A Relation for q-Pochhammer Symbol, q-Bracket, q-Factorial and q-Binomial Coefficient.

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"For now we see through a glass, darkly; but then face to face: now I know in part; but then shall I know even as also I am known." - 1 Corinthians 13:12.

Abstract. In this paper, we construct a relation involving q-Pochhammer symbol, q-bracket, q-factorial and q-binomial coefficient among other things.

1. Introduction

We demonstrated that

$$\left(1-\frac{a}{q}\right)\frac{(b/q;q)_\infty}{(b;q)_\infty}=\left(1-\frac{b}{q}\right)\frac{(a/q;q)_\infty}{(a;q)_\infty},$$

by Corollary, we have

$$\frac{(q^{n+1};q)_\infty^2}{(q^n;q)_\infty(q^{n+2};q)_\infty}=\frac{1-q^{n+1}}{1-q^n},$$

and beautiful identities

$$\frac{(q^2;q^2)_\infty(q^3;q^2)_\infty}{(q;q^2)_\infty(q^4;q^2)_\infty}=1+q,$$

$$\frac{(q^3;q^3)_\infty(q^5;q^3)_\infty}{(q^2;q^3)_\infty(q^6;q^3)_\infty}=\frac{1+q+q^2}{1+q},$$

$$\frac{(q^4;q^4)_\infty(q^7;q^4)_\infty}{(q^3;q^4)_\infty(q^8;q^4)_\infty}=\frac{1+q+q^2+q^3}{1+q+q^2}$$

and

$$\frac{(q^5;q^5)_\infty(q^9;q^5)_\infty}{(q^4;q^5)_\infty(q^{10};q^5)_\infty}=\frac{1+q+q^2+q^3+q^4}{1+q+q^2+q^3}.$$

2. Preliminary

Lemma 1. *If $a,b\in\mathbb{R}$ and $b\neq 0$, then*

$$\frac{a}{b}=\prod_{j=1}^{\infty}\frac{(a+j-1)(b+j)}{(a+j)(b+j-1)}.$$

Proof. I well-know the identity

$$\frac{a}{b}\cdot\frac{b!(a-1)!}{[b+(a-1)+1]!}=\frac{a!(b-1)!}{[a+(b-1)+1]!},\tag{1}$$

using the definition for beta function (4), I obtain

$$\frac{a}{b}=\frac{B(a+1,b)}{B(a,b+1)}.\tag{2}$$

On the other hand, the beta function have the following infinite product representation [5, p. 899]

$$(a+b+1)B(a+1,b+1)=\prod_{j=1}^{\infty}\frac{j(a+b+j)}{(a+j)(b+j)},\tag{3}$$

valid for $a,b\neq-1,-2,...$ Setting $a\to a-1$ and $b\to b-1$, respectively, in both members of (3), we find

$$B(a,b+1)=\frac{1}{a+b}\prod_{j=1}^{\infty}\frac{j(a+b+j-1)}{(a+j-1)(b+j)},\tag{4}$$

and

$$B(a+1,b)=\frac{1}{a+b}\prod_{j=1}^{\infty}\frac{j(a+b+j-1)}{(a+j)(b+j-1)}.\tag{5}$$

Substituting (4) and (5) into the right hand side of (2), we get

$$\frac{a}{b}=\frac{a+b}{a+b}\prod_{j=1}^{\infty}\frac{j(a+b+j-1)(a+j-1)(b+j)}{(a+j)(b+j-1)j(a+b+j-1)}.$$

Eliminate the same terms in the numerator and denominator of the above equation, and encounter

$$\frac{a}{b}=\prod_{j=1}^{\infty}\frac{(a+j-1)(b+j)}{(a+j)(b+j-1)},$$

which is the desired result. □

3. The q-Pochhammer Symbol: a Relation

3.1. A Relation for the q-Pochhammer Symbol.

Theorem 2. *If* $0<a,b,q<1$, *then*

$$\left(1-\frac{a}{q}\right)\frac{(b/q;q)_\infty}{(b;q)_\infty}=\left(1-\frac{b}{q}\right)\frac{(a/q;q)_\infty}{(a;q)_\infty},$$

where $(z;q)_\infty$ *denotes the q-Pochhammer Symbol.*

Proof. We know that [1, p. 85]

$$\lim_{q\to 1^-}\frac{1-q^v}{1-q}=v. \tag{6}$$

Applying (6) into the right hand side of Lemma 1, we find

$$\lim_{q\to 1^-}\frac{1-q^a}{1-q^b}=\lim_{q\to 1^-}\prod_{j=1}^{\infty}\frac{(1-q^{a+j-1})(1-q^{b+j})}{(1-q^{a+j})(1-q^{b+j-1})}$$
$$\Rightarrow \lim_{q\to 1^-}\frac{1-q^a}{1-q^b}=\lim_{q\to 1^-}\frac{(1-q^a)(q-q^b)(q^{a-1};q)_\infty(q^b;q)_\infty}{(1-q^b)(q-q^a)(q^{b-1};q)_\infty(q^a;q)_\infty}$$
$$\Rightarrow \lim_{q\to 1^-}\frac{1-q^{a-1}}{1-q^{b-1}}=\lim_{q\to 1^-}\frac{(q^{a-1};q)_\infty(q^b;q)_\infty}{(q^{b-1};q)_\infty(q^a;q)_\infty}.$$

