CONVERGENCE OF THE RATIO OF PERIMETER OF A REGULAR POLYGON TO THE LENGTH OF ITS LONGEST DIAGONAL AS THE NUMBER OF SIDES OF POLYGON INCREASES

Pawan Kumar Bishwakarma

Independent Researcher

Corresponding Author: bishowkarmapawan2020@gmail.com

ABSTRACT

A regular polygon is a planar geometrical structure with all sides of equal length and all angles of equal magnitude. The ratio of perimeter of any regular polygon to the length of its longest diagonal is a constant term and the ratio converges to the value of π as the number of sides of the polygon increases. The result has been shown to be valid by actually calculating the ratio for each polygon by using corresponding formula and geometrical reasoning. A computational calculation of the ratio has also been presented to validate the convergence. The values have been calculated up to 30 significant digits.

Keywords: Regular Polygon, π, convergence

INTRODUCTION

A regular polygon is a planar geometrical structure with equal sides and equal angles. For a regular polygon of n sides there are $\frac{n(n-3)}{2}$ diagonals. Each angle of a regular polygon of n sides is given by $180°\frac{n-2}{n}$ while the sum of interior angles is $180°(n-2)$. The angle made at the center of any polygon by lines from any two consecutive vertices (center angle) of a polygon of sides n is given by $\frac{360^o}{n}$. It is evident that each exterior angle of a regular polygon is always equal to its center angle.

The ratio of the perimeter to the longest diagonal or diameter is characteristic feature of any regular polygon. For a regular polygon of even number of sides the ratio is given by $n \sin\left[\frac{180^o}{n}\right]$, while for a regular polygon of odd number of sides the ratio is given by $2n\, sin\left[\frac{90^o}{n}\right]$. As the number of sides of a regular polygon becomes infinitesimally large, i.e. $n \rightarrow \infty$ the resulting polygon is called regular apeirogon which resembles a circle. The regular polygon at this state has countably infinite number of equal edges. The ratio of the perimeter to the longest diagonal reduces to $C\,\frac{sin\theta}{p}$, where C is circumference of the resembling circle, θ is angle opposite to the perpendicular arm and p is perpendicular arm of the right angled triangle whose hypotenuse is diameter of the circle.

For inclusion of all regular polygons equilateral triangle poses a significant hurdle. It has no recognizable diagonal or diameter. The ratio of the perimeter of the equilateral triangle to the length of one of its sides is taken in this case which is equal to 3.

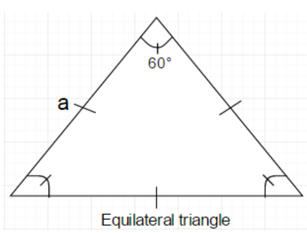
The formulas for the ratio of perimeter to the longest diagonal can be summarized as

I. BK = 3, for n=3
II. BK = f(n) = nsin[$\frac{180°}{n}$], for n is an even number
III. BK = f(n) = 2nsin[$\frac{90°}{n}$], for n is an odd number
IV. BK = $C\ \frac{sin\theta}{p}$, for n $\rightarrow \infty$

The ratio for individual regular polygon has been represented by a single term, BK; BK is short form of Bishwakarma. BK ratio for convenience.

CALCULATION OF BK RATIOS FOR SOME REGULAR POLYGONS

1. For n=3, representing an equilateral triangle



$$BK = \frac{total\ perimeter\ of\ equilateral\ triangle}{a\ side\ length} = \frac{3a}{a} = 3.$$

2. For a regular polygon with even number of sides, i.e. n is an even number and $n \geq 4$
   Square ($n = 4$), Hexagon ($n = 6$), Octagon ($n = 8$) are the basic examples of this types of polygons.
   The length of the longest diagonal d of these regular polygons is given as by $\frac{a}{\sin(\frac{180°}{n})}$ [ref. MM], where a is the length of the side of the polygon.
   If a regular polygon has even number of edges, then length of the longest diagonal of that polygon will be hypotenuse of a right angled triangle for which one of the sides of the polygon is the perpendicular side as illustrated in figure

