Generalized Painlevé-Gambier XVII equation and Applications

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Abstract

This work exhibits a generalized Painlevé-Gambier XVII equation and its applications in physics.

1. Introduction

A century later, after their elaboration, Painlevé-Gambier equations continue to be the subject of intense study since their analytical properties and applications are not fully investigated. Ince in his book [1] studied many aspects of the properties of these equations by determining the first integrals and corresponding analytical solutions. Recently other authors have studied these equations by computing other first integrals in addition to those found by Ince [1] in order to systematically express the exact analytical solutions by the generalized Sundman transformation [2]. The generalized Sundman transformation is a powerful mathematical tool in the linearization of nonlinear differential equations to facilitate their solving process by the usual methods [2-4]. This method of Sundman linearization was recently applied to detect for the first time the existence of a class of nonlinear differential equations of Liénard type whose solutions are trigonometric [3] but with amplitude dependent frequency or with isochronicity that is amplitude independent frequency. Thus it was allowed to show for the first time that some Painlevé-Gambier equations could admit explicit and exact general trigonometric periodic solutions [3].

Recently, this generalized Sundman transformation proposed by Akande et al. [3] was used to calculate the explicit and exact general periodic solutions of the cubic Duffing equation and of some Painlevé-Gambier equations [4].The present work is still interested in the Painlevé-Gambier equations, in particular in Painlevé-Gambier equation XVII. It must be noted, first of all, that the general solution of this equation has been given in the book of Ince [1]. So under these conditions, it remains in the analytical investigation of this equation to propose other elegant alternative ways for explicit and exact general solutions, generalized equations or evidence of other properties which are not directly studied by Ince [1]. Thus, the problem of interest in this work is to build a generalized Painlevé-Gambier XVII equation with isochronicity property [5, 6],which admits explicit and exact general solution of trigonometric form [4]and the linear harmonic oscillator equation, that is the prototype of isochronous systems, as special case with several applications in physics. Therefore, the question to be solved under these conditions can be announced as follows: Is there a generalized Painlevé-Gambier XVII equation admitting the linear harmonic oscillator equation as limiting case? It is asserted that this generalized Painlevé-Gambier XVII equation with isochronicity exists. The identification of this equation is of great importance given the numerous possibilities of applications it may offer in physics, especially in classical mechanics, quantum mechanics, mechanics of continuous media and nonlinear optics. The nonlinear differential equations with isochronicity may be of high interest in physics and engineering applications when the dynamical systems with amplitude independent frequency are concerned. In classical mechanics as well as in practical applications, the linear harmonic oscillator equation, as well known, is often used in the study of mechanical oscillations [7]. However, this structural model could not take into consideration, for example, the nonlinear phenomena of energy dissipation in heat and geometrical nonlinearities exhibited by real mechanical systems. As such, nonlinear differential equations with explicit and exact general periodic solutions with the damping and geometrical nonlinearity properties for which the linear harmonic oscillator equation consists of a limiting case are required in mathematical modeling of mechanical oscillations. In that, it has been observed that these dissipation phenomena and geometrical nonlinearities may be better captured by oscillators described by quadratic Liénard type equations [8-15]. These quadratic Liénard type equations may therefore be considered in the calculation of the quantum properties of dynamical systems. In addition such equations exhibit a position dependent mass dynamics, which has been of great importance in improving description of the classical and quantum properties of dynamical systems [11-14, 16]. Thus, these harmonic oscillators with a mass varying with distance have been used in many fields of quantum physics as well as engineering applications, and are still the subject of intense research activity [11-14, 16, 17]. However, the Schrödinger equation with a mass varying with distance becomes quickly more complicated to be formulated and solved than the usual Schrödinger equation with a constant mass. The formulation of such a Schrödinger equation is not easy because the Hamiltonian operator is no longer Hermitian. One way of overcoming this difficulty is to consider the expression of the Hermitian Hamiltonian of von Roos [18]. But this introduces another complication in the Schrödinger equation known under the ambiguity parameter problem, because there is no rule leading to a rational parametric choice. As soon as this problem is solved, it remains therefore the choice of the appropriate method of solving the Schrödinger eigenvalue equation with a position dependent mass. There is a multitude of techniques for solving such equations in the literature, which proves that there is no standard method that works in all cases of study. In this contribution, the quantization of the generalized Painlevé-Gambier XVII equation has been carried out in the formalism of the Schrödinger equation with a mass varying with the displacement to show its application in quantum mechanics. The mechanics of continuous media is devoted to study the loading behavior of structural materials assimilated to media with continuous properties as opposed to discrete considerations. In this sense, the continuous media are often modeled according to three schemes. The elastic medium which instantaneously covers its initial state after removal of the stress causing its deformation may be distinguished. The viscoelastic medium which covers its initial state in a delayed manner after removal of the involved stress and the viscoplastic medium which exhibits a permanent residual deformation after suppression of stress may also be noticed [15]. The viscoelastic media are essentially characterized by the nonlinear phenomena of creep under constant stress and relaxation of stress under constant deformation. Such phenomena are of central importance in the integrity of the structures designed in these materials during their service, which can threat the safety and comfort of the users. So one may note that the prediction and numerical simulation models of such phenomena is of considerable importance. In this perspective the generalized Painlevé-Gambier XVII equation is shown to have the ability to be used in the interpretation of stress relaxation curves in materials with viscoelastic behavior, that is to say, it can be considered as a simulation model of this phenomenon. On the other hand, it is well known in nonlinear optics that the dielectrics exhibit a polarization phenomenon when they are subjected to an electric field, which may be connected to the resulting strain. In general the strain response is viscoelastic due to the phenomena of dissipation of energy in heat which are observed in living materials as well as in engineering materials. In this respect, it is shown that the generalized Painlevé-Gambier XVII equation may be useful in describing the polarization dynamics of dielectrics. Therefore to perform the purpose of this work, the general theory needed to be considered [6] in the determination of the generalized Painlevé-Gambier XVII equation under study is briefly formulated (Section 2), as well as its explicit and exact general solutions (Section 3). Finally, the applications of this generalized Painlevé-Gambier XVII equation are exhibited (Section 4) as well as a conclusion for all the work.

