The difference of any real transcendental number and complex number e^i is always a complex transcendental number.

Charanjeet Singh Bansrao

bansrao357@gmail.com

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Section 1. From Euler's formula,

$$e^{ix} = \cos x + i\sin x$$

we can derive the following equation

$$ae^{ix} = 2c - a + 2i\sqrt{ca - c^2} \qquad (1)$$

Where , a, c and x are real numbers and can be defined as

$$x = \cos^{-1}\frac{2c-a}{a}$$

$c = \frac{a(\cos x+1)}{2}$ and a is the diameter of the circle on complex plane such that $a \leq c \leq 0 \leq c \leq a$ and

$0 \leq x \leq \pi$ for positive values of a and c and $\pi \leq x \leq 2\pi$ for negative values of a and c

The equation (1) can be obtained as follows. Consider the following circle on a complex plane with center O touching the imaginary axis at zero.

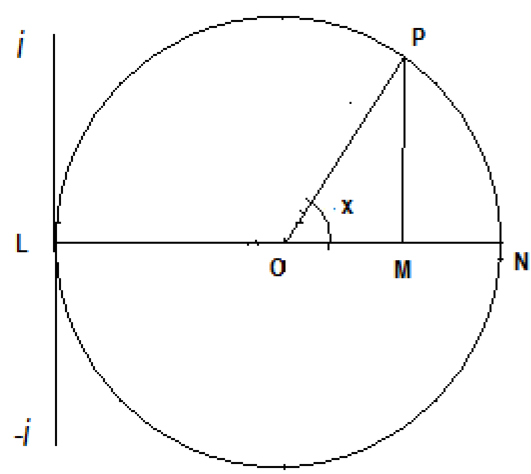

Diagram 1

Let length LM is c and MN is b then diameter LN should be c+b and radius OP or ON is $\frac{c+b}{2}$. In this way, the length OM is

$$OM = ON - MN$$

$$= \frac{c+b}{2} - b = \frac{c-b}{2}$$

Using Pythagoras theorem, the length PM can be obtained as follows:

$$PM = \sqrt{OP^2 - OM^2}$$

$$PM = \sqrt{\left[\frac{c+b}{2}\right]^2 - \left[\frac{c-b}{2}\right]^2} = \sqrt{cb}$$

Hence $$\sin x = \frac{PM}{OP} = \frac{2\sqrt{cb}}{c+b} \qquad (2)$$

and $$\cos x = \frac{OM}{OP} = \frac{c-b}{c+b} \qquad (3)$$

Insert (2) and (3) in Euler's formula to get

$$e^{ix} = \frac{c-b}{c+b} + i\frac{2\sqrt{cb}}{c+b}$$

$$(c+b)e^{ix} = c - b + 2i\sqrt{cb}$$

Let $c + b = a$ then $b = a - c$; so we have,

$$ae^{ix} = c - (a-c) + 2i\sqrt{c(a-c)}$$

Or $$ae^{ix} = 2c - a + 2i\sqrt{ca - c^2}$$

Section 2. Checking the validity of the equation:

(A) Let a = 1 then we have

$$e^{ix} = 2c - 1 + 2i\sqrt{c - c^2}$$

Now we check the equality of the equation for different values of c by solving R.H.S first then L.H.S