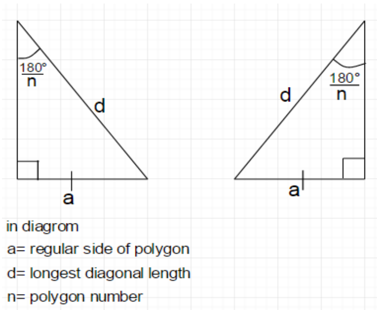


Figure 1: Illustration of the longest diagonal as the perpendicular of a right angled triangle

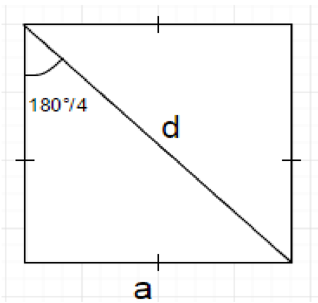
$$\sin(\frac{180°}{n}) = \frac{a}{d}$$

$$or, d = \frac{a}{\sin(\frac{180°}{n})}$$

ILLUSTRATION I

The values of length of the longest diagonal for some regular polygons of have been geometrically presented below. The calculation of the ratio of the perimeter of the corresponding polygon to the length of the longest diagonal has been presented alongside.

I. Square (n=4)



$$\sin(\frac{180°}{4}) = \frac{a}{d}$$

$$d = \frac{a}{\sin 45°}$$

$$= (\sqrt{2})a$$

The ratio of the perimeter of the square to the length of its longest diagonal is

From geometry,

$$BK = \frac{\text{perimeter of the square}}{length\ of\ the\ longest\ diagonal}$$

$$= \frac{4a}{\sqrt{2}\,a}$$

$$= 2\sqrt{2}$$

From formula,

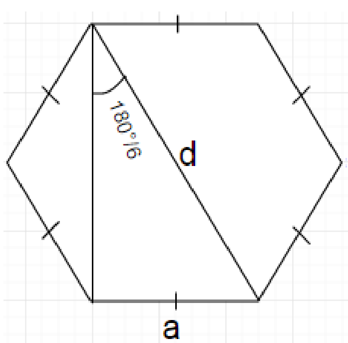
$$BK = n\sin\left[\frac{180°}{n}\right]$$

$$= 4\sin\left[\frac{180°}{4}\right]$$

$$= 4\sin 45°$$

$$= 2\sqrt{2}$$

II. Regular Hexagon (n=6)



$$\sin\left(\frac{180°}{6}\right) = \frac{a}{d}$$

$$d = \frac{a}{\sin 30°}$$

$$= 2a$$

The ratio of the perimeter of the square to the length of its longest diagonal is

From geometry,

$$BK = \frac{\text{perimeter of the hexagon}}{length\ of\ the\ longest\ diagonal}$$

$$= \frac{6a}{2a}$$

$$= 3$$

From formula,

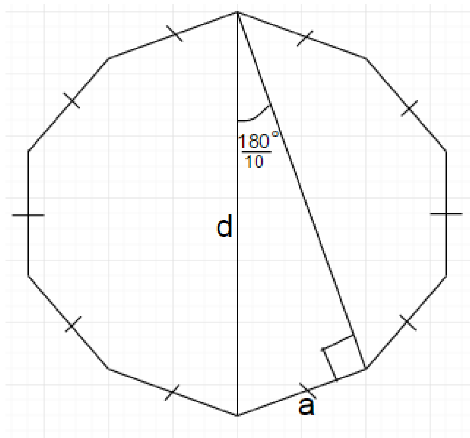
$$BK = n\sin\left[\frac{180°}{n}\right]$$

$= 6\ \sin\left[\frac{180°}{6}\right]$

$= 6\ \sin 30°$

$= 3$

Regular Decagon (n=10)