2. General theory

It has recently been shown by Monsia et al. [6] that some classes of mixed or quadratic Liénard type nonlinear differential equations can be explicitly and exactly solved by application of the Riccati variable transformation. In this work let us consider such a class of equations defined by [6]

$$\ddot{x}+\left(\frac{a}{h(x)}-\frac{h'(x)}{h(x)}\right)\dot{x}^2+\frac{4\omega_0{}^2-\lambda^2}{4a}h(x)=0 \tag{1}$$

in which $h(x)\neq 0$, is an arbitrary function, $a\neq 0$, ω_0 and λ being arbitrary parameters. The dot over a symbol designates a derivative with respect to time and prime means differentiation with respect to x. The problem to be solved under these conditions is to find the appropriate function $h(x)$ to be used to specify (1) as a generalized Painlevé-Gambier XVII equation with isochronicity, having the ability to ensure applications in physics. Imposing $h(x)=mx$, where $m\neq 0$, is an arbitrary parameter, the equation (1) leads to

$$\ddot{x}-\left(\frac{m-a}{m}\right)\frac{\dot{x}^2}{x}+\left(\frac{4\omega_0{}^2-\lambda^2}{4a}\right)mx=0 \tag{2}$$

Making

$$b=\left(\frac{4\omega_0{}^2-\lambda^2}{4a}\right)m \quad (3)$$

the equation (2) may be rewritten

$$\ddot{x}-\left(\frac{m-a}{m}\right)\frac{\dot{x}^2}{x}+bx=0 \qquad (4)$$

The equation (2) or (4) is the desired generalized Painlevé-Gambier XVII equation. It suffices to note that for the parametric choice $b=0$, that is for $4\omega_0{}^2-\lambda^2=0$, and $m\neq 0$, equation (4) reduces to

$$\ddot{x}-\frac{m-a}{m}\frac{\dot{x}^2}{x}=0 \qquad (5)$$

which, for $a=1$, gives

$$\ddot{x}-\frac{m-1}{m}\frac{\dot{x}^2}{x}=0 \qquad (6)$$

known as the Painlevé-Gambier XVII equation [1]. As such one may observe the generalized equation (4) as the Painlevé-Gambier XVII equation with linear external forcing function $-bx$. The equation (4) belongs to the general class of quadratic Liénard type equation for which a sufficient condition for exhibiting periodic oscillations has been established [5]. So knowing that the term bx is odd function, one may expect that the generalized Painlevé-Gambier XVII equation exhibits periodic oscillations, that is, the origin to be a center. According to the above the explicit and exact general solution to (4) may be computed and applied to show that the equation (4) may exhibit not only periodic oscillations but also an isochronous center.

3. Explicit and Exact general solution

This part is devoted to calculate the explicit and exact general solution to (4). This solution is used to show that the equation (4) may exhibit isochronicity property. It is also demonstrated that for an appropriate parametric choice, the general solution to the Painlevé-Gambier XVII equation given by Ince [1]is recovered by means of the linearizing transformation applied in this research work to find solutions.