(i). c = 0

$$e^{ix} = 0 - 1 + 0$$

$x = \cos^{-1}\frac{2c-a}{a}$

$x = \cos^{-1}\frac{-1}{a1} = -1$ therefore $x = \pi$

$e^{i\pi} = -1$, hence L.H.S = R.H.S

(ii). c = 0.25

$$e^{ix} = 0.5 - 1 + 2i\sqrt{0.25 - 0.0625}$$

$$= -0.5 + 2i\sqrt{0.1875}$$

$$= -0.5 + 0.866i$$

$x = \cos^{-1}\frac{0.5-1}{1}$

$x = \cos^{-1} -0.5$, therefore $x = \frac{2\pi}{3}$

$e^{i2\pi/3} = -0.5 + 0.866i$ hence L.H.S = R.H.S

(iii). c = 0.5

$$e^{ix} = 1 - 1 + 2i\sqrt{0.5 - 0.25}$$

$$= i$$

$x = \cos^{-1}\frac{1-1}{1}$, therefore $x = \frac{\pi}{2}$

$e^{i\pi/2} = i$

(iv) c= 0.75

$$e^{ix} = 1.5 - 1 + 2i\sqrt{0.75 - 0.5625}$$

$$= 0.5 + 2i\sqrt{0.1875}$$

$$= 0.5 + 0.866i$$

$x = \cos^{-1}\frac{1.5-1}{1}$

$x = \cos^{-1} 0.5$, therefore $x = \frac{\pi}{3}$

$e^{\pi/3} = 0.5 + 0.866i$ hence L.H.S = R.H.S

(v) c =1

$$e^{ix} = 2 - 1 + 2i\sqrt{1 - 1}$$

$= 1$

$x = \cos^{-1}\frac{2-1}{1}$, therefore x = 0

$e^{0} = 1$, hence L.H.S = R.H.S

(B) Let $a = -1$ then we have

$-e^{ix} = 2c + 1 + 2i\sqrt{-c - c^2}$ or

$e^{ix} = -2c - 1 - 2i\sqrt{-c - c^2}$

(i) c = 0

$e^{ix} = -1$

$x = \cos^{-1}\frac{0+1}{-1}$, therefore x = π

$e^{i\pi}$ = -1, hence L.H.S = R.H.S

(ii) c= -0.25

$e^{ix} = 0.5 - 1 - 2i\sqrt{0.25 - 0.0625}$

$= -0.5 - 2i\sqrt{0.1875}$

e^{ix} = -0.5 $-0.866i$

$x = \cos^{-1}\frac{-0.5+1}{-1}$, therefore $x = \frac{4\pi}{3}$

$e^{i4\pi/3}$ = -0.5 $-0.866i$, hence L.H.S = R.H.S

(iii) c= -0.5

$e^{ix} = 1 - 1 - 2i\sqrt{0.5 - 0.25}$

$e^{ix} = 1 - 1 - 2i\sqrt{0.25}$

$e^{ix} = -i$

$x = \cos^{-1}\frac{-1+1}{-1}$, $x = \frac{3\pi}{2}$

$e^{i\frac{3\pi}{2}} = -i$, hence L.H.S = R.H.S

(iv) c = -0.75

$e^{ix} = 1.5 - 1 - 2i\sqrt{0.75 - 0.5625}$

$e^{ix} = 1.5 - 1 - 2i\sqrt{0.1875}$

$e^{ix} = 0.5 - 0.866i$

x = $\cos^{-1}\frac{-1.5+1}{-1}$ = -0.5, therefore x= $\frac{5\pi}{3}$

$e^{i\frac{5\pi}{3}} = 0.5 - 0.866i$, hence L.H.S = R.H.S

(v) c= -1

$e^{ix} = 2 - 1 - 2i\sqrt{1 - 1}$

$e^{ix} = 1$

x = $\cos^{-1}\frac{-2+1}{-1}$ = 1, therefore x= 2π

$e^{i2\pi} = 1$

Therefore we can conclude that

$ae^{ix} = 2c - a + 2i\sqrt{ca - c^2}$ such that $a \leq$ c $\leq 0 \leq$ c $\leq a$ and

x ≤ π for positive values of a and c and

π ≤x ≤ 2π for negative values of a and c

Based on above values, the following diagram can be presented.

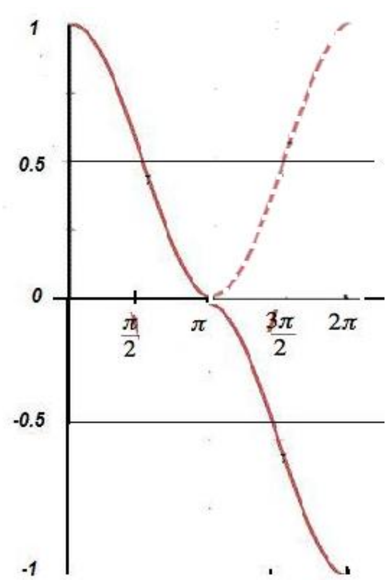

Diagram 2

From the diagram 2, we can see that equation 1 completes half cycle for positive values of a and c and completes other half for negative values of a and c. The other half cycle is the mirror image of dotted cosine wave. That means when c falls from 1 to 0 the circle flips to the negative side of the real number line and the value of c starts falling from 0 to -1. This behavior can be seen in the following diagram.