Regular decagon

$$\sin\left(\frac{180°}{10}\right) = \frac{a}{d}$$

$$d = \frac{a}{\sin 18°}$$

The ratio of the perimeter of the square to the length of its longest diagonal is

From geometry,

$$BK = \frac{\text{perimeter of the decagon}}{\textit{length of the longest diagonal}}$$

$$= \frac{10a}{\frac{a}{\sin 18°}}$$

$$= 3.090169$$

From formula,

$BK = n\sin\left[\frac{180°}{n}\right]$

$= 10\ \sin\left[\frac{180°}{10}\right]$

= 10 sin 18°

= 3.0901

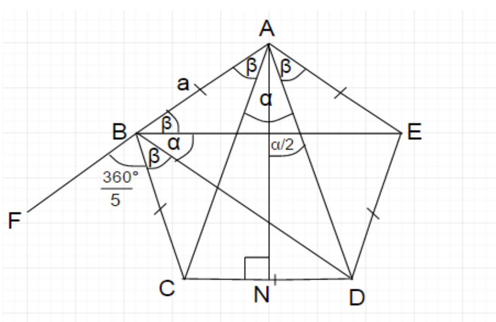
Similarly we can use BK= $n\sin\left[\frac{180°}{n}\right]$ formula for all regular polygons having even number of edges. Therefore BK= $n\sin\left[\frac{180°}{n}\right]$ is a valid formula for given domain.

3. For a regular polygon with odd number of sides, i.e. n is an odd number and $n \geq 4$
Pentagon ($n = 5$), Heptagon ($n = 7$), Nonagon ($n = 9$) are the basic examples of this types of polygons.
The length of the longest diagonal d of these regular polygons is given as by $\frac{a}{2\sin(\frac{90°}{n})}$ [ref. MM], where a is the length of the side of the polygon.

ILLUSTRATION 2

The value of length of the longest diagonal for some regular polygons of have been geometrically presented below. The calculation of the ratio of the perimeter of the corresponding polygon to the length of the longest diagonal has been presented alongside.

I. Regular Pentagon (n = 5)



Regular pentagon

Diagonals from D and E are drawn to B such that it subtends an angle α, and creates equal angles β on either side of the angle α. Similarly diagonals AC and AD subtend an angle α at A creating equal angles β on either side of the angle α.

Side AB is extended upto F and AN ⊥ CD such that

- Triangle ACD and BDE are isosceles
- Tirangle CAN and DAN are right angled triangles

From △ABD,

(α+ β) + (α+ β) + α = 180° (sum of interior angles of a triangle)

3α + 2β = 180°…………………….(i)

On straight line ABF,

∠ABE + ∠EBD + ∠ DBC + ∠CBF = 180° (angle add up on a straight line)

$$B + \alpha + \beta + \frac{360°}{n} = 180°$$

$$2\beta = 180° - (\alpha + \frac{360°}{n}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

From equation (i) and (ii)

$$180° - (\alpha + \frac{360°}{n}) = 180° - 3\alpha$$

$$\alpha = \frac{360°}{2n} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

In triangle ACD,line AN divides triagle ACD into equal two parts

$\angle CAN = \angle DAN = \frac{\alpha}{2}$ & CN = ND = a /2

From right angled triangle CAN and DAN, AC=AD, both AC and AD are longest diagonal(d) for pentagon and hypotenuse for right angled triangles CAN and DAN respectively.

$$\sin(\frac{\alpha}{2}) = \frac{a/2}{AC}$$

$$AC = d = \frac{(a/2)}{\sin(\frac{\alpha}{2})}$$

$$AC = \frac{a}{2\sin(\frac{\alpha}{2})} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

