# The quintic equation:

# $x^5+10x^3+20x-1=0$

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abstract

This note presents the roots (in radicals) of the equations:
$x^5+10x^3+20x-1=0$ , $x^5-20x^4-10x^2-1=0$
and related fractals.

## 1. Introduction

***Titans of the Quintic:*** *E. Tschirnhaus (1651-1708) , E.S. Bring (1736-1798) , Gian Francesco Malfatti (1731-1807), Paolo Ruffini (1765-1822) , Niels Henrik Abel (1802-1829) ,E.B. Jerrard (1804-1863), Évariste Galois (1811-1822) , Charles Hermite (1822-1901) , …*

- 1771 , Francesco Malfatti solve the quintic (Malfatti formulae) using a resolvent of sixth degree.
- 1803 , Paolo Ruffini gave a proof that the Quintic is not solvable with radicals.
- 1824 , Niels Henrik Abel gave a more rigorous proof, of the same thing.
- 1830 , Évariste Galois invented group theory , and also showed the same impossibility as Ruffini and Abel.
- 1858 , Charles Hermite solve the quintic via Jacobi theta functions.
- 1948 , G. Watson gave a procedure for solving a (solvable) quintic in radicals.
- 1991 , D.S. Dummit developed exact formulas for the roots of solvable quintics.

- Galois Criterion: A polynomial $f \in \mathbb{Q}[x]$ is solvable by radicals if and only if its Galois group $Gal(f)$ is soluble.

- Example 1.:

$$p(x) = x^5 + 10x^3 + 20x - 1 = 0 \tag{1}$$

$$Gal(p) = F_{20} \quad \text{, soluble (Maple)} \tag{2}$$

- Example 2.:

$$q(x) = x^5 - 20x^4 - 10x^2 - 1 = 0 \tag{3}$$

$$Gal(q) = F_{20} \quad \text{, soluble (Maple)} \tag{4}$$

- Remark: $q(x) = -x^5 p\left(\frac{1}{x}\right)$ .

## 2. The Quintic: $x^5 + 10x^3 + 20x - 1 = 0$ .

- Roots:

$$x^5 + 10x^3 + 20x - 1 = 0 \Rightarrow x = \{x_1, x_2, x_3, x_4, x_5\} \tag{5}$$

$$x_1 = \sqrt[5]{\frac{\sqrt{129}+1}{2}} - \sqrt[5]{\frac{\sqrt{129}-1}{2}} \tag{6}$$

$$\begin{aligned} x_2 = {} & \left(\frac{\sqrt{5}-1}{8}\right)\left(\sqrt[5]{16\sqrt{129}+16} - \sqrt[5]{16\sqrt{129}-16}\right) \\ & + \frac{i}{16}\left(\sqrt{5}+1\right)\sqrt{10-2\sqrt{5}}\left(\sqrt[5]{16\sqrt{129}+16} + \sqrt[5]{16\sqrt{129}-16}\right) \end{aligned} \tag{7}$$

$$\begin{aligned} x_3 = {} & \left(\frac{\sqrt{5}-1}{8}\right)\left(\sqrt[5]{16\sqrt{129}+16} - \sqrt[5]{16\sqrt{129}-16}\right) \\ & - \frac{i}{16}\left(\sqrt{5}+1\right)\sqrt{10-2\sqrt{5}}\left(\sqrt[5]{16\sqrt{129}+16} + \sqrt[5]{16\sqrt{129}-16}\right) \end{aligned} \tag{8}$$

$$x_4 = \left(\frac{\sqrt{5}+1}{8}\right)\left(\sqrt[5]{16\sqrt{129}-16} - \sqrt[5]{16\sqrt{129}+16}\right) + \frac{i}{8}\sqrt{10-2\sqrt{5}}\left(\sqrt[5]{16\sqrt{129}+16} + \sqrt[5]{16\sqrt{129}-16}\right) \tag{9}$$

$$x_5 = \left(\frac{\sqrt{5}+1}{8}\right)\left(\sqrt[5]{16\sqrt{129}-16} - \sqrt[5]{16\sqrt{129}+16}\right) - \frac{i}{8}\sqrt{10-2\sqrt{5}}\left(\sqrt[5]{16\sqrt{129}+16} + \sqrt[5]{16\sqrt{129}-16}\right) \tag{10}$$