Let $q^a\to a$ and $q^b\to b$; by quantization process, we eliminate the limit formula in previous equation and find

$$\frac{1-a/q}{1-b/q}=\frac{(a/q;q)_\infty(b;q)_\infty}{(b/q;q)_\infty(a;q)_\infty}$$
$$\Rightarrow\left(1-\frac{a}{q}\right)\frac{(b/q;q)_\infty}{(b;q)_\infty}=\left(1-\frac{b}{q}\right)\frac{(a/q;q)_\infty}{(a;q)_\infty},$$

which is the desired result. □

Corollary 3. *If* $0<q<1$ *and* $n\in\mathbb{N}$, *then*

$$\frac{(q^{n+1};q)_\infty^2}{(q^n;q)_\infty(q^{n+2};q)_\infty}=\frac{1-q^{n+1}}{1-q^n},$$

where $(z;q)_\infty$ *denotes the q-Pochhammer Symbol.*

Proof. Let $a=q^{n+2}$ and $b=q^{n+1}$. This completes the proof. □

Example 4. Setting $n=1$, in Corollary 3, we get

$$x:=\frac{(q^2;q)_\infty^2}{(q;q)_\infty(q^3;q)_\infty}=1+q.$$

Using the elementary property

$$(a;q)_\infty=\prod_{k=0}^{n-1}(aq^k;q^n)_\infty,$$

for $n=2$, in previous equation, we have

$$x:=\frac{(q^2;q^2)_\infty(q^3;q^2)_\infty}{(q;q^2)_\infty(q^4;q^2)_\infty}=1+q.$$

Example 5. Setting $n=2$, in Corollary 3, we get

$$y:=\frac{(q^3;q)_\infty^2}{(q^2;q)_\infty(q^4;q)_\infty}=\frac{1+q+q^2}{1+q}.$$

Using the elementary property

$$(a;q)_\infty=\prod_{k=0}^{n-1}(aq^k;q^n)_\infty,$$

for $n=3$, in previous equation, we have

$$y:=\frac{(q^3;q^3)_\infty(q^5;q^3)_\infty}{(q^2;q^3)_\infty(q^6;q^3)_\infty}=\frac{1+q+q^2}{1+q}.$$

Example 6. Setting $n=3$, in Corollary 3, we get

$$z:=\frac{(q^4;q)_\infty^2}{(q^3;q)_\infty(q^5;q)_\infty}=\frac{1+q+q^2+q^3}{1+q+q^2}.$$

Using the elementary property

$$(a;q)_\infty=\prod_{k=0}^{n-1}(aq^k;q^n)_\infty,$$

for $n=4$, in previous equation, we have

$$z:=\frac{(q^4;q^4)_\infty(q^7;q^4)_\infty}{(q^3;q^4)_\infty(q^8;q^4)_\infty}=\frac{1+q+q^2+q^3}{1+q+q^2}.$$

Example 7. Setting $n=4$, in Corollary 3, we get

$$t:=\frac{(q^5;q)_\infty^2}{(q^4;q)_\infty(q^6;q)_\infty}=\frac{1+q+q^2+q^3+q^4}{1+q+q^2+q^3}.$$

Using the elementary property

$$(a;q)_\infty=\prod_{k=0}^{n-1}(aq^k;q^n)_\infty,$$

for $n=5$, in previous equation, we have

$$t:=\frac{(q^5;q^5)_\infty(q^9;q^5)_\infty}{(q^4;q^5)_\infty(q^{10};q^5)_\infty}=\frac{1+q+q^2+q^3+q^4}{1+q+q^2+q^3}.$$

3.2. Equation Between x and y.

Theorem 8. *If* $0<q<1$ *and let*

$$x:=\frac{(q^2;q^2)_\infty(q^3;q^2)_\infty}{(q;q^2)_\infty(q^4;q^2)_\infty}$$

and

$$y:=\frac{(q^3;q^3)_\infty(q^5;q^3)_\infty}{(q^2;q^3)_\infty(q^6;q^3)_\infty},$$

then,

$$x^2-(y+1)x+1=0.$$

Proof. Let $q\to(1-v)/(1+v)$ in the right hand side of Example 4 and 5

$$x=\frac{2}{v+1} \tag{7}$$

and

$$y=\frac{v^2+3}{2(v+1)}. \tag{8}$$

Eliminate v from (7) and (8) and encounter

$$x^2-(y+1)x+1=0,$$

which is the desired result. □

3.3. Equation Between y and z.

Theorem 9. *If* $0<q<1$ *and let*

$$y:=\frac{(q^3;q^3)_\infty(q^5;q^3)_\infty}{(q^2;q^3)_\infty(q^6;q^3)_\infty}$$

and

$$z:=\frac{(q^4;q^4)_\infty(q^7;q^4)_\infty}{(q^3;q^4)_\infty(q^8;q^4)_\infty},$$

then,

$$zy^3-(z^2+2)y^2+(z+2)y-1=0.$$

Proof. Let $q\to(1-v)/(1+v)$ in the right hand side of Example 5 and 6

$$y=\frac{v^2+3}{2(v+1)} \tag{9}$$

and

$$z=\frac{4(v^2+1)}{(v+1)(v^2+3)}. \tag{10}$$

Eliminate v from (9) and (10) and encounter

$$zy^3-(z^2+2)y^2+(z+2)y-1=0,$$

which is the desired result. □

3.4. Equation Between z and t.

Theorem 10. *If* $0<q<1$ *and let*

$$z:=\frac{(q^4;q^4)_\infty(q^7;q^4)_\infty}{(q^3;q^4)_\infty(q^8;q^4)_\infty}$$

and

$$t:=\frac{(q^5;q^5)_\infty(q^9;q^5)_\infty}{(q^4;q^5)_\infty(q^{10};q^5)_\infty},$$