From equation (iii) and (iv)

$$AC = \frac{a}{2\sin(\frac{360°}{2n} \times \frac{1}{2})}$$

$$= \frac{a}{2\sin(\frac{90°}{n})}$$

For pentagon

$$BK = \frac{\text{perimeter of regular pentagon}}{\textit{longest diagonal length}}$$

$$= \frac{5a}{\frac{a}{2\sin(\frac{90°}{5})}}$$

= 3.090169…

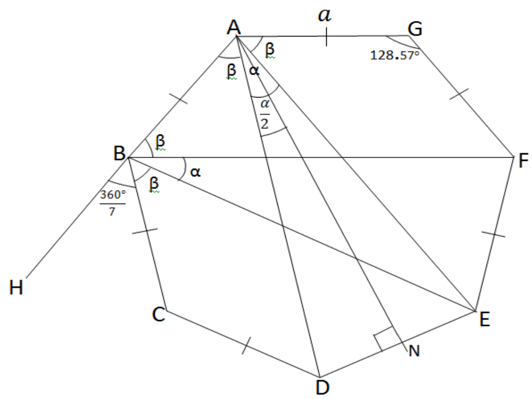
From formula,

$BK = 2n \sin[\frac{90°}{n}]$

$= 10 \sin\left[\frac{90°}{5}\right]$

$= 3.090169\ldots$

II. Regular Heptagon(n=7)



Regular Heptagon

From both vertex angle A and B of Heptagon an angle 'α' taken from the middle so that by symmetry all angles marked 'β' are equal in diagram.

Line AB extended upto H

Triangle ADE and BEF are isosceles

Tirangle DAN and EAN are right angled triangle

$\angle AEB = \angle DAE = \angle EBF = \alpha$ (isosceles triangles)

From ABE triangle,

$(\alpha + \beta) + (\alpha + \beta) + \alpha = 180°$ {∴ sum of interior angles of a triangle}

$3\alpha + 2\beta = 180°$……………………..(i)

On straight line ABH,

$\angle ABF + \angle FBE + \angle EBC + \angle CBH = 180°$

$B + \alpha + \beta + \frac{360°}{n} = 180°$

$2\beta = 180° - \left(\alpha + \frac{360°}{n}\right)$…………………..(ii)

From equation (i) and (ii)

$180° - \left(\alpha + \frac{360°}{n}\right) = 180° - 3\alpha$

$\alpha = \frac{360°}{2\,n}$ ……………………(iii)

In triangle ADE,line AN divides triagle ADE into equal two parts

$\angle DAN = \angle EAN = \frac{\alpha}{2}$ and $DN = NE = a/2$

From right angled triangle DAN and EAN, AD=AE, both AD and AE are longest diagonal(d) for heptagon and hypotenuse for right angled triangles DAN and EAN respectively.

Here in right angled triangle DAN

$\sin(\frac{\alpha}{2}) = \frac{a/2}{AD}$

$AD = d = \frac{a/2}{\sin(\frac{\alpha}{2})}$

$AD = \frac{a}{2\sin(\frac{\alpha}{2})}$ ………………… (iv)

From equation (iii) and (iv)

$AD = \frac{a}{2\sin(\frac{360°}{2n}\times\frac{1}{2})} = \frac{a}{2\sin(\frac{360°}{4n})} = \frac{a}{2\sin(\frac{90°}{n})}$

This beautiful rule holds all the regular polygons having odd number of sides

Since n=7 for heptagon

$BK = \frac{\text{perimeter of regular pentagon}}{longest\ diagonal\ length} = \frac{7a}{\frac{a}{2\sin(\frac{90°}{7})}} = 7\times 2\sin(\frac{90°}{7}) = 14\sin(\frac{90°}{7}) = 3.115293\ldots$

From formula,

$BK = 2n\sin[\frac{90°}{n}]$

$BK = (2\times 7)\sin[\frac{90°}{7}]$

$= 14\sin[\frac{90°}{7}] = 3.115293\ldots$

Similarly this formula gives an accurate ratio of perimeter to the diagonal of all polygons having odd number of sides.

4. For $n \to \infty$ representing a apeirogon

For a given perimeter, if n become very large (or countably infinite) then a regular polygon becomes regular apeirogon [ref]. Regular apeirogon is ultimate form of regular polygon and it has countably infinite number of equal edges.