## 3. The Quintic $x^5 - 20x^4 - 10x^2 - 1 = 0$ .

- Roots:

$$x^5 - 20x^4 - 10x^2 - 1 = 0 \Rightarrow x = \{x_1, x_2, x_3, x_4, x_5\} \tag{11}$$

$$x_1 = 4 + \left(\frac{\sqrt{129}+1}{8}\right)\sqrt[5]{16\sqrt{129}-16} + \frac{1}{2}\sqrt[5]{\left(16\sqrt{129}-16\right)^2} + \left(\frac{\sqrt{129}+1}{64}\right)\sqrt[5]{\left(16\sqrt{129}-16\right)^3} + \frac{1}{16}\sqrt[5]{\left(16\sqrt{129}-16\right)^4} \tag{12}$$

$$\begin{aligned} x_2 = {} & 4 + \left(\sqrt{5}-1\right)\left(\frac{\sqrt{129}+1}{32}\right)\sqrt[5]{16\sqrt{129}-16} - \left(\frac{\sqrt{5}+1}{8}\right)\sqrt[5]{\left(16\sqrt{129}-16\right)^2} \\ & - \left(\sqrt{5}+1\right)\left(\frac{\sqrt{129}+1}{256}\right)\sqrt[5]{\left(16\sqrt{129}-16\right)^3} + \left(\frac{\sqrt{5}-1}{64}\right)\sqrt[5]{\left(16\sqrt{129}-16\right)^4} \\ & + i\sqrt{10-2\sqrt{5}}\left\{\left(\sqrt{5}+1\right)\left(\frac{\sqrt{129}+1}{64}\right)\sqrt[5]{16\sqrt{129}-16} + \frac{1}{8}\sqrt[5]{\left(16\sqrt{129}-16\right)^2} \right. \\ & \left. - \left(\frac{\sqrt{129}+1}{256}\right)\sqrt[5]{\left(16\sqrt{129}-16\right)^3} - \left(\frac{\sqrt{5}+1}{128}\right)\sqrt[5]{\left(16\sqrt{129}-16\right)^4}\right\} \end{aligned} \tag{13}$$

$$
\begin{aligned}
x_3 = 4 &+ \left(\sqrt{5}-1\right)\left(\frac{\sqrt{129}+1}{32}\right)\sqrt[5]{16\sqrt{129}-16} - \left(\frac{\sqrt{5}+1}{8}\right)\sqrt[5]{\left(16\sqrt{129}-16\right)^2} \\
&- \left(\sqrt{5}+1\right)\left(\frac{\sqrt{129}+1}{256}\right)\sqrt[5]{\left(16\sqrt{129}-16\right)^3} + \left(\frac{\sqrt{5}-1}{64}\right)\sqrt[5]{\left(16\sqrt{129}-16\right)^4} \\
&- i\sqrt{10-2\sqrt{5}}\left\{\left(\sqrt{5}+1\right)\left(\frac{\sqrt{129}+1}{64}\right)\sqrt[5]{16\sqrt{129}-16} + \frac{1}{8}\sqrt[5]{\left(16\sqrt{129}-16\right)^2}\right. \\
&\left. - \left(\frac{\sqrt{129}+1}{256}\right)\sqrt[5]{\left(16\sqrt{129}-16\right)^3} - \left(\frac{\sqrt{5}+1}{128}\right)\sqrt[5]{\left(16\sqrt{129}-16\right)^4}\right\}
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
x_4 = 4 &- \left(\sqrt{5}+1\right)\left(\frac{\sqrt{129}+1}{32}\right)\sqrt[5]{16\sqrt{129}-16} + \left(\frac{\sqrt{5}-1}{8}\right)\sqrt[5]{\left(16\sqrt{129}-16\right)^2} \\
&+ \left(\sqrt{5}-1\right)\left(\frac{\sqrt{129}+1}{256}\right)\sqrt[5]{\left(16\sqrt{129}-16\right)^3} - \left(\frac{\sqrt{5}+1}{64}\right)\sqrt[5]{\left(16\sqrt{129}-16\right)^4} \\
&+ i\sqrt{10-2\sqrt{5}}\left\{\left(\frac{\sqrt{129}+1}{32}\right)\sqrt[5]{16\sqrt{129}-16} - \left(\frac{\sqrt{5}+1}{16}\right)\sqrt[5]{\left(16\sqrt{129}-16\right)^2}\right. \\
&\left. + \left(\sqrt{5}+1\right)\left(\frac{\sqrt{129}+1}{512}\right)\sqrt[5]{\left(16\sqrt{129}-16\right)^3} - \frac{1}{64}\sqrt[5]{\left(16\sqrt{129}-16\right)^4}\right\}
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
x_5 = 4 &- \left(\sqrt{5}+1\right)\left(\frac{\sqrt{129}+1}{32}\right)\sqrt[5]{16\sqrt{129}-16} + \left(\frac{\sqrt{5}-1}{8}\right)\sqrt[5]{\left(16\sqrt{129}-16\right)^2} \\
&+ \left(\sqrt{5}-1\right)\left(\frac{\sqrt{129}+1}{256}\right)\sqrt[5]{\left(16\sqrt{129}-16\right)^3} - \left(\frac{\sqrt{5}+1}{64}\right)\sqrt[5]{\left(16\sqrt{129}-16\right)^4} \\
&- i\sqrt{10-2\sqrt{5}}\left\{\left(\frac{\sqrt{129}+1}{32}\right)\sqrt[5]{16\sqrt{129}-16} - \left(\frac{\sqrt{5}+1}{16}\right)\sqrt[5]{\left(16\sqrt{129}-16\right)^2}\right. \\
&\left. + \left(\sqrt{5}+1\right)\left(\frac{\sqrt{129}+1}{512}\right)\sqrt[5]{\left(16\sqrt{129}-16\right)^3} - \frac{1}{64}\sqrt[5]{\left(16\sqrt{129}-16\right)^4}\right\}
\end{aligned}
\tag{16}
$$

