Division by Zero and the Arrival of *Ada*

I Gede Putra Yasa "Gus Satya"

Dedicated to: my son I Gede Sathya Nandishvara

Abstract

Division by 0 is not defined in mathematics. Mathematics suggests solutions by work around methods. However they give only approximate, not the actual or exact, results. Through this paper we propose methods to solve those problems. One characteristic of our solution methods is that they produce actual or exact results. They are also in conformity with, and supported by, physical or empirical facts. Other characteristic is their simplicity. We can do computations easily based on basic arithmetic or algebra or other computation methods we already familiar with.

1 Introduction

This work in essence is the duplicate of the paper titled One, Zero, *Ada*, and Cyclic Number System by I Gede Putra Yasa "Gus Satya" [16] with the title changed to the new one above and with expanded introduction section. The new title is needed so that it will be easier for scientific community and general public especially the ones who interested in works related to problems of division by 0, to find the paper online. And the expansion of introduction section mainly is to accommodate more explanations of the entity of fullness or allness including it's number representative.

We have problems, paradoxes, indeterminations, confusions, and so on related to division by 0. The source of those problems actually is the neglect of an entity that is fullness or allness, including it's number representative. Our solutions to those division by zero problems include the involvement of a number representing allness, A (IPA $\wedge$, pronounced like in c*u*p, l*u*ck) with it's value of $\frac{1}{0}$. For the purpose of discussions in this paper, let's call that number *Ada* (with IPA $\wedge$ pronunciation for it's vocal A or a) or it's abbreviation A. *Ada* is a word in Indonesian language that means exist.

We may feel comfortable to think and talk about emptiness or nothingness or it's number representative 0. We say 0 x any number is $= 0$, $0 +$ any number is $=$ any number, etc. However, seldom we see discussions or speeches on fullness or allness, and it's number representative is rarely if ever be mentioned, maybe because it is unknown or maybe because of it's usage is not pressingly needed at those moments. Perhaps for us to accept concept of nothingness as compared to allness is easier. Maybe imagining that there are 2 mangoes that become 1 mango because a subtraction by 1 mango and then become 0 mango when it subtracted again by 1 mango is easier than imagining to add 1 mango to 0 mango again and again until the whole existent is full of mangoes or is mango. It maybe so. Although allness is actually can be quite straightforward to imagine too, our self, as well as those outside and inside us.

There are 3 number brothers so to speak: One (1), Zero (0), and *Ada* (A). They respectively represent the self-the owner-or the intrinsic divisor, emptiness or nothingness, and fullness or allness. *Ada* is lost sometime somewhere and leaves 1 and 0 to fight all alone for all mathematical problems and calculations of the world. Everything was worked fine and perfect until more and more demanding mathematical questions, problems, and also some confusions that need a certain level of calculations or solutions to solve, emerged. Then it is time for *Ada* to be invited, accepted and embraced once again and reunite it again with it's beloved brothers One and Zero. And together they are able to solve the problems of mathematics including those currently unsolvables that include division by 0 .

Allness and Ada, just like the self-the owner-or the intrinsic divisor and 1 and emptiness or nothingness and 0 , is fundamental as an aspect of nature or reality and mathematics, and cannot be ignored, as shown in this work. The absent of A creates confusions and problems. Some examples, that n x 0 is assumed to be $= 0$ for any number n , then on assumption that $\frac{0}{0}$ is not $= 1$, problems called Zeno paradoxes and other problems or paradoxes, discontinuity of functions, on why Bhaskaracarya's rule regarding multiplication and division by 0 is not accurate, confusion on relation between 0 and ∞ , and so on.

In this work beside the acceptance of A we also propose an expanded number system, for the purpose of discussions in this paper, let's call it cyclic number system, in which it's members includes real numbers, with the addition of A and it's outcomes of operations with itself and other numbers. Let's call those members cyclic numbers. Along with cyclic number system and cyclic numbers, we also introduce their 2-dimensional coordinate system, let's call it cyclic coordinate system. Cyclic number system, cyclic numbers, and cyclic coordinate system, all accept the factual and mathematical validity and definability of A .

This work was started in author of this paper's college days in 1981 in Sepuluh Nopember Institute of Technology, located in Surabaya, East Java, Indonesia. In 1987, the previous version of this paper was reviewed and rejected by IBM Corp. headquarters in the US, as it deemed not applicably useful to the company, at that time the author of this paper was working at PT. USI/IBM, an Indonesian subsidiary of IBM Corp. The work was continued in 2007, then discontinued, and continued again on-and-off since 2013.

PT. Bali Ning, Jl. Raya Sading 141D, Kab. Badung, Bali, Indonesia, Email: *putragussatya@gmail.com.*
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1.1 Some works related to solutions to the problems of division by 0.

Some works by researchers related to solutions to the problems of division by 0 are available, such as by Setzer [5] and Carlstrom [6]; with two extra numbers, $\infty = \frac{1}{0}$ and $\perp = \frac{0}{0}$, be adjoined to the set of real numbers, Miller [7], that works on $\frac{0}{0}$ based on the presence of the so called subspace directions; Barukcic [8], that works on $\frac{0}{0}$ through Einstein's theory of special relativity; Abubakr [9], and Bhaskaracarya [10]. All those works support some form of definability of division by 0 or at least definability of $\frac{0}{0}$.

Abubakr's work [9] and Bhaskaracarya's work [10] interestingly have some statements that have similarities with some of our work's statements. Abubakr [9] statements that we are in conformity with:

- addition of cyclic numbers is not necessarily associative, we also in conformity with Abubakr [9] on following a convention to perform addition from left to right,
- addition of cyclic numbers is not necessarily commutative,
- multiplication of cyclic numbers is not necessarily commutative,
- for every number a, a - a = a(0),
- -1 + 1 = -0,
- acceptance to the validity of $\frac{1}{0}$, in Abubakr's [9] $\frac{1}{0}=$ devanagari "ka", in our work $\frac{1}{0} = A$,
- for every number a, a(0) = a(0).