then,

$$(t^2-t+1)z^4-(t^3-t^2+2t+2)z^3+(t^2+t+4)z^2-(t+3)z+1=0.$$

Proof. Let $q\to(1-v)/(1+v)$ in the right hand side of Example 6 and 7

$$z=\frac{4(v^2+1)}{(v+1)(v^2+3)} \tag{11}$$

and

$$t=\frac{v^4+10v^2+5}{4(v+1)(v^2+1)}. \tag{12}$$

Eliminate v from (11) and (12) and encounter

$$(t^2-t+1)z^4-(t^3-t^2+2t+2)z^3+(t^2+t+4)z^2-(t+3)z+1=0,$$

which is the desired result. □

3.5. Equation Between y and t.

Theorem 11. *If* $0<q<1$ *and let*

$$y:=\frac{(q^3;q^3)_\infty(q^5;q^3)_\infty}{(q^2;q^3)_\infty(q^6;q^3)_\infty}$$

and

$$t:=\frac{(q^5;q^5)_\infty(q^9;q^5)_\infty}{(q^4;q^5)_\infty(q^{10};q^5)_\infty}$$

then,

$$y^4-(3t+1)y^3+(2t^2+3t+1)y^2-(2t^2+t+1)y+t^2-t+1=0.$$

Proof. Let $q\to(1-v)/(1+v)$ in the right hand side of Examples 5 and 7

$$y=\frac{v^2+3}{2(v+1)} \tag{13}$$

and

$$t = \frac{v^4 + 10v^2 + 5}{4(v+1)(v^2+1)}. \tag{14}$$

Eliminate v from (13) and (14) and encounter

$$y^4 - (3t+1)y^3 + (2t^2 + 3t + 1)y^2 - (2t^2 + t + 1)y + t^2 - t + 1 = 0,$$

which is the desired result. □

Corollary 12. *If* $0 < q < 1$ *and* $n \in \mathbb{N}^+$*, then*

$$\frac{(q^{n+1}; q^{n+1})_\infty (q^{2n+1}; q^{n+1})_\infty}{(q^n; q^{n+1})_\infty (q^{2n+2}; q^{n+1})_\infty} = \frac{1 - q^{n+1}}{1 - q^n},$$

where $(z;q)_\infty$ *denotes the q-Pochhammer Symbol.*

Proof. Using the elementary property

$$(a;q)_\infty = \prod_{k=0}^{n-1} (aq^k; q^n)_\infty$$

setting $n \to n+1$

$$(a;q)_\infty = \prod_{k=0}^{n} (aq^k; q^{n+1})_\infty,$$

applying in $(q^{n+1}; q)_\infty$, $(q^n; q)_\infty$ and $(q^{n+2}; q)_\infty$, we encounter

$$(q^{n+1}; q)_\infty = \prod_{k=0}^{n} (q^{n+k+1}; q^{n+1})_\infty, \tag{15}$$

$$(q^n; q)_\infty = \prod_{k=0}^{n} (q^{n+k}; q^{n+1})_\infty \tag{16}$$

and

$$(q^{n+2}; q)_\infty = \prod_{k=0}^{n} (q^{n+k+2}; q^{n+1})_\infty \tag{17}$$

Substitute the rigth hand side from (15), (16) and (17) in the left hand side of Corollary 3, and obtain

$$\begin{aligned}\frac{(q^{n+1};q)_\infty^2}{(q^n;q)_\infty (q^{n+2};q)_\infty} &= \prod_{k=0}^{n} \frac{(q^{n+k+1}; q^{n+1})_\infty^2}{(q^{n+k}; q^{n+1})_\infty (q^{n+k+2}; q^{n+1})_\infty} \\ &= \frac{(q^{n+1}; q^{n+1})_\infty^2}{(q^n; q^{n+1})_\infty (q^{n+2}; q^{n+1})_\infty} \cdot \ldots \cdot \frac{(q^{2n+1}; q^{n+1})_\infty^2}{(q^{2n}; q^{n+1})_\infty (q^{2n+2}; q^{n+1})_\infty} \\ &= \frac{(q^{n+1}; q^{n+1})_\infty (q^{2n+1}; q^{n+1})_\infty}{(q^n; q^{n+1})_\infty (q^{2n+2}; q^{n+1})_\infty}.\end{aligned} \tag{18}$$

From (18) and Corollary 3, it follows the desired identity. □

Exercise 1. Prove that

$$\boldsymbol{\frac{(q^2;q)_\infty^2}{(q;q)_\infty (q^3;q)_\infty} = \frac{(q^2;q^2)_\infty (q^3;q^2)_\infty}{(q;q^2)_\infty (q^4;q^2)_\infty} = 1 + q = \frac{1-q^2}{1-q}}$$

$$\frac{(q^3;q)_\infty^2}{(q^2;q)_\infty (q^4;q)_\infty} = \frac{(q^3;q^2)_\infty (q^4;q^2)_\infty}{(q^2;q^2)_\infty (q^5;q^2)_\infty} = \frac{1+q+q^2}{1+q} = \frac{1-q^3}{1-q^2},$$

$$\frac{(q^4;q)_\infty^2}{(q^3;q)_\infty (q^5;q)_\infty} = \frac{(q^4;q^2)_\infty (q^5;q^2)_\infty}{(q^3;q^2)_\infty (q^6;q^2)_\infty} = \frac{1+q+q^2+q^3}{1+q+q^2} = \frac{1-q^4}{1-q^3},$$

