ON THE KAKEYA SET CONJECTURE

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ABSTRACT. In this article we will prove the Kakeya set conjecture. In addition we will prove that in the usual approach to the Kakeya maximal function conjecture we can assume that the tube-sets are maximal. Moreover, we will construct a tube- set were the well known L2 bound for the Kakeya maximal function is attained.

1. INTRODUCTION

The Kakeya maximal function conjecture and it‘s variations have gained considerable interest especially after an influential paper by Bourgain [1]. For example, it would follow from the conjecture that the Kakeya sets and the Nikodym sets have necessarily full dimensions [10, 11, 7]. However, the Nikodym set conjecture is implied by the Kakeya set conjecture [7, 11].The case $n = 2$ was proved by Davies see [4] and the finite field case by Dvir [5]. A Kakeya is a set that contains an unit line in every direction. For surveys see [15, 12, 2]. Almost all the necessary preliminaries for this paper can be found for example in [7], [10] and in [13]. Define the δ - tubes in standard way: for all $\delta > 0, \omega \in S^{n-1}$ and $a \in \mathbb{R}^n$, let

$$T_\omega^\delta(a) = \{x \in \mathbb{R}^n : |(x-a)\cdot\omega| \leq \frac{1}{2}, |proj_{\omega^\perp}(x-a)| \leq \delta\}.$$

Moreover, let $f \in L^1_{loc}(\mathbb{R}^n)$. Define the Kakeya maximal function $f^*_\delta : S^{n-1} \to \mathbb{R}$ via

$$f^*_\delta(\omega) = \sup_{a\in\mathbb{R}^n} \frac{1}{T_\omega^\delta(a)} \int_{T_\omega^\delta(a)} |f(y)| \mathrm{d}y.$$

In this paper any constant can depend on dimension n. In study of the Kakeya maximal function conjecture we are aiming at the following bounds

$$||f^*_\delta||_p \leq C_\epsilon \delta^{-n/p+1-\epsilon}, \tag{1}$$

for all $\epsilon > 0$. Remarkably, a bound of the form (1) follows from a bound of the form

$$|| \sum_{\omega\in\Omega} 1_{T_\omega(a_\omega)} ||_{p/(p-1)} \leq C_\epsilon \delta^{-n/p+1-\epsilon}, \tag{2}$$

for all $\epsilon > 0$, and for any set of δ-separated of δ - tubes. See for example [11] or [7]. We will prove that we need to consider only the case were the set Ω is maximal. As usual we define that "$A \lessapprox B$ iff for all $\epsilon > 0$ and for all $\delta > 0$, it holds that $A \leq C_\epsilon \delta^{-\epsilon} B$. We will prove the following theorem.

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Theorem 1. *Let Ω be a maximal set of δ - tubes, then*

$$|\bigcup_{\omega\in\Omega} T_\omega| \approx 1.$$

We define the Minkowski dimension $Dim_M(K)$ of any bounded set K as follows. Let $\delta > 0$ and let K_δ be the $\delta-$ neighbourhood of the set K, then

$$Dim_M(K) = n - \lim_{\delta\to 0} \frac{\log|K_\delta|}{\log\delta},$$

if the limit exists. We will prove that

Corollary 1. *Any Kakeya set has full Minkowski dimension.*

We define the (spherical) Hausdorff content $H^s(K)$ of a subset $K \subset \mathbb{R}^n$ as follows. Let $r > 0$ and let $0 < r_j < r$. Then we define

$$H^s_r(K) = \inf\{\sum_{j=i}^{\infty} r_j^s | K \subset \bigcup_{j=1}^{\infty} B(x_j, \frac{r_j}{2})\},$$

where each $B(x_j, \frac{r_j}{2})$ is a ball with a diameter strictly less than r. The $s-$ dimensional Hausdorff content of K is defined as $\lim_{r\to 0} H^s_r(K)$. We define the Hausdorff dimension as

$$Dim_H(K) = \inf\{s \geq 0 | H^s(K) = 0\}.$$

We will prove that

Corollary 2 (Kakeya conjecture)**.** *Any Kakeya set has full Hausdorff dimension.*