  Abubakr [9] states, "An observer or a machine performing mathematical calculations must not create or destroy information using zero," or about something related to "conservation of information".

  Bhaskaracarya [10] states "The product of cipher is nought : but it must be retained as a multiple of cipher, if any further operation impend. Cipher having become a multiplier, should nought afterwards become a divisor, the definite quantity must be understood to be unchanged."

  And we state in this paper "The product of 0 or A must be kept as multiple of 0 or A, or in the raw, original, or pristine form, except that, that is the last operation."

Considering as well those works, we continue to propose our solution methods to solve problems posed by division by 0 in general.

1.2 Definitions of words and notations.

For the purpose of discussions in this paper, let's define words and notations:

- *cyclically*, means with computation(s) that accept validity of cyclic number system,
- *cyclic calculation* or *cyclic operation*, means calculation, computation, or operation that accept validity of cyclic number system,
- for space efficiency especially in graphs and figures, let the notations of $\underline{n}$ indicate the value of $A+n$, so that $\underline{2}=A+2$, $\underline{+5}=$ A+5, $\underline{\text{-}3}=A-3$, and so on,
- let's define *infinity* (∞), as an infinitely large number, as a result of operation(s) that don't include division by 0,
- *Expression within brackets* "[]", means that it is the last operation, therefore it must not be in the raw form. For example, $[2(0)=0]$, $[A=-A]$. Note: we got inspiration to use this bracketing technique from Abubakr [9],
- *"Plain" expression or expression without brackets*, means that it is not the last operation, therefore it must be in the raw form. For example, $4(0)-0=3(0)$, $-1(0)=-0$. Explicit "non cyclically", "non cyclic calculation" or "non cyclic operation" statement related to the expression overrides this definition.

2 Cyclic number system

This paper proposes a new number system called cyclic number system.

Definition 2.1. *Cyclic number system is an extension of real number system to include as it's members, A and it's outcomes of operations with itself and other numbers. Members of cyclic number system are called cyclic numbers.*

As it's name suggest, cyclic number system has cyclic characteristic or nature. The cyclic nature of that number system is caused by the properties of 0 and A. In cyclic number system, 0 and A both act as turning points. Addition of positive or negative number to 0 will result in a number that is closer to A, and addition of positive or negative number to A will result in a number that is closer to 0.

2.1 Rules of cyclic number system.

If no additional information to the rules, that means those rules are or come from axioms. Other rules come from lemmas, corollaries, examples, propositions, and theorems.

Vital numbers that form cyclic number system are 1, 0, and A. We need to know their intrinsic and operational characteristics before we do calculations on cyclic number system. Those characteristics are also shown by the following rules.

Rule 1. 1 is the self, the owner, or the intrinsic divisor.

Rule 2. A number is actually a fraction with that number as nominator and 1 as denominator. So for example,

$$a = \frac{a}{1}, \quad 0.5 = \frac{0.5}{1}, \quad \text{and so on.}$$

Rule 3. For the question, "what is the quotient of any number divided by any number?", it means that for a fraction with numerator and denominator of any number, what is the quotient if the denominator is 1?

Statement $\frac{4}{2} = ?$ means that for a fraction with numerator of 4 and denominator of 2, what is the quotient if the denominator is 1? In this case the answer is $\frac{2}{1}$ or 2.

Rule 4. 0 is emptiness, or nothingness. 0 is not small nor large.

Before we can say something is small or large we must be able to sense it, at least by sight or touch. Because of emptiness or nothingness, we can't see or touch a thing therefore we can't say it is small or it's large.

Rule 5. 0 is not positive nor negative number, it's unsigned.
$+0 = +1(0)$, and $-0 = -1(0)$.

Rule 6. 0 is an even number.

Rule 7. $0 \neq \frac{1}{\infty}$, and $0 \neq \pm\frac{1}{\infty}$.

Rule 8. The product of 0 must be kept as multiple of 0, except if that is the last operation.

Lemma 2.2. *Rule 8 exist because for any numbers a and b, if $a \neq b$ then $a(0) \neq b(0)$.*

Proof.

$$\begin{aligned} \text{For } a \neq b \text{ then, } & a(1) \neq b(1), \\ \text{applying rule 11, therefore, } & a(0)(A) \neq b(0)(A), \\ \text{multiplying both side by 0 we get, } & a(0) \neq b(0). \end{aligned}$$

□

Corollary 2.3. *For every number a, $a(0) = 0$, if and only if $a = 1$.*

Proof. $a(0) = 0$, multiplying both side by A, we get, $a = 1$. □

Example 2.4. $3(0) = 3(0), but \neq 0$.

Corollary 2.5. *Proofs of lemma 2.2. and corollary 2.3. also apply to the rule expounded by Bhaskaracarya in his work Lilavati (translated by Colebrooke, with notes by H.C. Banerji) [10] in part of verses 44-45 that say "The product of cipher is nought : but it must be retained as a multiple of cipher, if any further operation impend. Cipher having become a multiplier, should nought afterwards become a divisor, the definite quantity must be understood to be unchanged."*

$$\textit{Example, by applying also rule 22, } \frac{a \times 0}{0} = a.$$

Please note that Bhaskaracarya's words of "multiple of cipher" and "if any further operation impend." give ideas of the wording of some of our rules.