$$\frac{(q^4;q^2)_\infty^2}{(q^2;q^2)_\infty (q^6;q^2)_\infty} = 1 + q^2 = \frac{1-q^4}{1-q^2},$$

$$\frac{(q^5;q)_\infty^2}{(q^4;q)_\infty (q^6;q)_\infty} = \frac{(q^5;q^2)_\infty (q^6;q^2)_\infty}{(q^4;q^2)_\infty (q^7;q^2)_\infty} = \frac{1+q+q^2+q^3+q^4}{1+q+q^2+q^3} = \frac{1-q^5}{1-q^4},$$

$$\frac{(q^6;q)_\infty^2}{(q^5;q)_\infty (q^7;q)_\infty} = \frac{(q^6;q^2)_\infty (q^7;q^2)_\infty}{(q^5;q^2)_\infty (q^8;q^2)_\infty} = \frac{1+q+q^2+q^3+q^4+q^5}{1+q+q^2+q^3+q^4} = \frac{1-q^6}{1-q^5},$$

$$\frac{(q^7;q)_\infty^2}{(q^6;q)_\infty (q^8;q)_\infty} = \frac{(q^7;q^2)_\infty (q^8;q^2)_\infty}{(q^6;q^2)_\infty (q^9;q^2)_\infty} = \frac{1+q+q^2+q^3+q^4+q^5+q^6}{1+q+q^2+q^3+q^4+q^5} = \frac{1-q^7}{1-q^6},$$

$$\frac{(q^8;q)_\infty^2}{(q^7;q)_\infty (q^9;q)_\infty} = \frac{(q^8;q^2)_\infty (q^9;q^2)_\infty}{(q^7;q^2)_\infty (q^{10};q^2)_\infty} = \frac{1+q+q^2+q^3+q^4+q^5+q^6+q^7}{1+q+q^2+q^3+q^4+q^5+q^6} = \frac{1-q^8}{1-q^7}.$$

Exercise 2. Prove that

$$\frac{(q^2;q)_\infty^2}{(q;q)_\infty(q^3;q)_\infty}=\frac{(q^2;q^3)_\infty(q^4;q^3)_\infty}{(q;q^3)_\infty(q^7;q^3)_\infty}=1+q=\frac{1-q^2}{1-q},$$

$$\boldsymbol{\frac{(q^3;q)_\infty^2}{(q^2;q)_\infty(q^4;q)_\infty}=\frac{(q^3;q^3)_\infty(q^5;q^3)_\infty}{(q^2;q^3)_\infty(q^6;q^3)_\infty}=\frac{1+q+q^2}{1+q}=\frac{1-q^3}{1-q^2},}$$

$$\frac{(q^3;q^3)_\infty(q^4;q^3)_\infty(q^5;q^3)_\infty}{(q;q^3)_\infty(q^6;q^3)_\infty(q^7;q^3)_\infty}=1+q+q^2=\frac{1-q^3}{1-q},$$

$$\frac{(q^4;q)_\infty^2}{(q^3;q)_\infty(q^5;q)_\infty}=\frac{(q^4;q^3)_\infty(q^6;q^3)_\infty}{(q^3;q^3)_\infty(q^7;q^3)_\infty}=\frac{1+q+q^2+q^3}{1+q+q^2}=\frac{1-q^4}{1-q^3},$$

$$\frac{(q^5;q)_\infty^2}{(q^4;q)_\infty(q^6;q)_\infty}=\frac{(q^5;q^3)_\infty(q^7;q^3)_\infty}{(q^4;q^3)_\infty(q^8;q^3)_\infty}=\frac{1+q+q^2+q^3+q^4}{1+q+q^2+q^3}=\frac{1-q^5}{1-q^4},$$

$$\frac{(q^6;q)_\infty^2}{(q^5;q)_\infty(q^7;q)_\infty}=\frac{(q^6;q^3)_\infty(q^8;q^3)_\infty}{(q^5;q^3)_\infty(q^9;q^3)_\infty}=\frac{1+q+q^2+q^3+q^4+q^5}{1+q+q^2+q^3+q^4}=\frac{1-q^6}{1-q^5},$$

$$\frac{(q^7;q)_\infty^2}{(q^6;q)_\infty(q^8;q)_\infty}=\frac{(q^7;q^3)_\infty(q^9;q^3)_\infty}{(q^6;q^3)_\infty(q^{10};q^3)_\infty}=\frac{1+q+q^2+q^3+q^4+q^5+q^6}{1+q+q^2+q^3+q^4+q^5}=\frac{1-q^7}{1-q^6},$$

$$\frac{(q^8;q)_\infty^2}{(q^7;q)_\infty(q^9;q)_\infty}=\frac{(q^8;q^3)_\infty(q^{10};q^3)_\infty}{(q^7;q^3)_\infty(q^{11};q^3)_\infty}=\frac{1+q+q^2+q^3+q^4+q^5+q^6+q^7}{1+q+q^2+q^3+q^4+q^5+q^6}=\frac{1-q^8}{1-q^7}.$$