## 4. Formulas and Fractals

❖ Linear Recurrence for $r=\sqrt[5]{\frac{\sqrt{129}+1}{2}}-\sqrt[5]{\frac{\sqrt{129}-1}{2}}$ :

$$a_{n+5}=20a_{n+4}+10a_{n+2}+a_n \quad ,n\in\mathbb{N} \tag{17}$$

$$a_n=\{1,20,400,8010,160400,...\} \tag{18}$$

$$\frac{a_n}{a_{n+1}}\to r \tag{19}$$

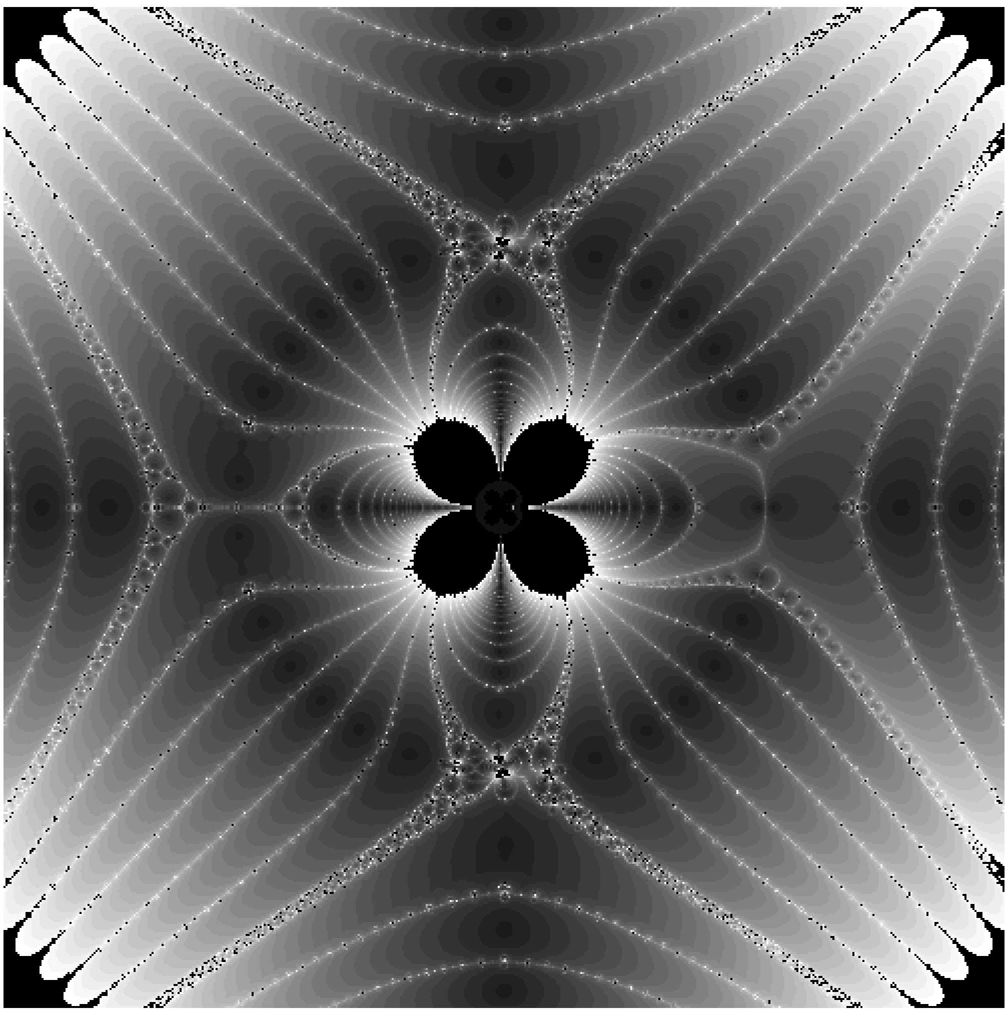

Fig. 1.

❖ Recurence for $r$ :

$$u_{n+1} = \frac{1}{20 + 10u_n^2 + u_n^4} \quad , u_1 = 0 \Rightarrow u_n \to r \tag{20}$$

$$u_{n+1} = \frac{1 - u_n^5}{20 + 10u_n^2} \quad , u_1 = 0 \Rightarrow u_n \to r \tag{21}$$