Remark 2.6. *But interestingly, the note provided by H.C. Banerji seems to counter the rule expounded by Bhaskaracarya, as part of the note says "The rule, viz., "cipher having become a multiplier, &c," is not accurate. For* $\frac{a \times 0}{0} = \frac{0}{0} =$ *indeterminate, and, not* $= a$*, as the rule says."*

Corollary 2.7. *Proofs of lemma 2.2. and corollary 2.3. also apply to the rule expounded by Bhaskaracarya [10] in part of verse 46 that says "and what number it is, which multiplied by chiper, and added to half itself, and multiplied by three, and divided by chiper, amounts to a given number sixty-three." The notation from that statement as provided by the note of H.C. Banerji is:*

$$\frac{0 \times (x + \frac{1}{2}x) \times 3}{0} = 63. \quad \textit{x=14 is the solution given by the note.}$$

Rule 9. $[+0 = -0]$.

If further operation impend, then $+0 \neq -0$.
$+0$ is unsigned 0 with a potential to confirm the sign of the result of future operation. -0 is unsigned 0 with a potential to inverse the sign of the result of future operation.

0 is unsigned 0 with no potential to confirm nor inverse the sign of the result of future operation. For practicality however, it is considered that $0 = +0$.

Rule 10. A is fullness, or allness. A is not large nor small.

Before we can say something is large or small we must be able to sense it, at least by sight or touch. Because of fullness or allness, we can't see or touch a thing, because no space and no material for witness, since allness engulfs all. Without witness, we can't see or touch a thing, therefore we can't say it is large or small.

Another supporting proof that A is not large nor small is, 0 is not small nor large, $A = \frac{1}{0}$, therefore A is not large nor small.

Rule 11. $A = \frac{1}{0}$, therefore $A(0) = 1$, and $\frac{1}{A} = 0$.

Remark 2.8. *Theorem 3.1 in section 3 also gives proof to this rule.*

Rule 12. A is not positive nor negative number, it's unsigned.
$+A = +1(A)$, and $-A = -1(A)$.

Rule 13. A is an even number.

Rule 14. $A \neq \infty$, and $A \neq \pm\infty$.

Rule 15. A and 0 are number *poles*.

Remark 2.9. *A and 0 characteristics includes:*

a). in balance (related to rule 4 and 10),
- *not big nor small, unsigned,*

b). as a turning point,
- *addition of positive or negative number to 0 will result in a number that is closer to A,*
- *addition of positive or negative number to A will result in a number that is closer to 0,*

c). in opposition,
- *two opposites or contrasted qualities.*

Rule 16. The product of A should be kept as multiple of A, except if that is the last operation.

Lemma 2.10. *For any numbers a and b, If $a \neq b$ then $a(A) \neq b(A)$.*

Proof.

$$\begin{aligned} \text{For } a \neq b \text{ then, } a(1) &\neq b(1), \\ a(0)(A) &\neq b(0)(A), \\ \text{multiplying both side by } A \text{ we get, } a(A) &\neq b(A). \end{aligned}$$

□

Corollary 2.11. *For every number a, $a(A) = A$, if and only if $a = 1$.*

Proof. $a(A) = A$, multiplying both side by 0 we get, $a = 1$. □

Example 2.12. $3(A) = 3(A), but \neq A$.

Rule 17. $[+A = -A]$.

If further operation impend, then $+A \neq -A$.
$+A$ is unsigned A with a potential to confirm the sign of the result of future operation. $-A$ is unsigned A with a potential to inverse the sign of the result of future operation.

A is unsigned A with no potential to confirm nor inverse the sign of the result of future operation. For practicality however, it is considered that $A = +A$.

Rule 18. For every number a, $a - a = a(0)$.

Lemma 2.13. *For every number a, $a - a = a(0)$.*

Proof.
$$a - a = a(1-1) = a(0).$$
□

Example 2.14. *Examples:*

a) 1-1 = 1(1-1) = 1(0) = 0,
b) -1+1 = (-1)-(-1) = -1(1-1) = -0,
c) 2-2 = 2(0),
d) -2+2 = -2(0),
e) A-A = 1,
f) -A+A = -1,
g) (1-1)+1 = 0+1,
h) 1+(-1+1) = 1-0.

Rule 19. Commutative property of addition is not totally extendable to cyclic numbers.

Remark 2.15. *Addition of cyclic numbers is not necessarily commutative. This is especially true for addition involving additive inverse numbers, as shown by example 2.14, points a) through f).*

Rule 20. Associative property of addition is not totally extendable to cyclic numbers.

Remark 2.16. *Addition of cyclic numbers is not necessarily associative. This is especially true for addition involving additive inverse numbers, as shown by example 2.14, points g) and h).*

Rule 21. Commutative property of multiplication is not totally extendable to cyclic numbers.

Remark 2.17. *Multiplication of cyclic numbers is not necessarily commutative, as shown by equation (9) and (10) in sub-subsection 5.2.2.*

Rule 22. $\frac{0}{0} = 1$.

Lemma 2.18. *This is related to rule 2 and 3.*

The equation $\frac{0}{0} = ?$ *, has a meaning that in a fraction, if the numerator is* 0 *and the denominator is also* 0 *, what is the value of the numerator if the denominator is* 1 *? In this particular case, since the numerator equal the denominator, the answer is* 1 *. So that*

$$\frac{0}{0} = \frac{1}{1} = 1.$$

But, let say that the answer of $\frac{0}{0}$ *is* n *. Is it correct to say that* n *can be any number since any number multiplied by* 0 *equal to* 0 *? The answer is no.*

Proof.

$$\text{Let's say } \frac{0}{0} = n, \text{ therefore } 0 = 0(n),$$
$$\text{multiplying both side by} A,\ A(0) = A(0)n, 1 = n.$$
$$\text{Therefore, } \frac{0}{0} = n \text{ if and only if } n = 1.$$

□

Rule 23. $\frac{A}{A} = 1$.