Exercise 3. Prove that

$$\frac{(q^2;q)_\infty^2}{(q;q)_\infty(q^3;q)_\infty}=\frac{(q^2;q^4)_\infty(q^5;q^4)_\infty}{(q;q^4)_\infty(q^6;q^4)_\infty}=1+q=\frac{1-q^2}{1-q},$$

$$\frac{(q^3;q)_\infty^2}{(q^2;q)_\infty(q^4;q)_\infty}=\frac{(q^3;q^4)_\infty(q^6;q^4)_\infty}{(q^2;q^4)_\infty(q^7;q^4)_\infty}=\frac{1+q+q^2}{1+q}=\frac{1-q^3}{1-q^2},$$

$$\boldsymbol{\frac{(q^4;q)_\infty^2}{(q^3;q)_\infty(q^5;q)_\infty}=\frac{(q^4;q^4)_\infty(q^7;q^4)_\infty}{(q^3;q^4)_\infty(q^8;q^4)_\infty}=\frac{1+q+q^2+q^3}{1+q+q^2}=\frac{1-q^4}{1-q^3},}$$

$$\frac{(q^5;q)_\infty^2}{(q^4;q)_\infty(q^6;q)_\infty}=\frac{(q^5;q^4)_\infty(q^8;q^4)_\infty}{(q^4;q^4)_\infty(q^9;q^4)_\infty}=\frac{1+q+q^2+q^3+q^4}{1+q+q^2+q^3}=\frac{1-q^5}{1-q^4},$$

$$\frac{(q^6;q)_\infty^2}{(q^5;q)_\infty(q^7;q)_\infty}=\frac{(q^6;q^4)_\infty(q^9;q^4)_\infty}{(q^5;q^4)_\infty(q^{10};q^4)_\infty}=\frac{1+q+q^2+q^3+q^4+q^5}{1+q+q^2+q^3+q^4}=\frac{1-q^6}{1-q^5},$$

$$\frac{(q^7;q)_\infty^2}{(q^6;q)_\infty(q^8;q)_\infty}=\frac{(q^7;q^4)_\infty(q^{10};q^4)_\infty}{(q^6;q^4)_\infty(q^{11};q^4)_\infty}=\frac{1+q+q^2+q^3+q^4+q^5+q^6}{1+q+q^2+q^3+q^4+q^5}=\frac{1-q^7}{1-q^6},$$

$$\frac{(q^8;q)_\infty^2}{(q^7;q)_\infty(q^9;q)_\infty}=\frac{(q^8;q^4)_\infty(q^{11};q^4)_\infty}{(q^7;q^4)_\infty(q^{12};q^4)_\infty}=\frac{1+q+q^2+q^3+q^4+q^5+q^6+q^7}{1+q+q^2+q^3+q^4+q^5+q^6}=\frac{1-q^8}{1-q^7}.$$

Exercise 4. Prove that

$$\frac{(q^2;q)_\infty^2}{(q;q)_\infty(q^3;q)_\infty}=\frac{(q^2;q^5)_\infty(q^6;q^5)_\infty}{(q;q^5)_\infty(q^7;q^5)_\infty}=1+q=\frac{1-q^2}{1-q},$$

$$\frac{(q^3;q)_\infty^2}{(q^2;q)_\infty(q^4;q)_\infty}=\frac{(q^3;q^5)_\infty(q^7;q^5)_\infty}{(q^2;q^5)_\infty(q^8;q^5)_\infty}=\frac{1+q+q^2}{1+q}=\frac{1-q^3}{1-q^2},$$

$$\frac{(q^3;q^5)_\infty(q^6;q^5)_\infty}{(q;q^5)_\infty(q^8;q^5)_\infty}=1+q+q^2=\frac{1-q^3}{1-q},$$

$$\frac{(q^4;q)_\infty^2}{(q^3;q)_\infty(q^5;q)_\infty}=\frac{(q^4;q^5)_\infty(q^8;q^5)_\infty}{(q^3;q^5)_\infty(q^9;q^5)_\infty}=\frac{1+q+q^2+q^3}{1+q+q^2}=\frac{1-q^4}{1-q^3},$$

$$\boldsymbol{\frac{(q^5;q)_\infty^2}{(q^4;q)_\infty(q^6;q)_\infty}=\frac{(q^5;q^5)_\infty(q^9;q^5)_\infty}{(q^4;q^5)_\infty(q^{10};q^5)_\infty}=\frac{1+q+q^2+q^3+q^4}{1+q+q^2+q^3}=\frac{1-q^5}{1-q^4},}$$

$$\frac{(q^6;q)_\infty^2}{(q^5;q)_\infty(q^7;q)_\infty}=\frac{(q^6;q^5)_\infty(q^{10};q^5)_\infty}{(q^5;q^5)_\infty(q^{11};q^5)_\infty}=\frac{1+q+q^2+q^3+q^4+q^5}{1+q+q^2+q^3+q^4}=\frac{1-q^6}{1-q^5},$$

$$\frac{(q^7;q)_\infty^2}{(q^6;q)_\infty(q^8;q)_\infty}=\frac{(q^7;q^5)_\infty(q^{11};q^5)_\infty}{(q^6;q^5)_\infty(q^{12};q^5)_\infty}=\frac{1+q+q^2+q^3+q^4+q^5+q^6}{1+q+q^2+q^3+q^4+q^5}=\frac{1-q^7}{1-q^6},$$

$$\frac{(q^8;q)_\infty^2}{(q^7;q)_\infty(q^9;q)_\infty}=\frac{(q^8;q^5)_\infty(q^{12};q^5)_\infty}{(q^7;q^5)_\infty(q^{13};q^5)_\infty}=\frac{1+q+q^2+q^3+q^4+q^5+q^6+q^7}{1+q+q^2+q^3+q^4+q^5+q^6}=\frac{1-q^8}{1-q^7}.$$