2. A reduction to the case where the tube-sets are maximal

Let Ω' be any set of δ - separated directions. We will prove that

$$||\sum_{\omega'\in\Omega'} 1_{T'_\omega(a_{\omega'})}||_{p/(p-1)} \leq ||\sum_{\omega\in\Omega} 1_{T_\omega(a_\omega)}||_{p/(p-1)},$$

where Ω is maximal. We construct the set Ω as follows. Let Ω' be the original direction-set and let $\Omega' \subset \Omega'''$ be maximal. Define

$$\Omega'' := \{\omega'' \in S^{n-1} | \omega'' \in \Omega'''/\{\Omega'\}\}.$$

Moreover, let

$$\Omega := \Omega' \cup \Omega''.$$

Clearly, Ω is maximal. We choose the tubes corresponding to directions in Ω' to have origo as their center of masses. Thus, what we do is that we add tubes to the original tube-set so it becomes maximal. Now, we can estimate:

$$||\sum_{\omega'\in\Omega'} 1_{T'_\omega(a_{\omega'})}||_{p/(p-1)} \leq ||\sum_{\omega'\in\Omega'} 1_{T'_\omega(a_{\omega'})} + \sum_{\omega''\in\Omega''} 1_{T''_\omega(0)}||_{p/(p-1)}$$
$$= ||\sum_{\omega\in\Omega} 1_{T_\omega(a_\omega)}||_{p/(p-1)}.$$

Thus, we need only to consider the cases where the tube sets are maximal.

3. Previously known results

We will use the following bound for the pairwise intersections of δ - tubes:

Lemma 1 (Corbòda)**.** *For any pair of directions $\omega_i, \omega_j \in S^{n-1}$ and any pair of points $a, b \in \mathbb{R}^n$, we have*

$$|T^\delta_{\omega_i}(a) \cap T^\delta_{\omega_j}(b)| \lesssim \frac{\delta^n}{|\omega_i - \omega_j|}.$$

A proof can be found for example in [7].

For any (spherical) cap $\Omega \subset S^{n-1}, |\Omega| \gtrsim \delta^{n-1}, \delta > 0$, deïňĄne its δ-entropy $N_\delta(\Omega)$ as the maximum possible cardinality for an $\delta-$separated subset of Ω.

Lemma 2. *In the notation just defined*

$$1 \leq N_\delta(\Omega) \sim \frac{|\Omega|}{\delta^{n-1}}.$$

Again, a proof can essentially be found in [7].

4. The L_2 estimate

The next estimate leads to a well known $L_2(\mathbb{R}^n)$ estimate. Let Ω be any collection of δ - tubes. We will show that

$$||\sum_{T_\Omega} 1_{T_\omega}||_2 \lessapprox \delta^{(2-n)/2}. \tag{3}$$

After rising everything to the power of two and using Fubini we need to show that

$$\int \sum_{\Omega} \sum_{\Omega'} 1_{T_\omega} 1_{T_{\omega'}} \lessapprox \delta^{2-n}.$$

It suffices to show that for every $T_{\omega'}$

$$\sum_{\Omega} |T_{\omega'} \cap T_\omega| \lessapprox \delta. \tag{4}$$

Split the sum over angle of separation between ω' and ω. So the estimate (4) becomes

$$\sum_{k=0}^{\log(1/\delta)} \sum_{T_\omega : \theta(\omega', \omega) \sim 2^{-k}, T_\omega \cap T_\omega \neq \emptyset} |T_{\omega_1} \cap T_{\omega_2}| \lessapprox \delta.$$

Notice that we do not need to consider the term where $\omega_1 = \omega_2$. We use lemma 1 to bound the intersection of $T_{\omega'}$ and T_ω by $2^k \delta^n$. So after a rearrangement of the previous inequality, we reduce to showing that

$$\begin{aligned} &\sum_{k=0}^{\log(1/\delta)} \#\{T_\omega : \theta(\omega, \omega') \sim 2^{-k}, T_{\omega'} \cap T_\omega \neq \emptyset\} \\ &\lesssim \sum_{k=0}^{\log(1/\delta)} 2^k \delta^n 2^{-k(n-1)} \delta^{1-n} \lessapprox \delta. \end{aligned} \tag{5}$$

The directions in (5) belong to a cap of size $\lesssim 2^{-k(n-1)}$. So we can δ -separate the cap via 2 and get the inequality (5). Now we have proved (3). Next, we prove that the bound is tight. Split the domain of integration via dyadic decomposition:

$$E_{2^k} := \{x|2^k \leq \sum_{\Omega} 1_{T_\omega}(x) \leq 2^{k+1}\}.$$

Suppose that each tube has it‘s center of mass in the origo. Now, the set contains an origo centered δ - ball. Let $2^i \leq |E_{\#(\Omega)}| \leq 2^{i+1}$. Then via lemma 1

$$||\sum_{\omega\in\Omega} 1_{T_{\omega(0)}}||_2 \gtrsim \#(\Omega)|E_{2^i}|^{1/2} \gtrsim \delta^{1-n}\delta^{n/2} = \delta^{(2-n)/2}.$$

Thus, the bound (3) is essentially tight.