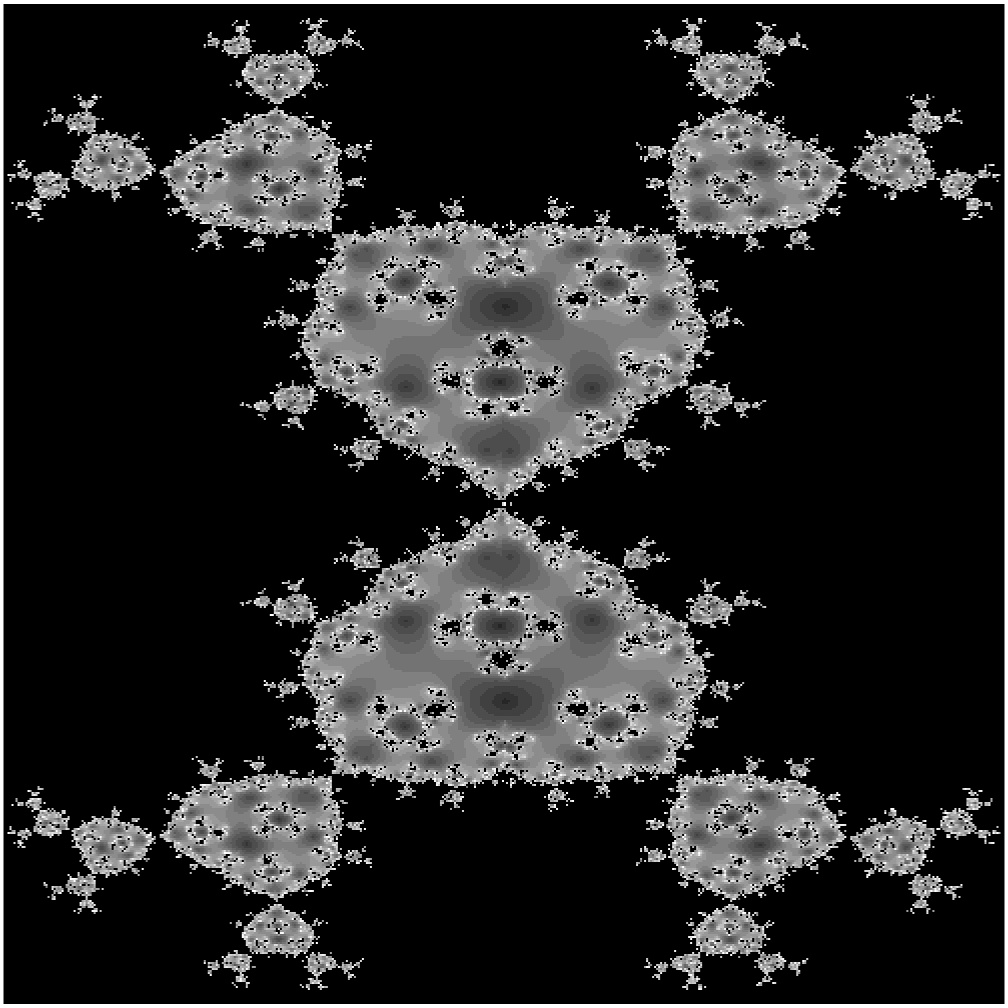

Fig. 2.

❖ Pi formula:

$$\pi = 2\sqrt{3}\sum_{n=0}^{\infty}\frac{3^{-n}r^{2n+1}}{2n+1}c_n \tag{22}$$

$$c_n = -\left(400c_{n-1} + 1200c_{n-2} + 1260c_{n-3} + 