{1269\alpha^2 - 432\alpha - 6588} + 8 - 12\alpha} - \frac{1}{2}\sqrt[3]{4\sqrt{1269\alpha^2 - 432\alpha - 6588} - 8 + 12\alpha} \tag{15}$$

❖ Real roots r_1, r_2 :

$$r_1 = \frac{\alpha}{4} - \frac{\sqrt{\alpha^2 + 4\beta}}{4} + \frac{\sqrt{2\alpha^2 - 4\beta - 2\alpha\sqrt{\alpha^2 + 4\beta} + 8\sqrt{\beta^2 + 12\alpha^2 - 4\alpha - 36}}}{4} \tag{16}$$

$$r_2 = \frac{\alpha}{4} - \frac{\sqrt{\alpha^2 + 4\beta}}{4} - \frac{\sqrt{2\alpha^2 - 4\beta - 2\alpha\sqrt{\alpha^2 + 4\beta} + 8\sqrt{\beta^2 + 12\alpha^2 - 4\alpha - 36}}}{4} \tag{17}$$

4. Real roots: iterative method

❖ The sequence $x_n \in \mathbb{R}$:

$$x_{n+1} = \sqrt[3]{1 - 3x_n} \quad , x_1 = 0 \tag{18}$$

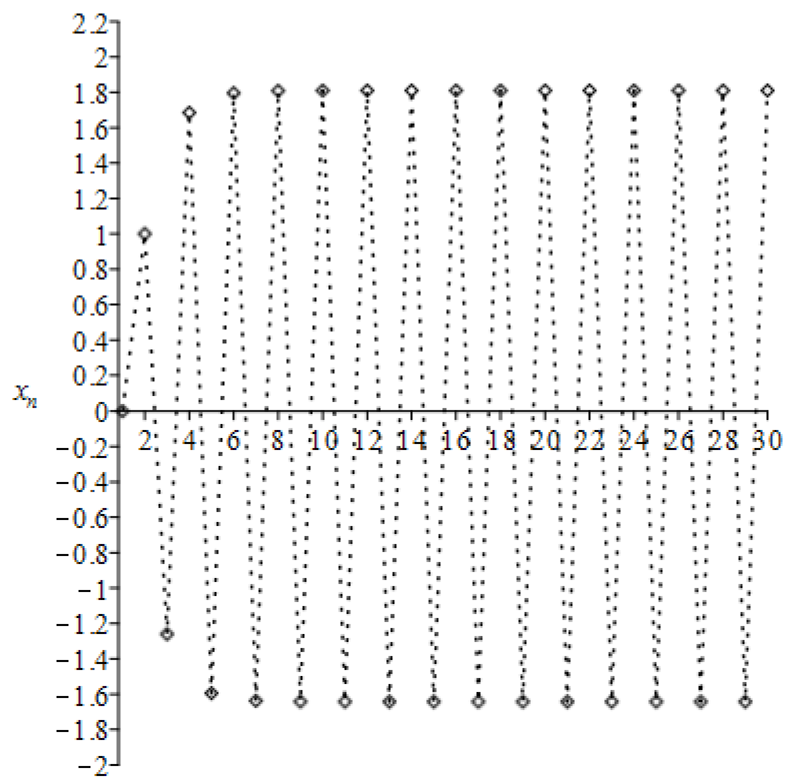

Fig. 3. x_n , $n = 1..30$.

$$x_{2n} \to r_1 \quad , \quad x_{2n-1} \to r_2 \tag{19}$$

$$|x_{2n+2} - r_1| \sim 0.11322 |x_{2n} - r_1| \tag{20}$$

$$|x_{2n+1} - r_2| \sim 0.11322 |x_{2n-1} - r_2| \tag{21}$$

References

1. G. Butler and J. McKay, The transitive groups of degree up to eleven, Comm. Algebra 30 (1983), 863-911.
2. D.S. Dummit, Solving solvable quintics, Math. Comp. 57 (1991),387-401.
3. T.R. Hagedorn, General Formulas for Solving Solvable Sextic Equations. Journal of Algebra 233 , 704-757 (2000).
4. P. Lefton, Galois resolvents of permutation groups , Amer. Math. Monthly 84, No. 8 (1977), 642-644.