4. The q-Bracket or q-Number: From finite at Infinite

4.1. An Newsworthy Lemma.

Lemma 13. *If* $0<z,q<1$*, then*

$$\frac{1-q}{1-zq}=\frac{(q;q)_\infty(zq^2;q)_\infty}{(q^2;q)_\infty(zq;q)_\infty},$$

where $(a;q)_\infty$ *denotes the q-Pochhammer Symbol.*

Proof. In previous paper [7], we proved that

$$\begin{aligned}\left(1-\frac{a}{q}\right)\frac{(b/q;q)_\infty}{(b;q)_\infty}&=\left(1-\frac{b}{q}\right)\frac{(a/q;q)_\infty}{(a;q)_\infty}\\ \Rightarrow\frac{1-\frac{a}{q}}{1-\frac{b}{q}}&=\frac{(a/q;q)_\infty(b;q)_\infty}{(a;q)_\infty(b/q;q)_\infty}.\end{aligned}\tag{19}$$

Replacing q^2 by a and zq^2 by b and encounter

$$\frac{1-q}{1-zq}=\frac{(q;q)_\infty(zq^2;q)_\infty}{(q^2;q)_\infty(zq;q)_\infty},$$

which is the desired result. □

Theorem 14. *We have*

$$\frac{(q^3;q)_\infty^2}{(q^2;q)_\infty(q^4;q)_\infty}=\frac{1-q^3}{1-q^2}=\frac{1+q+q^2}{1+q},$$

where $(a;q)_\infty$ *denotes the q-Pochhammer Symbol.*

Proof. In previous Lemma replace q^2 by z

$$\frac{1-q}{1-q^3}=\frac{(q;q)_\infty(q^4;q)_\infty}{(q^2;q)_\infty(q^3;q)_\infty}\Rightarrow\frac{1-q^3}{1-q}=\frac{(q^2;q)_\infty(q^3;q)_\infty}{(q;q)_\infty(q^4;q)_\infty}\tag{20}$$

and replace q by z

$$\frac{1-q}{1-q^2}=\frac{(q;q)_\infty(q^3;q)_\infty}{(q^2;q)_\infty(q^2;q)_\infty}.\tag{21}$$

Multiplying (20) by (21), we obtain

$$\frac{(q^3;q)_\infty^2}{(q^2;q)_\infty(q^4;q)_\infty}=\frac{1+q+q^2}{1+q}=\frac{1-q^3}{1-q^2},$$

which is the desired result. □

4.2. The *q*-Bracket or *q*-Number.

Theorem 15. *If* $q\in\mathbb{C}-\{1\}$*, then*

$$[k]_q=\frac{(q^2;q)_\infty(q^k;q)_\infty}{(q;q)_\infty(q^{k+1};q)_\infty},$$

where $[k]_q$ *denotes the q-bracket or q-number and* $(a;q)_\infty$ *denotes the q-Pochhammer Symbol.*

Proof. By previous Lemma, we have

$$\frac{1-zq}{1-q}=\frac{(q^2;q)_\infty(zq;q)_\infty}{(q;q)_\infty(zq^2;q)_\infty}.$$

Replacing q^{k-1} by z in previous equation, we find

$$\frac{1-q^k}{1-q}=\frac{(q^2;q)_\infty(q^k;q)_\infty}{(q;q)_\infty(q^{k+1};q)_\infty}.\tag{22}$$

On the other hand, I know that [2]

$$[k]_q:=\frac{1-q^k}{1-q}.\tag{23}$$

From (22) and (23), it follows that

$$[k]_q = \frac{(q^2;q)_\infty (q^k;q)_\infty}{(q;q)_\infty (q^{k+1};q)_\infty},$$

which is the desired result. □

5. The q-Factorial Function: From finite at Infinite

5.1. The q-Factorial Function.

Theorem 16. *If* $0<q<1$ *and* $n \in \mathbb{N}_{\geqslant 0}$*, then*

$$[n]_q! = \frac{(q^2;q)_\infty^n}{(q;q)_\infty^{n-1}(q^{n+1};q)_\infty},$$

where $[n]_q!$ *denotes the q-factorial function and* $(a;q)_\infty$ *denotes the q-Pochhammer Symbol.*

Proof. I know that [3, p. 491, Eq. 2.2]

$$[n]_q! = \prod_{k=1}^{n} [k]_q. \tag{24}$$

Replacing the right hand side of the Theorem 15 in the right hand side of (24), we obtain

$$\begin{aligned}[n]_q! &= \prod_{k=1}^{n} \left[\frac{(q^2;q)_\infty (q^k;q)_\infty}{(q;q)_\infty (q^{k+1};q)_\infty}\right] \\ &= \frac{(q^2;q)_\infty^n}{(q;q)_\infty^{n-1}(q^{n+1};q)_\infty},\end{aligned}$$

which is the desired result. □

6. The q-Binomial Coefficient or Gaussian Binomial Coefficient: From finite at Infinite

6.1. The q-Binomial Coefficient.

Theorem 17. *If* $0<q<1$ *and* $n, r \in \mathbb{N}$*, then*

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{(q^{r+1};q)_\infty (q^{n-r+1};q)_\infty}{(q;q)_\infty (q^{n+1};q)_\infty},$$

where $\begin{bmatrix} n \\ r \end{bmatrix}_q$ *denotes the q-binomial coefficient or Gaussian binomial coefficient and* $(a;q)_\infty$ *denotes the q-Pochhammer Symbol.*

Proof. I know that [2]

$$\begin{bmatrix} n \\ r \end{bmatrix}_q := \frac{[n]_q!}{[r]_q![n-r]_q!}. \tag{25}$$

