THE KAKEYA TUBE CONJECTURE IMPLIES THE KAKEYA CONJECTURE

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Abstract. In this article we will give a proof that the Kakeya tube conjecture implies the Kakeya conjecture.

1. Introduction

We define the \( \delta \)-tubes in standard way: for all \( \delta > 0 \), \( \omega \in S^{n-1} \) and \( a \in \mathbb{R}^n \), let

\[
T^\delta_{\omega}(a) = \{ x \in \mathbb{R} : |(x - a) \cdot \omega| \leq \frac{1}{2} |\text{proj}_{\omega \perp}(x - a)| \leq \delta \}.
\]

In this paper any constant can depend on dimension \( n \). A Kakeya set is a compact set that contains an unit line in every direction. We will give a proof that the result \( \bigcup_{\omega \in \Omega} T_{\omega} \approx 1 \) for maximal set of \( \delta \)-tubes implies the Kakeya conjecture.

Theorem 1 (Kakeya conjecture). Any Kakeya set has full Hausdorff dimension.

2. The Proof

For our definition of Hausdorff content see for example [6]. Let \( K \) be a Kakeya set, that is, a set that contains an unit line in every direction. Let \( \bigcup_{j=1}^\infty B_j \) be a cover of \( K \) with balls of diameters less than \( 1 > \beta > 0 \). Let \( n > n - \alpha > 0 \) be such that

\[
\sum_{j=1}^\infty r_j^{n-\alpha} < 1.
\]

If the hausdorff content is zero that kind of cover exists. By compactness of the Kakeya set we can take a subcover with diameters such that \( 1 > \beta > r_j \geq \delta > 0 \), where at least one \( r_j \sim \delta \). Now, assume

\[
\sum_{j=1}^M r_j^n \gtrsim |\bigcup_{j=1}^M B_j| \gtrsim |\bigcup_{i=1}^N T_i| \gtrsim 1.
\]

The second inequality above follows because the balls cover the middle lines of the tubes, so there exists a constant such that the second inequality above is valid. Using inequality (1) and (2) we obtain

\[
C_{\alpha/k}^{\delta-\alpha/k} \sum_{j=1}^M r_j^n > \sum_{j=1}^M r_j^{n-\alpha}.
\]

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Thus,

\begin{equation}
\sum_{j=1}^{M} r_j^n (C_{\alpha/k} \delta^{-\alpha/k} - r_j^{-\alpha}) > 0.
\end{equation}

It follows that for the average value of a power of diameters it holds that

\begin{equation}
C_{\alpha/k} \delta^{-\alpha/k} > \frac{1}{M} \sum_{j=1}^{M} r_j^{-\alpha} \geq \frac{1}{M-\alpha} \left( \sum_{j=1}^{M} r_j \right)^{-\alpha},
\end{equation}

where we used Jensen’s inequality. Thus,

\begin{equation}
(\frac{c_{\alpha}}{M} \sum_{j=1}^{M} r_j) > \delta^{1/k}.
\end{equation}

From above it follows that

\begin{equation}
(\frac{c_{\alpha}}{M})^n \left( \sum_{j=1}^{M} r_j^n \right) \geq \left( \frac{c_{\alpha}}{M} \right)^n \left( \sum_{j=1}^{M} r_j \right)^n > \delta^{n/k},
\end{equation}

where we used Jensen’s inequality again. Thus, from above and inequality (1)

\begin{equation}
C_{\alpha} > M\delta^{n/k}.
\end{equation}

It follows from above that

\begin{equation}
\delta^{-n/k} C_{\alpha} > M
\end{equation}

We can do the steps (3), (4) and (5) again for \( \epsilon = \alpha/2 \) and obtain

\begin{equation}
C_{\alpha/2} \delta^{-\alpha/2} > \frac{1}{M} \sum_{j=1}^{M} r_j^{-\alpha}.
\end{equation}

Let \( k \) and a small \( \delta \) be such that

\begin{equation}
\delta^{-\alpha/3} > C_{\alpha} \delta^{-n/k}.
\end{equation}

From above and inequalities (7) and (8) we obtain

\begin{equation}
C_{\alpha/2} \delta^{-\alpha/2} > \delta^{\alpha/3} \sum_{j=1}^{M} r_j^{-\alpha} > \delta^{\alpha/3} \delta^{-\alpha} = \delta^{-2/3},
\end{equation}

which is a contradiction when \( \delta \) is small.

REFERENCES
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