540c_{n-4} + 81c_{n-5}\right),\ n \geq 5 \tag{23}$$

$$c_0 = 20, c_1 = -7910, c_2 = 3140045, c_3 = -1246551200, c_4 = 494862381800 \tag{24}$$

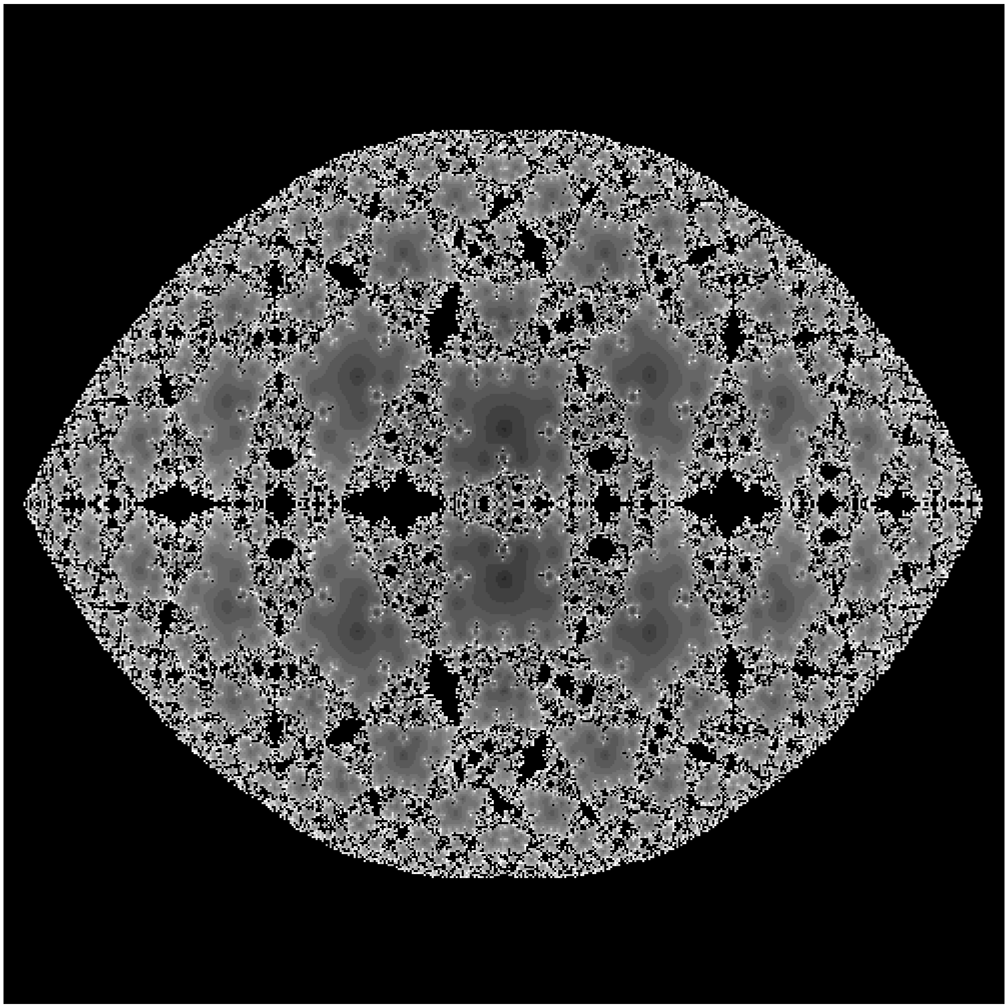

Fig. 3.