The ratio of the perimeter of this polygon to the length of its largest diagonal is calculated by the limiting value as $n \to \infty$. The limiting value is calculated to be π.

Proof

For polygons of even number of sides

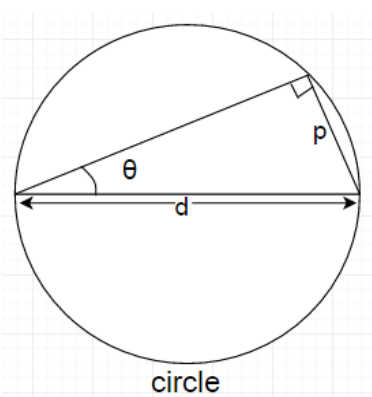
$$BK_{n\to\infty} = \lim_{n\to\infty} n\,sin\left[\frac{180°}{n}\right]$$
$$= \pi$$

For polygons of odd number of sides

$$BK_{n\to\infty} = \lim_{n\to\infty} 2n\sin\left[\frac{90°}{n}\right]$$
$$= \pi$$

ANALOGY TO CIRCLE

By analogy to regular polygons with large number of edges, regular apeirogon resembles a circle. In the figure below we can see that C is circumference of circle, θ is angle opposite to the perpendicular arm and p is length of the perpendicular arm of the right angled triangle whose hypotenuse is diameter of the circle.



If two lines from diametrically opposite points of the circle intersect at any point on the circumference then angle formed on the circumference by these two lines is always 90°.

ILLUSTRATION

From ΔABC

$$\sin\theta = \frac{p}{d}$$

or, $d = \frac{p}{\sin\theta}$

The BK ratio is then given by

$$BK = \frac{C}{d}$$

$$= \frac{C}{\frac{p}{\sin\theta}}$$

$$= C\frac{\sin\theta}{p}$$

$$\therefore BK = C\frac{\sin\theta}{p}$$

COMPUTATIONAL ILLUSTRATION

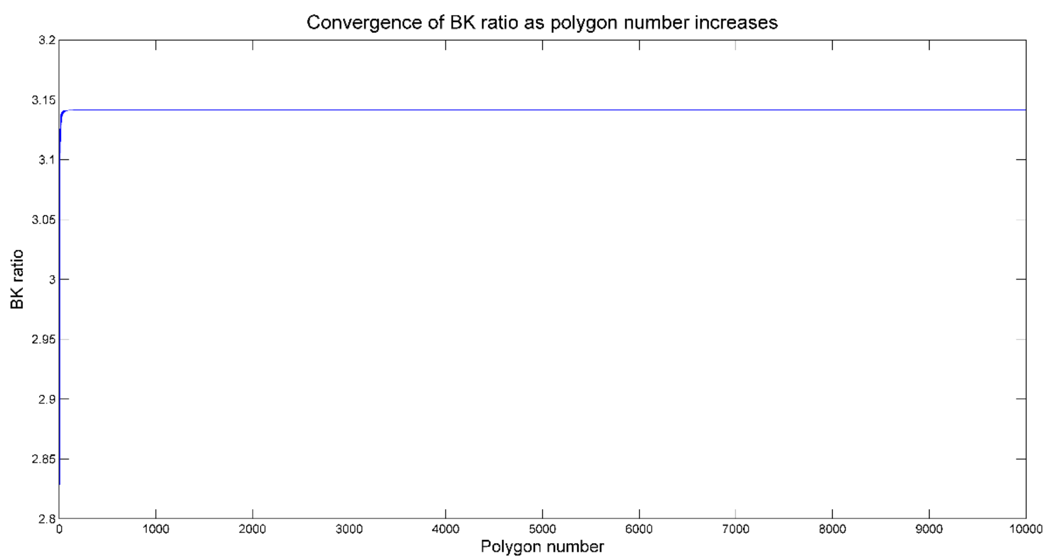


Figure 2: Convergence of the BK ratio as the number of sides of a regular polygon increases