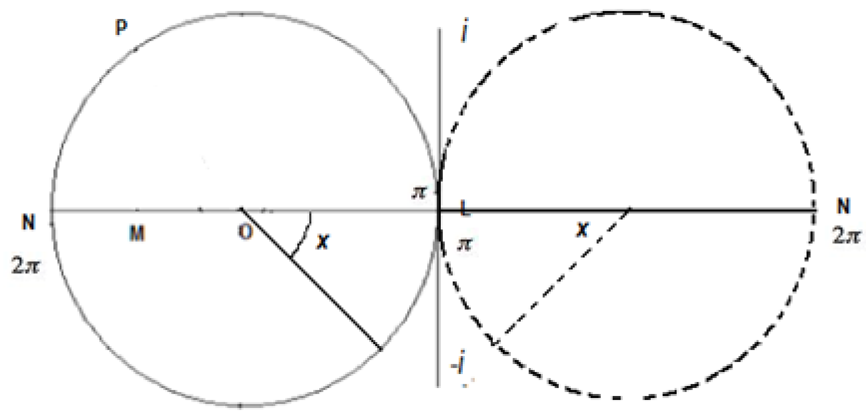

Diagram 3

The left circle is flipped circle or the mirror image of dotted circle and therefore rotating clockwise, the angle x is increasing from π to 2π and the value of c is falling from 0 to -1. The dotted circle depicts the dotted cosine wave in diagram 2 and is rotating anticlockwise.

Section 3 : If c is transcendental then c-e^i or –c+e^i is also transcendental

Lemma 1. If x is algebraic and $x \in \left\{\cos^{-1}\frac{2c-a}{a}\right\}$ then a is algebraic and c is transcendental.

Proof: Consider equation 3

$$\cos x = \frac{c-b}{c+b}$$

Or $\cos x = \frac{c-b}{a} = \frac{2c-a}{a}$ because $b = a - c$.

Therefore $x = \cos^{-1}\frac{2c-a}{a}$

According to Lindemann's theorem, for all algebraic values of x the trigonometric function $\cos x$ is transcendental. But $\frac{2c-a}{a}$ can always be transcendental only if a is algebraic and c is transcendental. This implies that set of algebraic values of x is subset of $\left\{\cos^{-1}\frac{2c-a}{a}\right\}$ when a is algebraic and c is transcendental.

It is not impossible to make number a algebraic of any desired value by adding some unknown transcendental number b in c. Similarly we can obtain any desired algebraic value of x by adjusting the value of a.

Lemma 2. $2i\sqrt{ca-c^2}$ is always transcendental if a is algebraic and c is transcendental.

Proof. Let $ca-c^2$ = y

Where we assume y is algebraic. We get the following quadratic equation:

$$c^2 \quad -ac + \text{y} = 0$$

Therefore $$c = \frac{-(-a) \pm\sqrt{(-a)^2 -4y}}{2} \qquad (4)$$

Since c is transcendental and a is algebraic then equation (4) can only be transcendental if y is transcendental. Therefore our assumption that y or $ca - c^2$ is algebraic is wrong .Hence term $2i\sqrt{ca-c^2}$ is transcendental.

Proposition: $c-e^i$ or $-c+e^i$ is a complex transcendental number where c is any real transcendental number.

Proof: We can re-write equation 1 as follows:

$$a = 2i\sqrt{ca-c^2} + 2c - ae^{ix} \qquad (5)$$

Let a is algebraic and c is transcendental.

We have $x = \cos^{-1}\frac{2c-a}{a}$

According to lemma 1 we can make x alegbraic of value of one radian by adjusting the value of a. Hence equation 5 becomes—

$$a = 2i\sqrt{ca-c^2} + 2c - ae^{i}$$

Since a is algebraic therefore both the terms $2i\sqrt{ca - c^2}$ and $2c - ae^i$ should be algebraic or transcendental.

But from lemma 2 we know $2i\sqrt{ca - c^2}$ is transcendental therefore $2c - ae^i$ is also transcendental.

We can put some algebraic number n in place of 2 and a without affecting the value of the term $2c - ae^i$. Hence we can write --

$$n(c - e^i) = 2c - ae^i$$

If c is negative then a is also negative therfore above equation becomes as follows

$$n(-c + e^i) = -2c + ae^i$$

In this way we conclude that the difference $c - e^i$ or $-c + e^i$ is a complex transcendental number where c is any real transcendental number.