- Some radicals:

$$r=\sqrt[10]{32+\sqrt{32+\sqrt{32+\sqrt{32+\ldots}}}}-\sqrt[10]{32-\sqrt{32-\sqrt{32+\sqrt{32-\ldots}}}} \qquad (25)$$

$$r=\sqrt[5]{1+\sqrt{32-\sqrt{32-\ldots}}}-\sqrt[5]{\sqrt{32-\sqrt{32-\sqrt{32-\ldots}}}} \qquad (26)$$

$$r=\sqrt[5]{\sqrt{32+\sqrt{32+\sqrt{32+\ldots}}}}-\sqrt[5]{-1+\sqrt{32+\sqrt{32+\ldots}}} \qquad (27)$$

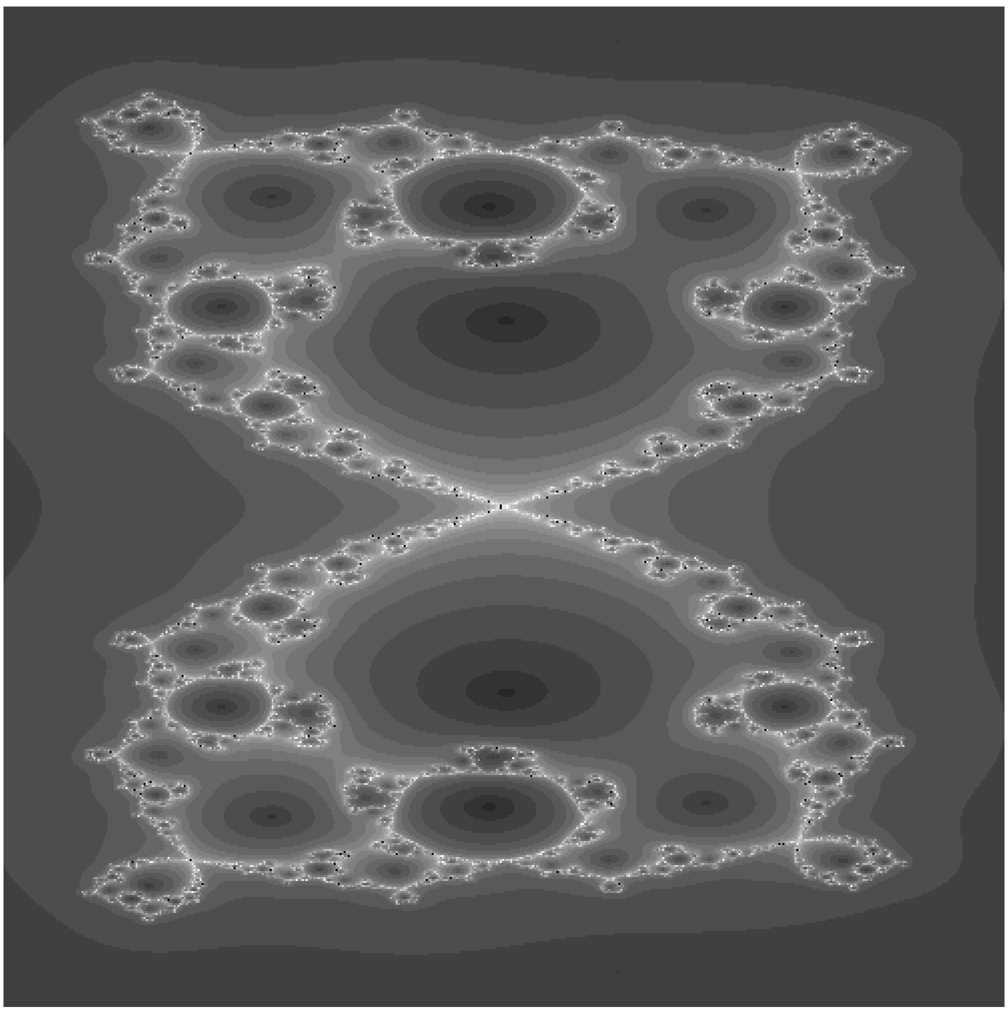

Fig. 4.

