Fermat's *zero* theorem

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Abstract
Fermat's *zero* theorem is stated as follows: *It is impossible to separate a square of a **difference** of two natural numbers into two squares of **differences**, or a cube power of a **difference** into two cube powers of **differences**, or a fourth power of a **difference** into two fourth powers, or in general, any power higher than the first, into two like powers of **differences**.*

MSC numbers: 11Axx, 11Dxx, 11D04, 11D09, 11A41, 11N80[2]
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Introduction

Around 1637, Fermat wrote his Last Theorem in the margin of his copy of the *Arithmetica* next to Diophantus' sum-of-squares problem:

> *It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second, into two like powers. I have discovered a truly marvelous proof of this, which this margin is too narrow to contain.*

Fermat proofed the case $n = 4$, as described in the section *Proofs for specific exponents*.
Andrew Wiles proofed the Fermat last theorem using a 20th-century technique.

Discussion

Fermat's zero theorem is stated as follows: There are no triple of ***distinct*** natural numbers (a,b,c) with $(a < b < c)$ satisfying $(c-a) = (c-b) + (b-a)$ also satisfying $(c-a)^n = (c-b)^n + (b-a)^n$ for a natural number $n > 1$.
I discuss proofs for natural numbers $n = 1,2$ and $n = 3$ of the equation

$$(c-a)^n = (c-b)^n + (b-a)^n$$

where (a,b,c) with $(a < b < c)$ are natural numbers.

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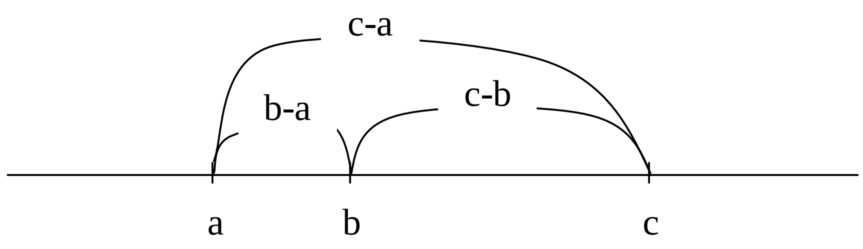
[2] https://cran.r-project.org/web/classifications/MSC.html
http://www.ams.org/msc/msc2010.html

(1) $n=1$

It is obvious that every triple of natural numbers (a,b,c) with $(a<b<c)$, satisfy

$$(c-a)^1=(c-b)^1+(b-a)^1$$



Natural numbers line

(2) Proof of $n=2$

We have

$$(c-a)^2=(c-b)^2+(b-a)^2$$

Expanding both sides

$$\begin{aligned} c^2-2ac+a^2 &= (c^2-2bc+b^2)+(b^2-2ab+a^2) \\ &= c^2-2bc+b^2+b^2-2ab+a^2 \\ &= c^2-2bc+2b^2-2ab+a^2 \\ &= c^2-2(c+a)b+2b^2+a^2 \end{aligned}$$

Cancelling similar terms on both sides and rearranging

$$2ac-2(c+a)b+2b^2=0$$

Dividing both sides by 2, and rearranging

$$b^2-(a+c)b+ac=0$$

It is a quadratic equation which has solutions

$$\begin{aligned} b &= \frac{+(a+c)\pm\sqrt{(a+c)^2-4ac}}{2} \\ &= \frac{+(a+c)\pm\sqrt{a^2+2ac+c^2-4ac}}{2} \\ &= \frac{+(a+c)\pm\sqrt{a^2-2ac+c^2}}{2} \\ &= \frac{+(a+c)\pm\sqrt{(a-c)^2}}{2} \\ &= \frac{+(a+c)\pm(a-c)}{2} \end{aligned}$$

$b=\dfrac{+(a+c)+(a-c)}{2}=\dfrac{2a}{2}=a$, or, $b=\dfrac{+(a+c)-(a-c)}{2}=\dfrac{2c}{2}=c$.

For $n=2$ triples of natural numbers (a,b,c) with $(a<b<c)$ are not ***distinct***.

Practical example for $n=2$

It is true that $(5)^2=(3)^2+(4)^2$, putting $(c-a)=5$, $(c-b)=3$ and $(b-a)=4$, gives the result $5=7$.

(3) Proof of $n=3$

We have

$$(c-a)^3=(c-b)^3+(b-a)^3$$

Expanding both sides

$$\begin{aligned} c^3-3c^2a+3ca^2-a^3 &= (c^3-3c^2b+3cb^2-b^3)+(b^3-3b^2a+3ba^2-a^3) \\ &= c^3-3c^2b+3cb^2-b^3+b^3-3b^2a+3ba^2-a^3 \\ &= c^3-3c^2b+3cb^2-b^3+b^3-3b^2a+3ba^2-a^3 \\ &= c^3-3(c^2-a^2)b+3(c-a)b^2-a^3 \end{aligned}$$

Cancelling similar terms on both sides and rearranging

$$3c^2a-3ca^2-3(c^2-a^2)b+3(c-a)b^2=0$$

$$3ca(c-a)-3(c^2-a^2)b+3(c-a)b^2=0$$

$$3ca(c-a)-3(c-a)(c+a)b+3(c-a)b^2=0$$

Dividing both sides by $3(c-a)$, and rearranging

$$b^2-(c+a)b+ca=0$$

This is the same as in case $n=2$. The values of b are either a or c.
For $n=3$ triples of natural numbers (a,b,c) with $(a<b<c)$ are not ***distinct***.

(4) Proof for $n=4$

We have

$$\begin{aligned} c^4-4c^3a+6c^2a^2-4ca^3+a^4 &= (c^4-4c^3b+6c^2b^2-4cb^3+b^4)+(b^4-4b^3a+6b^2a^2-4ba^3+a^4) \\ &= c^4-4c^3b+6c^2b^2-4cb^3+b^4+b^4-4b^3a+6b^2a^2-4ba^3+a^4 \\ &= c^4-4(c^3+a^3)b+6(c^2+a^2)b^2-4(c+a)b^3+2b^4+a^4 \end{aligned}$$

Cancelling similar terms on both sides and rearranging

$$4c^3a-6c^2a^2+4ca^3-4(c^3+a^3)b+6(c^2+a^2)b^2-4(c+a)b^3+2b^4=0$$

Dividing both sides by 2, and rearranging

$$b^4-2(c+a)b^3+3(c^2+a^2)b^2-2(c^3+a^3)b+2c^3a-3c^2a^2+2ca^3=0$$

It is a quartic equation which has same solutions as for the cases of $n=2$ and $n=3$ as can be easily checked by direct substitution of $b=a$ or $b=c$ in the quartic equation above.
All higher powers give the same result.

Conclusion

The differences of natural numbers have interesting properties as the natural numbers themselves.

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