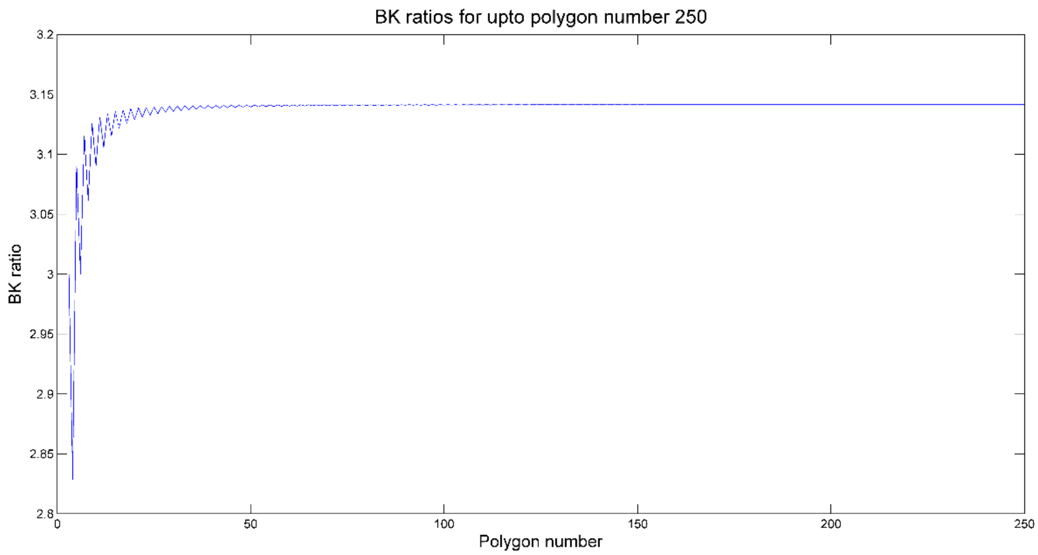


Figure 3: Fluctuations of the BK ratio for the number of sides of a regular polygon from 3 to 250

Figure 2 shows the convergence of the BK ratios as the polygon number increases. However, at lower polygon number there are significant fluctuations in the value of the ratio. As the polygon number exceeds 250 the fluctuation is approximately negligible and approaches the value of π.

APPENDIX

The numerical value of the BK Ratios have been presented below:

Serial number of regular polygon	**Name of regular polygon**	**BK Ratio**	**BK ratio with precision up to 30 decimal digits**
1	Equilateral triangle	BK = 3	3.000000000000000000 00000000000
2	Square	BK = $n\sin[\frac{180°}{n}]$= $4\sin[\frac{180°}{4}]$ = 4sin45°	2.828427124746190097 60337744842
3	Pentagon	BK= 2nsin[$\frac{90°}{n}$]=2× 5sin[$\frac{90°}{5}$] = 10sin18°	3.090169943749474241 02293417183
4	Hexagon	BK = 6 sin 30°	3.000000000000000000 00000000000
5	Heptagon	BK = 14sin(90°/7)	3.115293075388401660 04463590295
6	Octagon	BK = 8sin(180°/8)	3.061467458920718173 82767987224
7	Nonagon	BK = 18sin 10°	3.125667198004746279 33089928185