❖ Pi formula:

$$\frac{1}{\pi}=\frac{1}{16}\sum_{n=0}^{\infty}\left(\frac{5r}{2^{10}}\right)^{n} f_n \qquad (28)$$

$$f_n=\sum_{k=0}^{[n/3]}5^k\sum_{m=k}^{2k}\binom{n-2m}{k}\binom{k}{m-k}\binom{2n-4m}{n-2m}^3(42n-84m+5)\left(\frac{2^{19}}{125}\right)^m \qquad (29)$$

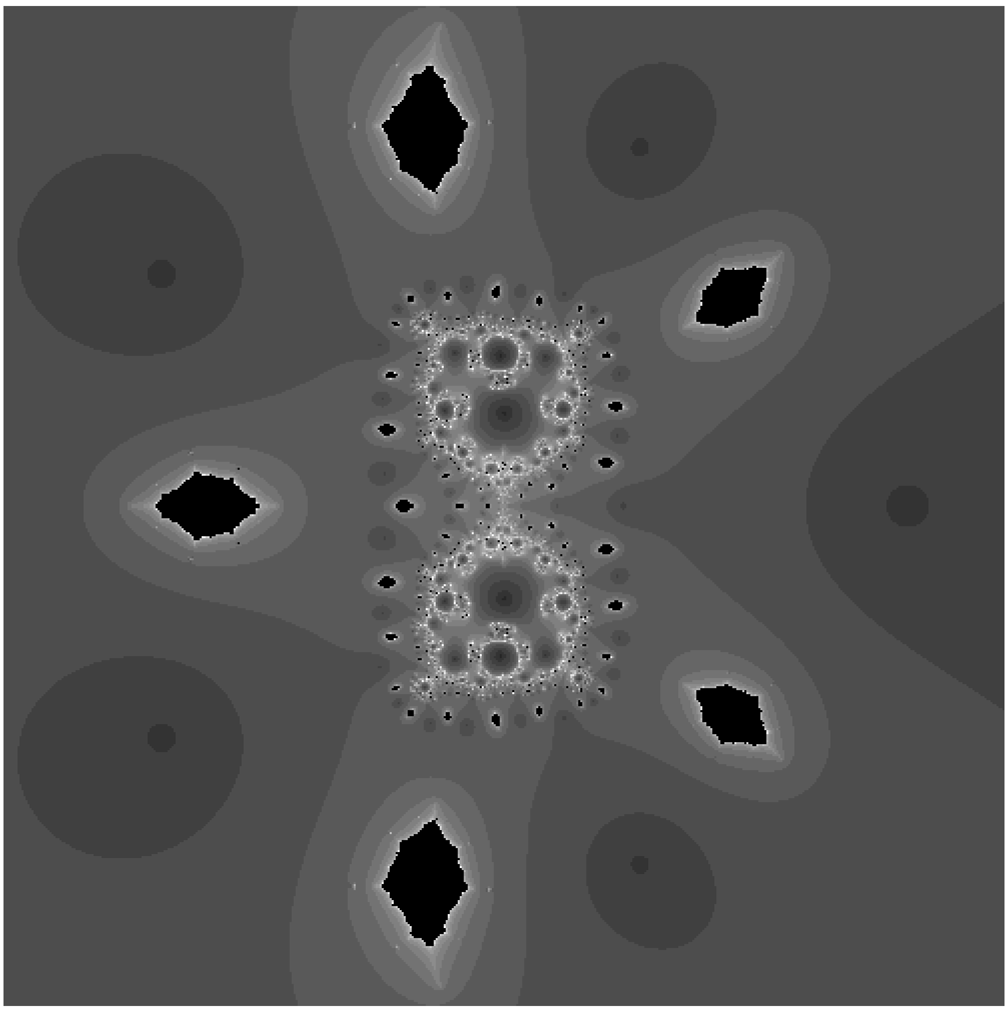

Fig. 5.