Replacing the right hand side of the Theorem 16 in the right hand side of (25), we obtain

$$\begin{aligned}\begin{bmatrix} n \\ r \end{bmatrix}_q &= \frac{(q^2;q)_\infty^n (q;q)_\infty^{r-1}(q^{r+1};q)_\infty (q;q)_\infty^{n-r-1}(q^{n-r+1};q)_\infty}{(q;q)_\infty^{n-1}(q^{n+1};q)_\infty (q^2;q)_\infty^r (q^2;q)_\infty^{n-r}} \\ &= \frac{(q^{r+1};q)_\infty (q^{n-r+1};q)_\infty}{(q;q)_\infty (q^{n+1};q)_\infty},\end{aligned}$$

which is the desired result. □

Corollary 18. *We have*

$$\frac{(q^2;q)_\infty}{(q;q)_\infty} = \frac{1}{1-q}.$$

Proof. In [4], for $n \in \mathbb{N}$, we have

$$[n]_q! = \Gamma_q(n+1). \tag{26}$$

and, in [5],

$$\Gamma_q(n) = \frac{(q;q)_\infty}{(q^n;q)_\infty}(1-q)^{1-n}. \tag{27}$$

Replacing $n+1$ by n in both members of (27), we find

$$\Gamma_q(n+1) = \frac{(q;q)_\infty}{(q^{n+1};q)_\infty(1-q)^n}. \tag{28}$$

From the Theorem 16 and (28), we obtain

$$\frac{(q;q)_\infty}{(q^{n+1};q)_\infty(1-q)^n} = \frac{(q^2;q)_\infty^n}{(q;q)_\infty^{n-1}(\,q^{n+1};q)_\infty}$$
$$\Rightarrow \frac{1}{(1-q)^n} = \frac{(q^2;q)_\infty^n}{(q;q)_\infty^{n-1}} \Leftrightarrow \frac{1}{1-q} = \frac{(q^2;q)_\infty}{(q;q)_\infty},$$

which is the desired result. □

Note 19. Letting $q \rightarrow q^2$ in previous Corollary, we get:

$$\frac{(q^4;q^2)_\infty}{(q^2;q^2)_\infty} = \frac{1}{1-q^2}. \tag{29}$$

Similarly,

$$\frac{(q^6;q^3)_\infty}{(q^3;q^3)_\infty} = \frac{1}{1-q^3}. \tag{30}$$

Corollary 20. *We have*

$$\frac{(q^2;q)_\infty^n}{(q;q)_\infty^{n-1}(\,q^{n+1};q)_\infty} = \frac{(q;q)_n}{(1-q)^n}.$$

Proof. In [4], for $n \in \mathbb{N}$, we have

$$[n]_q! = \frac{(q;q)_n}{(1-q)^n}. \tag{31}$$

From the Theorem 16 and (31), we obtain

$$\frac{(q;q)_n}{(1-q)^n} = \frac{(q^2;q)_\infty^n}{(q;q)_\infty^{n-1}(\,q^{n+1};q)_\infty},$$

which is the desired result. □

Example 21. Setting $n=2$ in previous Corollary, we get

$$\frac{(q^2;q)_\infty^2}{(q;q)_\infty(\,q^3;q)_\infty} = 1+q.$$

Setting $q \rightarrow q^2$ in previous Corollary, we get

$$\frac{(q^4;q^2)_\infty^2}{(q^2;q^2)_\infty(\,q^6;q^2)_\infty} = 1+q^2. \tag{32}$$

Setting $q \rightarrow q^3$ in previous Corollary, we get

$$\frac{(q^6;q^3)_\infty^2}{(q^3;q^3)_\infty(\,q^9;q^3)_\infty} = 1+q^3. \tag{33}$$