8	Decagon	BK =10sin 18°	3.09016994374947424102293417183
9	Hendecagon/Undecagon	BK = 22sin(90°/11)	3.13092644201227308976343870956
10	Dodecagon	BK = 12sin 15°	3.10582854123024914818678605149
11	Tridecagon	BK = 26sin(90°/13)	3.13395368663839938707575987377
12	Tetradecagon	BK = 14sin sin(180°/14)	3.11529307538840166004463590295
13	Pentadecagon	BK = 30sin 6°	3.13585389802960414199502464407
14	Hexadecagon	BK = 16sin(180°/16)	3.12144515225805228557255789563
15	Heptadecagon	BK = 34sin(90°/17)	3.13712422175226783814813764325
16	Octadecagon	BK = 18sin 10°	3.12566719800474627933089928185
17	Enneadecagon	BK = 38sin(90°/19)	3.13801512794862833481306950102
18	Icosagon	BK = 20sin(180°/20)	3.12868930080461738020210638934
19	Icosihenagon	BK = 42sin(90°/21)	3.13866393062981868021946932086
20	Icosikaidigon	BK = 22sin(180°/22)	3.13092644201227308976343870956
21	Icosikaitrigon	BK = 46sin(90°/23)	3.13915101477486489237467003503
22	Icosikaitetragon	BK = 24sin(180°/24)	3.13262861328123819716174946949
23	Icosikaipentagon	BK = 50sin(90°/25)	3.13952597646566880380891122828
24	Icosikaihexagon	BK = 26sin(180°/26)	3.13395368663839938707575987377
25	Icosikaiheptagon	BK = 54sin(90°/27)	3.13982076116569474109239290974
26	Icosikaioctagon	BK = 28sin(180°/28)	3.13500533089262003712376618762
27	Icosikaienneagon	BK = 58 sin(90°/29)	3.14005669795421651669468290652
28	Tricontagon	BK = 30sin 6°	3.13585389802960414199502464407
29	Tricontakaihenagon	BK = 62sin(90°/31)	3.14024846800018816128261516408
30	Tricontakaidigon	BK = 32sin(180°/32)	3.13654849054593926381425804444
31	Tricontakaitrigon	BK = 66sin(90°/33)	3.14040644436699163168595810608
32	Tricontakaitetragon	BK = 34sin(180°/34)	3.13712422175226783814813764325

33	Tricontakaipentagon	BK = 70sin(90°/70)	3.14053812453604478206632357613
34	Tricontakaihexagon	BK = 36sin 5°	3.13760673891569424809031375015
35	Tricontakaiheptagon	BK = 74sin(90°/37)	3.14064903651497463497711987755
36	Tricontakaioctagon	BK = 38sin(180°/38)	3.13801512794862833481306950102
37	Tricontakaienneagon	BK = 78sin(90°/39)	3.14074332853438118223244911686
38	Tetracontagon	BK = 40sin(180°/40)	3.13836382911379780131840983974
39	Tetracontakaihenagon	BK = 82sin(90°/41)	3.14082416258289860185044758480
40	Tetracontakaidigon	BK = 42sin(180°/42)	3.13866393062981868021946932086
41	Tetracontakaitrigon	BK = 86sin(90°/43)	3.14089398295865980729056484846
42	Tetracontakaitetragon	BK = 44sin(180°/44)	3.13892406076622297440286111695
43	Tetracontakaipentagon	BK = 90sin 2°	3.14095470322508744813956634628
44	Tetracontakaihexagon	BK = 46sin(180°/46)	3.13915101477486489237467003503
45	Tetracontakaiheptagon	BK = 94sin(90°/47)	3.14100783872140966227484961792
46	Tetracontakaioctagon	BK = 48sin(180°/48)	3.13935020304686720713514682121
47	Tetracontakaienneagon	BK = 98sin(90°/49)	3.14105460202220707488795148387
48	Pentacontagon	BK = 50sin(180°/50)	3.13952597646566880380891122828
49	Pentacontakaihenagon	BK = 102sin(90°/51)	3.14109597272937609520561928758
50	Pentacontakaidigon	BK = 52sin(180°/52)	3.13968186595887477877704164220
.	.	.	.
99	hectohenagan	BK = 202sin(90°/101)	3.14146600791087655005140024123
100	Hectokaidigon	BK = 102sin(180°/102)	3.14109597272937609520561928757
. .	. .	. .	. .
198	Dihectogon	BK = 200sin(180°/200)	3.14146346236413515065907066198
199	Dihectohenagon	BK = 402sin(90°/201)	3.14156067605829806847912624424
200	Dihectokaidigom	BK = 202sin(180°/202)	3.14146600791087655005140024123