Lemma 2.19. *This is related to rule 2 and 3.*

The expression $\frac{A}{A} = ?$ *, has a meaning that if the numerator is* A *and the denominator is also* A *, what is the value of the numerator if the denominator is* 1 *? Since the numerator equal the denominator, the answer is* 1 *.*

Proof.

$$\frac{A}{A} = \frac{\frac{1}{0}}{\frac{1}{0}} = \frac{1}{0}\left(\frac{0}{1}\right) = \frac{0}{0},$$
$$\text{applying rule 22, } \frac{A}{A} = 1.$$

□

3 An empirical and mathematical validity of A

Now let's discuss an empirical and mathematical case, in which it's solution supports the factual and mathematical validity and definability of A. That case is related to one of Zeno's problems called Achilles and the tortoise paradox.

The problem says that in a race, the pursuer with higher speed will never reach and overtake the pursued with lower speed because each time the pursuer reach the previous position of the pursued, the pursued has move ahead although with smaller distance than that of the pursuer. This situation repeats indefinitely, therefore the pursuer will never reach and overtake the pursued.

The problem can be rephrased as, the pursuer with higher speed will never reach and overtake the pursued with lower speed, since the magnitude of their last distance to each other is never $= 0$.

In factual reality however, that assertion is not true since the pursuer with higher speed will reach and overtake the pursued with lower speed. This is a valid empirical and mathematical problem.

Theorem 3.1. *In a race, the pursuer with higher speed will reach and overtake the pursued with lower speed.*

Let's examine mathematically a situation related to that Achilles and the tortoise problem.

Suppose that the speeds of object 1 and object 2 respectively is 1 meter (m) per second and $\frac{1}{2}$ m per second. And the head start distance given to object 2 is 1 m.

3.1 Non cyclically.

The approximate distance from starting point of object 1 to the point of the place object 1 will reach object 2 is shown by (1). Let's define step as a beginning of race for each term, with step number = term number - 1. Each term value of (1) represents the distance between the 2 objects at each step, so that the distance between the two objects at step 0 is = magnitude of term $1, = 1$ m.

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^\infty} \text{ m}, \tag{1}$$

$$\approx 2 \text{ m (approximately)}.$$

Let's see in (1) the problem faced related to the above Achilles and the tortoise paradox situation. If the quotient of the last term of it (i.e. $\frac{1}{2^\infty}$) is not 0 then how is it possible for object 1 to reach object 2? In order for object 1 to reach object 2, this last term's fraction's magnitude must be 0.

With ∞ as exponent of base 2 as part of the denominator, the fraction's magnitude of the last term of (1) which is $\frac{1}{2^\infty}$ is approaching 0, but not$= 0$.

3.2 Cyclically.

There is solution to that problem, so that the pursuer with higher speed will reach and overtake the pursued with lower speed.

Proof. To conform to factual reality, mathematically for the pursuer with higher speed to reach and overtake the pursued with lower speed, their magnitude of last distance or the magnitude of last term of the series must be 0.

Replacing ∞ with $\log_2 A$ in (1), we get,

$$\sum_{k=0}^{\log_2 A} \frac{1}{2^k} = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{\log_2 \frac{A}{4}}} + \frac{1}{2^{\log_2 \frac{A}{2}}} + \frac{1}{2^{\log_2 A}} \text{ m}, \tag{2}$$

$$= 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{\frac{A}{4}} + \frac{1}{\frac{A}{2}} + \frac{1}{A} \text{ m},$$

$$= 1 + \frac{1}{2} + \frac{1}{4} + \cdots + 4(0) + 2(0) + 0 \text{ m},$$

Last term of (2) is term $\log_2 A + 1$, it's value is $\frac{1}{2^{\log_2 A}} = \frac{1}{A} = 0$ m.

Object 1 reaches object 2 at step $\log_2 A$, and at the next step, it overtakes object 2.

Together with the empirical fact, this calculation proves, that the last distance of 0 is achievable, and proves as well theorem 3.1. This proves factual as well as mathematical validity and definability of a fraction with 0 denominator and non 0 nominator, this also proves factual as well as mathematical validity and definability of A, with it's value of $\frac{1}{0}$. □

Corollary 3.2. *Position of object 1 and object 2 when object 1 reaches object 2, and after. From (2),*

$$\sum_{k=0}^{\log_2 \mathrm{A}} \frac{1}{2^k} = p = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^{\log_2 \frac{A}{4}}} + \frac{1}{2^{\log_2 \frac{A}{2}}} + \frac{1}{2^{\log_2 \mathrm{A}}} \; m, \tag{3}$$

$$p = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{\frac{A}{4}} + \frac{1}{\frac{A}{2}} + \frac{1}{A} \; m, \tag{4}$$

$$\frac{p}{2} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{\frac{A}{2}} + \frac{1}{A} + \frac{1}{2A} \ m,$$
$$\left(p - \frac{p}{2}\right) = 1 - \frac{1}{2A} \ m, \Rightarrow \frac{p}{2} = 1 - \frac{0}{2} \ m,$$
$$p = 2 - 0 \ m.$$

The position of object 1 when it reaches object 2 is $2-0$ *m from object 1's starting point.*

When object 1 start leaving point $2-0$*, we apply (5) which is the horizontal flip of (4) to calculate it's positions.*

$$q = 0 + 2(0) + 4(0) + \cdots + \frac{1}{4} + \frac{1}{2} + 1 \ m. \tag{5}$$

In the next step or with addition of term 1 of (5) after overtaking object 2, object 1 reaches the point of 2 *m, and object 2 reaches the point of* $2-\frac{0}{2}$ *m. In the next step or with addition of term 2 of (5) object 1 reaches* $2+2(0)$ *m and object 2 reaches* $2+\frac{0}{2}$ *m.*

Theorem 3.3. *There is a first distance to move.*

Now let's see another mathematical problem. The situation is similar to another Zeno's problem called Dichotomy paradox.