5. A proof of the Kakeya set conjecture

In this section we will prove 1. Consider the integral

$$\int \sum_{\Omega} 1_{T_\omega} = \sum_{\Omega} |T_\omega|.$$

Split the domain of integration via dyadic decomposition:

$$E_{2^k} := \{x|2^k \leq \sum_{\Omega} 1_{T_\omega}(x) \leq 2^{k+1}\}.$$

Integrating inequality

$$2^k \leq \sum_{\Omega} 1_{T_\omega}(x) \leq 2^{k+1} \tag{6}$$

over the domain E_{2^k} we obtain

$$2^k|E_{2^k}| \leq \sum_{\Omega} |T_\omega \cap E_{2^k}| \leq 2^{k+1}|E_{2^k}|.$$

Let $\#(\Omega) = N$. Now, $k \in [0, .., C\log N]$. Notice that there exists k such that

$$1 \sim \sum_{\Omega} |T_\omega| \lesssim \log N \sum_{\Omega} |T_\omega \cap E_{2^k}| \sim \log N 2^k |E_{2^k}|. \tag{7}$$

Now, consider the terms $|T_\omega \cap E_{2^k}|$ in the above sum. We wan‘t to prove that we can essentially take them to be $\approx \delta^{n-1}$. Split the sum in two parts where $|T_{\omega'} \cap E_{2^k}| \geq \frac{c}{\log N}\delta^{n-1}$ and $|T_{\omega''} \cap E_{2^k}| < \frac{c}{\log N}\delta^{n-1}$,

$$1 \lesssim \log N \sum_{\omega'\in\Omega'} |T_{\omega'} \cap E_{2^k}| + \log N \sum_{\omega''\in\Omega''} |T_{\omega''} \cap E_{2^k}|.$$

It‘s clear that because the number of terms in the sums is $\lesssim \delta^{n-1}$ the last sum above is neglible. Next, we wan‘t to prove that if $|T_\omega \cap E_{2^k}| \approx \delta^{n-1}$, for essentially every $\omega \in \Omega$, then $2^k \approx 1$.
We first notice that if $k \geq 2$ and if

$$\sum_{\omega'\in\Omega_\omega} |T_{\omega'} \cap T_\omega \cap E_2^k| = |T_\omega \cap E_{2^k}|, \tag{8}$$

then

$$\begin{aligned}
2^k\delta^{n-1}\#(\Omega_\omega) &\approx \sum_{\omega'\in\Omega_\omega}\sum_{\omega\in\Omega}|T_{\omega'}\cap E_{2^k}\cap T_\omega| \\
&\sim \sum_{\omega'\in\Omega_\omega}\sum_{\omega\in\Omega}\sum_{\omega''\in\Omega}2^{-k}|E_{2^k}\cap T_{\omega'}\cap T_{\omega''}\cap T_\omega| \\
&= \sum_{\omega''\in\Omega}\sum_{\omega'\in\Omega_\omega}\sum_{\omega\in\Omega}2^{-k}|E_{2^k}\cap T_{\omega'}\cap T_{\omega''}\cap T_\omega| \\
&= \sum_{\omega''\in\Omega}\sum_{\omega'\in\Omega_\omega}\sum_{\omega'''\in\Omega}2^{-k}|E_{2^k}\cap T_{\omega'}\cap T_{\omega''}\cap T_\omega\cap T_{\omega'''}| \\
&= \sum_{\omega''\in\Omega}\sum_{\omega'\in\Omega_\omega}|E_{2^k}\cap T_{\omega'}\cap T_{\omega''}\cap T_\omega| \\
&= \sum_{\omega''\in\Omega}|E_{2^k}\cap T_{\omega''}\cap T_\omega| \\
&= 2^k|E_{2^k}\cap T_\omega| \\
&\approx 2^k\delta^{n-1}.
\end{aligned}$$

Thus,

$$\#(\Omega_\omega)\approx 1. \tag{9}$$

Because each term in the sum in the equation (8) is an intersection of $\sim 2^k$ tubes it follows from above (9) that $T_\omega\cap E_{2^k}$ is an intersection of $\approx 2^k$ tubes. Next, let‘s suppose that $2^k \gtrapprox \delta^{-\beta}, 0<\beta\le n-1$, and let‘s suppose that tube $T_{\omega'}$ intersecting $T_\omega\cap E_{2^k}$ has it‘s direction outside of a cap of size $\sim\delta^{n-1+\beta}$ on the unit sphere. Then the angle between T_ω and $T_{\omega'}$ is greater than $\sim\delta^{1+\beta/(n-1)}$. Thus, by equalities (5) and (9) and lemma 1:

$$|T_\omega\cap E_{2^k}|\lessapprox|T_\omega\cap T_{\omega'}\cap E_{2^k}|\le|T_\omega\cap T_{\omega'}|\lesssim\delta^{n-1-\beta/(n-1)},$$

which is a contradiction. So we can suppose that the directions in the intersection $E_{2^k}\cap T_\omega\cap T_{\omega''}$ belong to a cap of size $\sim\delta^{n-1+\beta}$. If we δ - separate the cap via lemma 2 we get that the cap can contain at most ~ 1 tube-directions, which is a contradiction.