❖ Some relations:

$$r=\sqrt{-5+\sqrt{5+r^{-1}}} \tag{30}$$

$$1=r\left(r^2+5-\sqrt{5}\right)\left(r^2+5+\sqrt{5}\right) \tag{31}$$

$$r=\cfrac{1}{20+\cfrac{1}{40+\cfrac{1}{11+\ldots}}}=\left[0;20,40,11,7,1,1,1,4,4,1,1,1,5,1,1,2,9,24,\ldots\right] \tag{32}$$

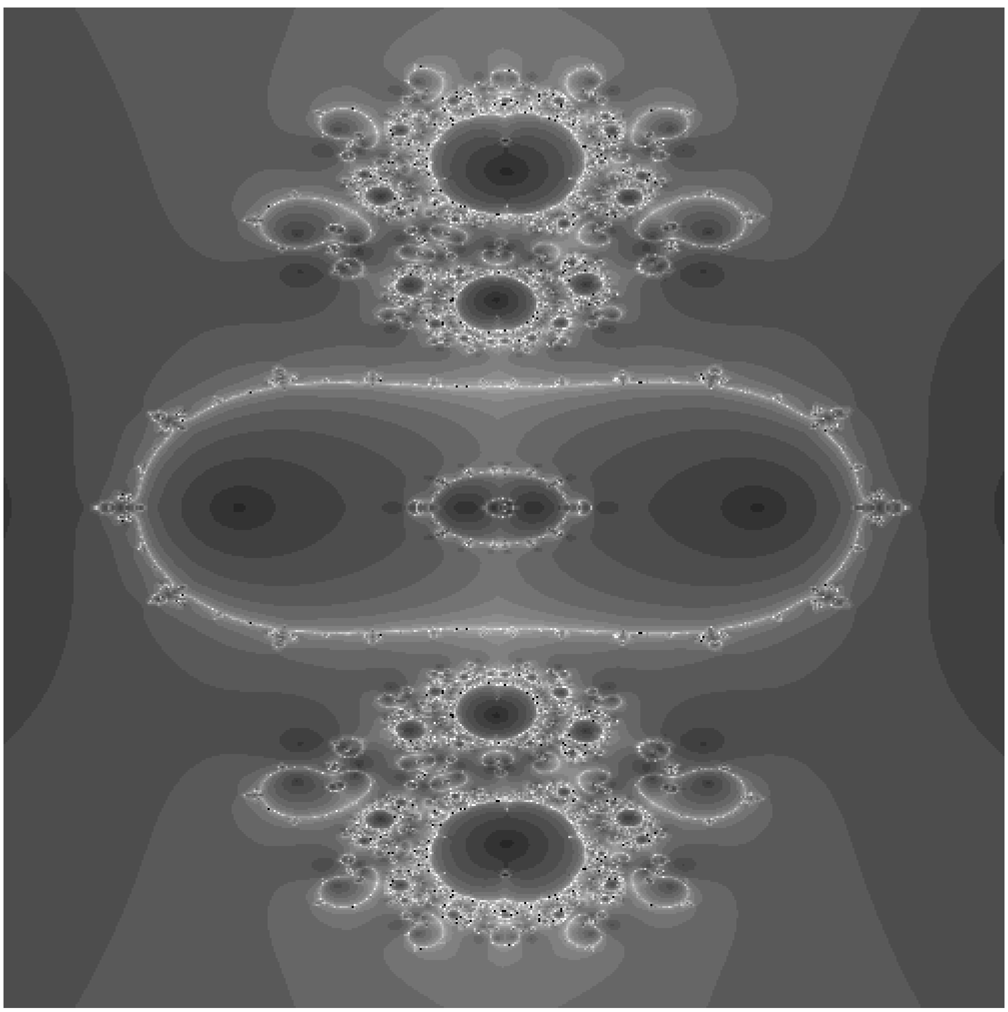

Fig. 6.

# References


1. D.S. Dummit,Solving solvable quintics, Mathematics of Computation, 57 (1991),387-401.(corrigenda, Math.Comp.,59 (1992),309).
2. M.J. Lavalle,B.K. Spearman,K.S. Williams, Watson Method of Solving a Quintic Equation.JP Jour. Algebra, Number Theory & Appl. 5(1) (2005) , 49-73.