Suppose there is an object to move from first to second point. Before it reaches second point, the object must reach the half distance, before that it must reach a quarter, before that it must reach one eight, and so on indefinitely to the $\frac{1}{\infty}$. According to this problem an object cannot move to even cover it's first distance, since the magnitude of supposed first distance is unknown because the distance can always be divided by 2.

For a question, is there a first distance for an object to move, and what is it's magnitude? The answer is yes, and 0.

Proof. Cyclically, we can continue to divide the distance of the object to move by 2 to the magnitude of

$$\frac{1}{2^{\log_2 A}} = \frac{1}{A} = 0.$$

Just as 0 is the magnitude of the last distance as shown by theorem 3.1, 0 is the magnitude of the first distance as well, for an object to move. This is also indicated by corollary 3.2.

The distance of 0 multiplied by multiple of A steps becomes an observable distance, such as $\frac{1}{\infty}$, $\frac{1}{1000}$, $\frac{1}{2}$, 1, 2, and so on. □

4 Cyclic coordinate system

Proposition 4.1. *Cyclic coordinate system.*

Definition 4.2. *Cyclic coordinate system can be visualized as 2 Euclidean 2-dimensional Cartesian coordinate systems, one that has 0,0 as it's origin and the other that has* A, A *as it's origin, wrapped to a spherical form opposite to each other, it's not to scale.*

In this paper, darker colored lines indicate that they are in the nearer side of the sphere, the lighter colored lines indicate that they are in the farther side of the sphere.

For simplicity, cyclic coordinate system can also be in the form of 2 separate Euclidean 2-dimensional Cartesian coordinate system planes, one that has $0, 0$ *as it's origin (let's call it* 0 *plane), and the other that has* A, A *as it's origin (let's call it* A *plane). These plane forms are also not to scale.*

Cyclic coordinate system is a supporting tool to visually represent cyclic number system. It shows some properties of relations between cyclic numbers.

Here's the representation of cyclic coordinate system,

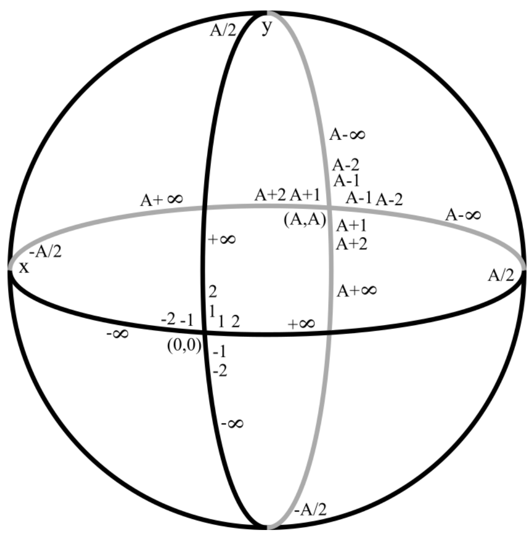

Figure 1: Cyclic coordinate system.

In cyclic coordinate system we can observe that:

- 0 *and* A *are respectively placed between odd numbers in number lines.* 0 *is even number, and so also* A *.*

- 0 *and* A *are respectively placed between positive and negative numbers in number lines.* 0 *is unsigned number, and so also* A *.*

- *0 and A can be understood as poles, their properties include unsigned and as a turning points. A can be seen as the opposite pole of 0 and vice versa.*

Proposition 4.3. *Graphs of functions in cyclic number system. Some examples;*

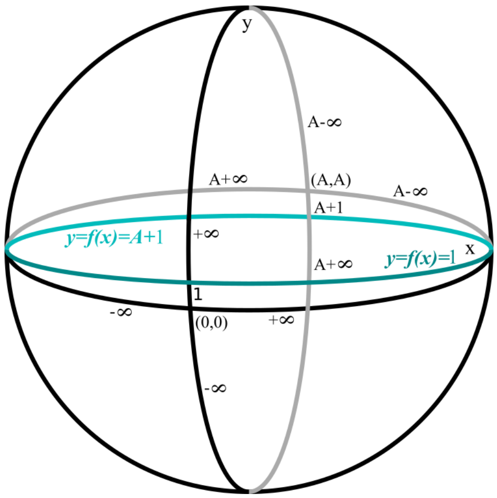

Figure 2: Graph of $y = 1$ and $y = A + 1$.

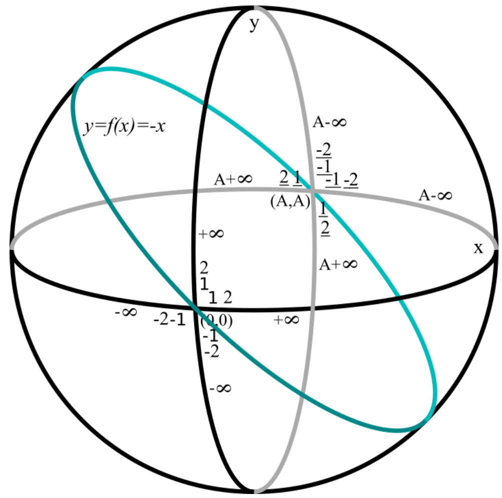

Figure 3: Graph of $y = -x$.

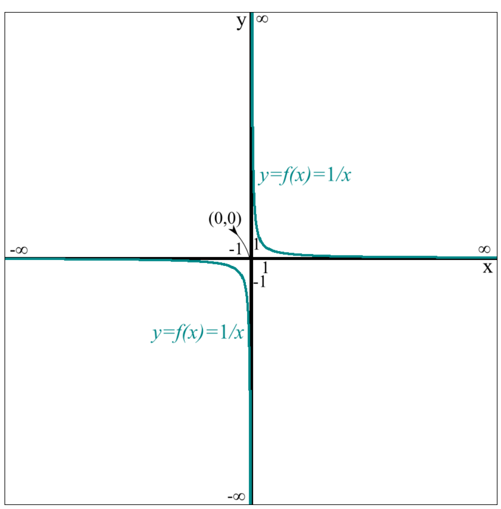

Figure 4: Graph of $y = \frac{1}{x}$, in 0 plane.