From (29) and (32), we get

$$\frac{(q^4;q^2)_\infty}{(q^6;q^2)_\infty} = 1-q^4.$$

From (30) and (33), we get

$$\frac{(q^6;q^3)_\infty}{(q^9;q^3)_\infty} = 1-q^6.$$

Exercise 5. Prove that

$$\frac{(q;q)_\infty}{(q^2;q)_\infty}=\frac{(q;q^2)_\infty}{(q^3;q^2)_\infty}=\frac{(q;q^3)_\infty}{(q^4;q^3)_\infty}=\frac{(q;q^4)_\infty}{(q^5;q^4)_\infty}=\frac{(q;q^5)_\infty}{(q^6;q^5)_\infty}=1-q,$$
$$\frac{(q^2;q)_\infty}{(q^3;q)_\infty}=\frac{(q^2;q^2)_\infty}{(q^4;q^2)_\infty}=\frac{(q^2;q^3)_\infty}{(q^5;q^3)_\infty}=\frac{(q^2;q^4)_\infty}{(q^6;q^4)_\infty}=\frac{(q^2;q^5)_\infty}{(q^7;q^5)_\infty}=1-q^2,$$
$$\frac{(q^3;q)_\infty}{(q^4;q)_\infty}=\frac{(q^3;q^2)_\infty}{(q^5;q^2)_\infty}=\frac{(q^3;q^3)_\infty}{(q^6;q^3)_\infty}=\frac{(q^3;q^4)_\infty}{(q^7;q^4)_\infty}=\frac{(q^3;q^5)_\infty}{(q^8;q^5)_\infty}=1-q^3,$$
$$\frac{(q^4;q)_\infty}{(q^5;q)_\infty}=\frac{(q^4;q^2)_\infty}{(q^6;q^2)_\infty}=\frac{(q^4;q^3)_\infty}{(q^7;q^3)_\infty}=\frac{(q^4;q^4)_\infty}{(q^8;q^4)_\infty}=\frac{(q^4;q^5)_\infty}{(q^9;q^5)_\infty}=1-q^4,$$
$$\frac{(q^5;q)_\infty}{(q^6;q)_\infty}=\frac{(q^5;q^2)_\infty}{(q^7;q^2)_\infty}=\frac{(q^5;q^3)_\infty}{(q^8;q^3)_\infty}=\frac{(q^5;q^4)_\infty}{(q^9;q^4)_\infty}=\frac{(q^5;q^5)_\infty}{(q^{10};q^5)_\infty}=1-q^5,$$
$$\frac{(q^6;q)_\infty}{(q^7;q)_\infty}=\frac{(q^6;q^2)_\infty}{(q^8;q^2)_\infty}=\frac{(q^6;q^3)_\infty}{(q^9;q^3)_\infty}=\frac{(q^6;q^4)_\infty}{(q^{10};q^4)_\infty}=\frac{(q^6;q^5)_\infty}{(q^{11};q^5)_\infty}=1-q^6,$$
$$\frac{(q^7;q)_\infty}{(q^8;q)_\infty}=\frac{(q^7;q^2)_\infty}{(q^9;q^2)_\infty}=\frac{(q^7;q^3)_\infty}{(q^{10};q^3)_\infty}=\frac{(q^7;q^4)_\infty}{(q^{11};q^4)_\infty}=\frac{(q^7;q^5)_\infty}{(q^{12};q^5)_\infty}=1-q^7.$$

Exercise 6. Prove that

$$\sum_{n=0}^{\infty}\left[\frac{(n+1)q^n}{1-q^{n+1}}-\frac{(n+2)q^{n+1}}{1-q^{n+2}}\right]=\frac{1}{1-q},$$
$$\sum_{n=0}^{\infty}\left[\frac{(n+z)q^{n+z-1}}{1-q^{n+z}}-\frac{(n+z+1)q^{n+z}}{1-q^{n+z+1}}\right]=\frac{z}{q^{1-z}(1-q^z)}.$$

Exercise 7. Prove that: if

$$A:=\frac{(q;q)_\infty}{(q^2;q)_\infty}=\frac{(q;q^2)_\infty}{(q^3;q^2)_\infty}=\frac{(q;q^3)_\infty}{(q^4;q^3)_\infty}=\frac{(q;q^4)_\infty}{(q^5;q^4)_\infty}=\frac{(q;q^5)_\infty}{(q^6;q^5)_\infty},$$
$$B:=\frac{(q^2;q)_\infty}{(q^3;q)_\infty}=\frac{(q^2;q^2)_\infty}{(q^4;q^2)_\infty}=\frac{(q^2;q^3)_\infty}{(q^5;q^3)_\infty}=\frac{(q^2;q^4)_\infty}{(q^6;q^4)_\infty}=\frac{(q^2;q^5)_\infty}{(q^7;q^5)_\infty},$$
$$C:=\frac{(q^3;q)_\infty}{(q^4;q)_\infty}=\frac{(q^3;q^2)_\infty}{(q^5;q^2)_\infty}=\frac{(q^3;q^3)_\infty}{(q^6;q^3)_\infty}=\frac{(q^3;q^4)_\infty}{(q^7;q^4)_\infty}=\frac{(q^3;q^5)_\infty}{(q^8;q^5)_\infty},$$

then,

$$A^2-2A=-B,\ A(B-1)=B-C,\ B^3-3B^2+3B=(2-C)C \text{ and } A(C-1)=B^2-2B+C.$$

Prove that: if

$$D:=\frac{(q^4;q)_\infty}{(q^5;q)_\infty}=\frac{(q^4;q^2)_\infty}{(q^6;q^2)_\infty}=\frac{(q^4;q^3)_\infty}{(q^7;q^3)_\infty}=\frac{(q^4;q^4)_\infty}{(q^8;q^4)_\infty}=\frac{(q^4;q^5)_\infty}{(q^9;q^5)_\infty},$$
$$E:=\frac{(q^5;q)_\infty}{(q^6;q)_\infty}=\frac{(q^5;q^2)_\infty}{(q^7;q^2)_\infty}=\frac{(q^5;q^3)_\infty}{(q^8;q^3)_\infty}=\frac{(q^5;q^4)_\infty}{(q^9;q^4)_\infty}=\frac{(q^5;q^5)_\infty}{(q^{10};q^5)_\infty},$$
$$F:=\frac{(q^6;q)_\infty}{(q^7;q)_\infty}=\frac{(q^6;q^2)_\infty}{(q^8;q^2)_\infty}=\frac{(q^6;q^3)_\infty}{(q^9;q^3)_\infty}=\frac{(q^6;q^4)_\infty}{(q^{10};q^4)_\infty}=\frac{(q^6;q^5)_\infty}{(q^{11};q^5)_\infty},$$

then,

$D^3-3D^2+3D=(2-F)F$, $D(F-1)=F+E^2-2E$, $D(E^4-4E^3+6E^2-4E+1)=-F^4+4F^3-6F^2+4F+E^4-4E^3+6E^2-4E$, $E^6-6E^5+15E^4-20E^3+15E^2-6E=F(-F^4+5F^3-10F^2+10F-5)$, $D^2(E^2-2E+1)+D(-2E^2+4E-2)=-F^3+3F^2-3F-E^2+2E$.

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