.	.	.	.
498	Pentahectogon	BK = 500sin(180°/500)	3.14157198277947562486765507897
499	Pentahectohenagon	BK = 1002sin(90°/501)	3.14158750648854657344911867315
500	Pentahectokaidigon	BK = 502sin(180°/502)	3.14157214715870277377552989356
.	.	.	.
998	Chilliagon	BK = 1000sin(180°/1000)	3.14158748587956335193322703549
999	Chilliahenagon	BK = 2002sin(90°/1001)	3.14159136424174274196381045095
1000	Chilliaidigon	BK = 1002sin(180°/1002)	3.14158750648854657344911867315
.	.	.	.
4998	Pentakischilliagon	BK = 5000sin(180°/5000)	3.14159244688128611672626766426
4999	Pentakischilliahenagon	BK = 10002sin(90°/5001)	3.14159260193333034429355391453
5000	Pentakischilliadigon	BK = 5002sin(180°/5002)	3.14159244704655375197152596898
.	.	.	.
9998	Myriagon	BK = 10000sin(180°/10000)	3.14159260191266569297934647928
9999	Myriahenagon	BK = 20002sin(90°/10001)	3.14159264067309477313310688367
10000	Myriakaidigon	BK = 10002sin(180°/10002)	3.14159260193333034429355391453
.	.	.	.
99998	Centachilliagon	BK = 100000sin(180°/100000)	3.14159265307302196048314802075
99999	Centachilliahenagon	BK = 200002sin(90°/100001)	3.14159265346060300278062061722
100000	Centachilliakaidigon	BK = 100002sin(180°/100002)	3.14159265307304263071415718326
.	.	.	.
999998	Megagon	BK = 1000000sin(180°/1000000)	3.14159265358462552568259596341
999999	Megahenagon	BK = 2000002sin(90°/1000001)	3.14159265358850131285148356440
1000000	Megakaidigon	BK = 1000002sin(180°/1000002)	3.14159265358462554635338507120
.	.	.	.
10^{12}-2	Teragon(10^{12} sides)	BK = 10^{12} sin(180°/10^{12})	3.14159265358979323846263821556
10^{12} -1	(10^{12}+1)-gon	BK = 2(10^{12}+1)sin(90°/(10^{12}+1))	3.14159265358979323846264209135

10^{12}	$(10^{12}+2)$-gon	$BK = (10^{12}+2)\sin(180°/(10^{12}+2))$	3.141592653589793238 46263821556 6722854897991273463
.	.	.	.
10^{15}-2	Petagon(10^{15} sides)	$BK = 10^{15}\sin(180°/10^{15})$	3.141592653589793238 46264338327 43351714171194293458
10^{15}-1	$(10^{15}+1)$-gon	$BK = 2(10^{15}+1)\sin(90°/(10^{15}+1))$	3.141592653589793238 46264338327 82109560021569094516
10^{15}	$(10^{15}+2)$-gon	$BK = (10^{15}+2)\sin(180°/(10^{15}+2))$	3.141592653589793238 46264338327 43351714171194500168
.	.	.	.
∞	Regular Apeirogon (Precise Circle) It is ultimate regular polygon	$BK = C\frac{\sin\theta}{p} = \pi$	3.141592653589793238 46264338327

REFERENCES

[1] "Murderous Maths, The LOngest Diagonal Formula!," [Online]. Available: http://www.murderousmaths.co.uk/books/longdiag.htm.

[2] [Online]. Available: http://faculty.uoit.ca/kay/courses/curs4140/lessons/lesson7/9_gr7gazebo.pdf.

[3] "Polygons and Ratios," [Online]. Available: https://mathteachme.wordpress.com/2015/10/05/polygons-and-ratio/.