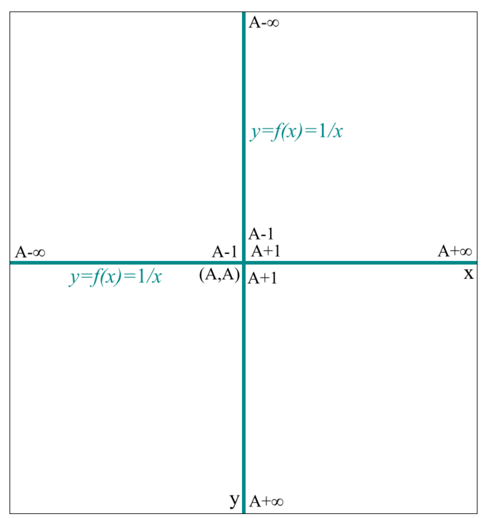

Figure 5: Graph of $y = \frac{1}{x}$, in A plane.

Please also note that cyclically, as also shown in Figure 4 and 5, $y = \frac{1}{x}$ is definable at $x = 0$, with $y = A$, therefore function $y = \frac{1}{x}$ is also continuous.

5 More examples of cyclic calculations

5.1 The case of Grandi's series.

$$\text{Series } S=\sum_{n=0}^{\infty}(-1)^n=1-1+1-1+1-1+\cdots \tag{6}$$

is also known as Grandi's series. Let's find sum of that series, if it exists.

5.1.1 Non cyclically.

With (6),

$$S=\sum_{n=0}^{\infty}(-1)^n=1-1+1-1+1-1+\cdots$$

$$\begin{aligned}
&\text{1st method, } S=(1-1)+(1-1)+(1-1)+\cdots=0+0+0+\cdots=0,\\
&\text{2nd method, } S=1+(-1+1)+(-1+1)+\cdots=1+0+0+\cdots=1,\\
&\text{3rd method, } 1-S=1-(1-1+1-1+\cdots)=1-1+1-1+\cdots=S\\
&\qquad 1=2S, \text{therefore } S=\frac{1}{2}.
\end{aligned} \tag{7}$$

There are more answers with certain ways of efforts to find it's sum. In mathematics, this series is considered has no sum.

5.1.2 Cyclically.

Lemma 5.1. *In cyclic number system we can work that series out, to find it's sum.*

Proof. Grandi's series in cyclic number system,

$$S=\sum_{n=0}^{A-1}(-1)^n=1-1+1-1+\cdots-1\leftarrow \text{ term } A. \tag{8}$$

Series (8) has A number of terms, that means an even number of terms.

$$\begin{aligned}
\text{1st method: } S &= (1-1)+(1-1)+\cdots+(1-1) \leftarrow \text{term}\left(\frac{1}{2}A\right)\\
S &= 0\left(\frac{1}{2}A\right)=\frac{1}{2},\\
\text{2nd method: } S &= 1+1+1+\cdots+1 \leftarrow \text{term}\left(\frac{1}{2}A\right)\\
&\quad + -1-1-1-\cdots-1 \leftarrow \text{term}\left(\frac{1}{2}A\right)\\
\text{applying rule 18, } S &= \left(\frac{1}{2}A\right)-\left(\frac{1}{2}A\right)=\left(\frac{1}{2}A\right)0=\frac{1}{2},\\
\text{3rd method: } 1-S &= 1-(1-1+1-1+\cdots-1 \leftarrow \text{ term } A)=1-\frac{1}{2}\\
S &= \frac{1}{2}.
\end{aligned}$$

These calculations show that the series has a sum, with value of $\frac{1}{2}$. □

Corollary 5.2. *Considering rule 19 and 20, meanwhile, the bracketing technique applied to (7) changes the original series S to a different new series T, which is actually the additive inverse of the original.*

$$\begin{aligned}
T &= 1+(-1+1)+(-1+1)+\cdots+ \mathit{term}\left(\frac{1}{2}A-1\right)\rightarrow(-1+1)+(-1),\\
T &= (-1+1)+(-1+1)+\cdots+(-1+1)\leftarrow \mathit{term}\left(\frac{1}{2}A\right)=-\frac{1}{2}.
\end{aligned}$$

Corollary 5.3. *Calculating sum of an example series S with certain odd number of terms.*

$$S=\sum_{n=0}^{A-2}(-1)^n=1-1+1-1+\cdots+1\leftarrow \mathit{term}\ (A\text{-}1)=1\frac{1}{2}.$$

5.2 Working with and without limit.

5.2.1 Limit case 1.

Non cyclically.

As x approaches ∞ the limit approaches the value of 2.

$$\lim_{x\to\infty} \frac{2x-1}{x} \approx 2.$$

Cyclically.

$$\text{Let } x = A \Rightarrow \frac{2A-1}{A} = \frac{A(2-0)}{A} = 2-0.$$

5.2.2 Limit case 2.

$$f(x) = \frac{x^2-1}{x-1}.$$

Non cyclically, that function has no definition on $x = 1$ (because of the division by 0), instead, on x approaching 1 we use limit,

$$\lim_{x\to 1} \frac{x^2-1}{x-1} \approx 2, \text{ when } x \text{ approaches 1, the outcome approaches 2.}$$