Thus, $2^k\approx 1$. From that and inequality (7) it follows that

$$1\sim\sum_{\Omega}|T_\omega|\lesssim\log N\sum_{\Omega}|T_\omega\cap E_{2^k}|\sim\log N 2^k|E_{2^k}|\lessapprox|E_{2^k}|\le|\bigcup_{\omega\in\Omega}T_\omega|.$$

So, we have the theorem 1. For the first corollary note that

$$1\approx|\bigcup_{\omega\in\Omega}T_\omega|\le|K_\delta|,$$

where K_δ is a δ - neighbourhood of a Kakeya set. Thus,

$$n=n-\lim_{\delta\to 0}\frac{\log|\bigcup_{\omega\in\Omega}T_\omega|}{\log\delta}\le n-\lim_{\delta\to 0}\frac{\log|K_\delta|}{\log\delta}.$$

6. A proof of the Kakeya conjecture

Let K be a Kakeya set, that is, a compact set that contains an unit line in every direction. let $\bigcup_{j=1}^{\infty} B_j$ be a cover of K with balls of diameters less than $1 > r > r_j > 0$. Let $n > n - \alpha > 0$ be such that

$$\sum_{j=1}^{\infty} r_j^{n-\alpha} < 1. \tag{10}$$

If the $n-\alpha-$ dimensional Hausdorff content is zero that kind of cover exists. By compactness of the Kakeya set we can take a subcover with diameters such that $1 > r > r_j \geq \delta > 0$, where at least one $r_j = \delta$. Now, we have proved that

$$\sum_{j=1}^{M} r_j^n \gtrsim |\bigcup_{j=1}^{M} B_j| \gtrsim |\bigcup_{i=1}^{N} T_i| \gtrapprox 1. \tag{11}$$

The second inequality above follows because the balls cover the middle lines of the tubes, so there exists a constant such that the second inequality above is valid. Using inequality (10) and (11) we obtain

$$C_{\alpha/k}\delta^{-\alpha/k} \sum_{j=1}^{M} r_j^n > \sum_{j=1}^{M} r_j^{n-\alpha}. \tag{12}$$

Thus,

$$\sum_{j=1}^{M} r_j^n (C_{\alpha/k}\delta^{-\alpha/k} - r_j^{-\alpha}) > 0. \tag{13}$$

It follows that for the average value of a power of diameters it holds that

$$C_{\alpha/k}\delta^{-\alpha/k} > \frac{1}{M}\sum_{j=1}^{M} r_j^{-\alpha} \geq \frac{1}{M^{-\alpha}}(\sum_{j=1}^{M} r_j)^{-\alpha}, \tag{14}$$

where we used Jensen‘s inequality. Thus,

$$c_\alpha \frac{1}{M}\sum_{j=1}^{M} r_j > \delta^{1/k}.$$

From above it follows that

$$\frac{(c_\alpha)^n}{M}(\sum_{j=1}^{M} r_j^n) \geq (\frac{c_\alpha}{M})^n(\sum_{j=1}^{M} r_j)^n > \delta^{n/k},$$

where we used Jensen‘s inequality again. Thus, from above and inequality (10)

$$C_\alpha > M\delta^{n/k}.$$

It follows from above that

$$\delta^{-n/k} C_\alpha > M \tag{15}$$

We can do the steps (12), (13) and (14) again for $\epsilon = \alpha/2$ and obtain

$$C_{\alpha/2}\delta^{-\alpha/2} > \frac{1}{M}\sum_{j=1}^{M} r_j^{-\alpha}. \tag{16}$$

Let k and a small δ be such that

$$\delta^{-\alpha/3} > C_\alpha \delta^{-n/k}.$$

From above and inequalities (15) and (16) we obtain

$$C_{\alpha/2}\delta^{-\alpha/2} > \delta^{\alpha/3}\sum_{j=1}^{M} r_j^{-\alpha} > \delta^{\alpha/3}\delta^{-\alpha} = \delta^{-2/3\alpha},$$

which is a contradiction when δ is small enough.

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