Cyclically,

$$(x+1)(x-1) = x^2 + x - x - 1 = (x^2-1) + 0x,$$

$$(x^2-1) = (x+1)(x-1) - 0x, \tag{9}$$

$$(x-1)(x+1) = x^2 - x + x - 1 = (x^2-1) - 0x,$$

$$(x^2-1) = (x-1)(x+1) + 0x, \tag{10}$$

$$\text{from (9), } \frac{x^2-1}{x-1} = (x+1) - \frac{0x}{x-1}, \tag{11}$$

$$\text{from (10), } \frac{x^2-1}{x-1} = (x+1) + \frac{0x}{x-1}, \tag{12}$$

$$(x+1) - \frac{0x}{x-1} = (x+1) + \frac{0x}{x-1}, \text{ so that } -\frac{0x}{x-1} = +\frac{0x}{x-1}.$$

From rule 9 and 17, since 0 and A share a unique quality to have properties of being equal to their respective additive inverses i.e $[0 = -0]$ and $[A = -A]$, therefore

$$\frac{0x}{x-1} = 0, \text{ and } A,$$
$$\text{from (11) and (12), } \frac{x^2-1}{x-1} = (x+1) \pm 0 \text{ and } \pm A,$$
$$\text{for x=1 } \Rightarrow \frac{x^2-1}{x-1} = 2, \text{ and } A+2.$$

Those calculations show that function

$$f(x) = \frac{x^2-1}{x-1}$$

is definable at $x=1$, and it is a continuous function.

5.2.3 Limit case 3.

Calculating Euler's number e.

Non cyclically.

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.718281828\dots. \tag{13}$$

Cyclically.

$$\text{With n=}A \text{ on (13)} \Rightarrow e = (1+0)^A.$$

Let's look at the Pascal's triangle in Figure 6, with n denotes row number and k denotes diagonal column number.

With terms of row A of the Pascal's triangle as binomial coefficients,

$$e = 1\left(1^A\right)\left(0^0\right) + A\left(1^{A-1}\right)\left(0^1\right) + \frac{A(A-1)}{2}\left(1^{A-2}\right)\left(0^2\right)$$
$$+\left(\frac{A(A-1)}{2}\right)\left(\frac{A-2}{3}\right)\left(1^{A-3}\right)\left(0^3\right) + \left(\frac{A(A-1)}{2}\right)\left(\frac{A-2}{3}\right)\left(\frac{A-3}{4}\right)\left(1^{A-4}\right)\left(0^4\right)\cdots$$
$$= 1+1+\frac{A^2-A}{2}0^2 + \frac{A^3-3A^2+2A}{6}0^3 + \frac{A^4-6A^3+11A^2-6A}{24}0^4\cdots \text{ and continuing the pattern,}$$
$$= 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\cdots+\frac{1}{3628800}+\frac{1}{39916800}\cdots = 2.718281826\dots.$$

$$
\begin{array}{lll}
\text{n} & & \text{k} \rightarrow \ 0\ 1\ 2\ 3\ 4\ 5 \cdots \infty \cdots A \\
\downarrow & & \swarrow \\
0 & \rightarrow & 1 \\
1 & \rightarrow & 1 \quad 1 \\
2 & \rightarrow & 1 \quad 2 \quad 1 \\
3 & \rightarrow & 1 \quad 3 \quad 3 \quad 1 \\
4 & \rightarrow & 1 \quad 4 \quad 6 \quad 4 \quad 1 \\
5 & \rightarrow & 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
\vdots & & \iddots \quad \iddots \quad \iddots \quad \iddots \\
\infty & \rightarrow & \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\vdots & & \iddots \quad \iddots \quad \iddots \quad \iddots \\
A & \rightarrow & 1 \quad A \quad \frac{A(A-1)}{2} \quad \left(\frac{A(A-1)}{2}\right)\left(\frac{A-2}{3}\right)\cdots
\end{array}
$$

Figure 6: Pascal's triangle in cyclic number system.

5.3 Thompson Lamp Problem.

The problem is similar to the following situation. At the start of term 1 the lamp is switched on, after 1 second at the start of term 2 it is switched off, after $1\frac{1}{2}$ seconds at the start of term 3 it is switched on, after $1\frac{3}{4}$ seconds at the start of term 4 it is switched off, and so on. The question is at the total time of 2 seconds, is the lamp on or off? Non cyclically, no answer for the problem.

5.3.1 Cyclically.

The problem relates to (3), this time the unit is second (s),

$$\sum_{k=0}^{\log_2 A} \frac{1}{2^k} = t = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^{\log_2 \frac{A}{4}}} + \frac{1}{2^{\log_2 \frac{A}{2}}} + \frac{1}{2^{\log_2 A}} \text{ s},$$

$$t = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{\frac{A}{4}} + \frac{1}{\frac{A}{2}} + \frac{1}{A} \text{ s.} \tag{14}$$

Let's see (14), the lamp is on at the start of odd terms, and off at the start of even terms which is right after odd terms. The value of last term of (14) is $\frac{1}{A}$ or 0, at the term number of $log_2 A - 1$, $log_2 A - 1$ is an odd number.

From (3) we get the sum of the series is $2 - 0$. Therefore at the time of $2 - 0$ second, the lamp is off, and at the time of 2 second, the lamp is on.

5.4 Real numbers in cyclic number system.

5.4.1 Countability of real numbers.

Theorem 5.4. *Real numbers are countable.*

Proof. Let define *face value*, *value of the place*, and *place value* as follows;
for the number 234.567,
it's *face value* of 2 is 2
it's *face value* of 5 is 5
it's *value of the place* of 2 is 100
it's *value of the place* of 5 is $\frac{1}{10}$
it's *place value* of 2 is 200.
it's *place value* of 5 is $\frac{5}{10}$.

In cyclic number system we can construct fractional parts of real numbers up to value of the place of 0.

With n denotes row number and k denotes column number or term, let's examine the list of fractional parts of real numbers as follows:

n, k $\rightarrow$	1	2	3	4	5	$\dots$			$(\log \frac{A}{10^2})$	$(\log \frac{A}{10^1})$	$(\log A)$
$\downarrow$											
1	0	0	0	0	0	$\dots$	0	0	0	0	0
2	0	0	0	0	0	$\dots$	0	0	0	0	1
3	0	0	0	0	0	$\dots$	0	0	0	0	2
4	0	0	0	0	0	$\dots$	0	0	0	0	3
5	0	0	0	0	0	$\dots$	0	0	0	0	4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\underline{-4}$	9	9	9	9	9	$\dots$	9	9	9	9	5
$\underline{-3}$	9	9	9	9	9	$\dots$	9	9	9	9	6
$\underline{-2}$	9	9	9	9	9	$\dots$	9	9	9	9	7
$\underline{-1}$	9	9	9	9	9	$\dots$	9	9	9	9	8
A	9	9	9	9	9	$\dots$	9	9	9	9	9

Figure 7. Fractional parts of real numbers in cyclic number system.

Figure 7 represents the fractional parts of real numbers in decimals. We choose term $\log A$ as the last term to cover variations of fractional parts of real numbers, because that last term's value of the place is 0. So that, series formed by value of the places of row n=1, k = 1 to $\log A$;

$$\sum_{k=1}^{\log A} \frac{1}{10^k} = \frac{1}{10^1} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots + \frac{1}{10^{\log \frac{A}{10^2}}} + \frac{1}{10^{\log \frac{A}{10}}} + \frac{1}{10^{\log A}}$$
$$= \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots + \frac{100}{A} + \frac{10}{A} + \frac{1}{A}.$$

Series formed by place values of row n=1, k = 1 to $\log A$;

$$= 0\left(\frac{1}{10}\right) + 0\left(\frac{1}{100}\right) + 0\left(\frac{1}{1000}\right) + \cdots + 0\left(\frac{100}{A}\right) + 0\left(\frac{10}{A}\right) + 0\left(\frac{1}{A}\right).$$

and, series formed by place values of row n= A, k = 1 to $\log A$;

$$= 9\left(\frac{1}{10}\right) + 9\left(\frac{1}{100}\right) + 9\left(\frac{1}{1000}\right) + \cdots + 9\left(\frac{100}{A}\right) + 9\left(\frac{10}{A}\right) + 9\left(\frac{1}{A}\right).$$

Table in Figure 7 and the above series also show that all unique variations of fractional parts of real numbers up to value of the place of 0, are there in the list. Therefore we can't create combination of numbers to form fractional part of real number, including by the method of Georg Cantor's diagonal argument, so that it will not listed in that list.

We can calculate quantity of unique variations of fractional parts of real numbers as A. The non negative integer parts of real numbers can be formed by duplicating the fractional parts of them. Quantity of real numbers including both non negative integer and fractional parts is A^2.

This proves the countability of real numbers, since they can be mapped one to one to natural numbers. □

Corollary 5.5. $0.999..., 1.000..., 1$.

$$0.999... = n = \frac{9}{10^1} + \frac{9}{10^2} + \frac{9}{10^3} + \cdots + \frac{9}{10^{\log \frac{A}{10^2}}} + \frac{9}{10^{\log \frac{A}{10}}} + \frac{9}{10^{\log A}},$$
$$n = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \ldots + \frac{9}{\frac{A}{10^2}} + \frac{9}{\frac{A}{10}} + \frac{9}{A},$$
$$\frac{n}{10} = \frac{9}{10^2} + \frac{9}{10^3} + \frac{9}{10^4} + \ldots + \frac{9}{\frac{A}{10}} + \frac{9}{A} + \frac{9}{10A},$$
$$n - \frac{n}{10} = \frac{9}{10} - \frac{9}{10A} \Rightarrow \frac{9n}{10} = \frac{9}{10} - \frac{9}{10A},$$
So $n = 0.999... = 1 - 0.$

$$
\begin{aligned}
0.000... = m &= \frac{0}{10^1} + \frac{0}{10^2} + \frac{0}{10^3} + \cdots + \frac{0}{10^{\log \frac{A}{10^2}}} + \frac{0}{10^{\log \frac{A}{10}}} + \frac{0}{10^{\log A}}, \\
m &= \frac{0}{10} + \frac{0}{10^2} + \frac{0}{10^3} + \ldots + \frac{0}{\frac{A}{10^2}} + \frac{0}{\frac{A}{10}} + \frac{0}{A}, \\
\frac{m}{10} &= \frac{0}{10^2} + \frac{0}{10^3} + \frac{0}{10^4} + \ldots + \frac{0}{\frac{A}{10}} + \frac{0}{A} + \frac{0}{10A}, \\
m - \frac{m}{10} &= \frac{0}{10} - \frac{0}{10A} \Rightarrow \frac{9m}{10} = \frac{0}{10} - \frac{0}{10A} \Rightarrow m = \frac{0}{9} - \frac{0^2}{9}, \\
\textit{So } 1 + m &= 1.000... = 1 + \frac{0}{9} - \frac{0^2}{9}. \\
\textit{And, } 1 &= 1.
\end{aligned}
$$

So non cyclically, they all are the same numbers, cyclically, however, they all are different numbers.

Also we know that cyclically, $1.000...$ *to* 1 *is closer then* $0.999...$ *to* 1 *.*

Only 1 *is equal to* 1 *.*

6 Conclusion

We conclude that division by 0, that includes implementation of A, is factually and mathematically valid and definable. As shown by some examples of cyclic calculations in this paper, computation methods that involve A and/or cyclic number system are useful to mathematics, from solving the previously unsolvable problems, simplifying the solution methods and processes, as well as uncovering new mathematical facts and information. Also the exact results produced by some examples of cyclic calculations in this paper, is because, now, we are in control.

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