3.1 Painlevé-Gambier XVII equation

To solve explicitly and exactly the Painlevé-Gambier XVII equation, let us consider the Riccati transformation [6]

$$\frac{\dot{y}}{y} = a\frac{\dot{x}}{h(x)} \tag{7}$$

By application of $h(x) = mx$, the equation (7) reduces to the equation of the free particle motion

$$\ddot{y} = 0 \tag{8}$$

with the general solution

$$y = k_1 t + k_2 \tag{9}$$

where k_1 and k_2 are arbitrary parameters, so that the general solution to (5) becomes

$$x(t) = (k_1 t + k_2)^{\frac{m}{a}} \tag{10}$$

In this regard, the general solution to the Painlevé-Gambier XVII equation may, knowing that $a = 1$, be written

$$x(t) = (k_1 t + k_2)^{m} \tag{11}$$

The solution (11) is the same as that given by Ince [1]. This solution is also obtained recently by other authors by applying a generalized Sundman transformation [19]. It is interesting to note that here, imposing $m = \frac{1}{2}$, and $a = 1$, the equation (6) reduces to the inverted Painlevé-Gambier XI equation

$$\ddot{x} + \frac{\dot{x}^2}{x} = 0$$

(12)

which admits the general solution

$$x(t) = (k_3 t + k_4)^{\frac{1}{2}} \tag{13}$$

where k_3 and k_4 are constants of integration. Now one may consider the explicit and exact general periodic solution to the generalized Painlevé-Gambier XVII equation of interest.

3.2 Generalized Painlevé-Gambier XVII equation

By applying the Riccati transformation (7) with $h(x)=mx$, the explicit and exact general periodic solution of (4) may be expressed as

$$x(t)=A^{\frac{m}{a}}\sin^{\frac{m}{a}}\left(\sqrt{b}\,t+K\right) \tag{14}$$

where A and K are arbitrary parameters, and $b\succ 0.$ Such a solution is periodic with amplitude independent frequency so that for $b\succ 0$, the generalized Painlevé-Gambier equation defined by (4) exhibits isochronicity property. For $\frac{m}{a}=1,$ the equation (4) reduces to that of the harmonic oscillator for $\lambda=0,$ such that the explicit and exact general periodic solution (14) becomes a trigonometric periodic solution with fixed frequency, that is with amplitude independent frequency. An interesting case is also where the exponent $\frac{m}{a}$ in the expression (14) is a positive integer greater than or equal to two, making it possible to write the solution (14) as a linear combination of terms having the form $\sin(qt+\varphi)$ or $\cos(qt+\varphi)$ where q denotes a positive integer, and φ is an arbitrary parameter. Under these conditions, the solution (14) assumes for $\frac{m}{a}=2,$ a very interesting expression

$$x(t)=\frac{A^2}{2}-\frac{A^2}{2}\cos\left(2\sqrt{b}\;t+C\right) \tag{15}$$

where $C=2K$. The solution (15) is a trigonometric solution like that of the harmonic oscillator but with a shift factor $\frac{A^2}{2}$. Some applications in physics of equations previously developed may now be shown in the sequel of this work.

4. Applications

As mentioned in the above, some applications of the generalized Painlevé-Gambier XVII equation developed are shown in this section to illustrate its physical interest.

4.1 Isochronous oscillations

In this subsection some graphical examples are exhibited to illustrate the isochronicity characteristic of the equation (4) under the condition that $b\succ 0.$

Figure 1 shows the isochronous oscillations of (4) with $\frac{m}{a}=2,\ b=0.005,$ and the initial conditions $x=0.2$, and $\dot{x}=0.5$, at $t=0$. The solid line represents the solution (15) and the circles denote the solution obtained by numerical integration of (4). Figure 2 shows the phase periodic orbits obtained from the analytical solution (solid line) and numerical solution (circles line) of equation (4), corresponding to the preceding initial conditions and parameter values.

It is interesting now to show that the generalized Painlevé-Gambier XVII equation may be quantized in the perspective that it may be used in physical and practical applications to compute the discrete bound state energy spectrum of dynamical systems.

4.2 Discrete bound state solutions

The objective of this part is to compute the exact discrete bound state solutions for the equation (4). This will be carried out under the formalism of Schrödinger equation with position-dependent mass by application of the well known Nikiforov-Uvarov method. Given the equation (4), the mass distribution function $M(x)$ may be calculated as [10-12]

$$M(x)=m_0 x^{2\left(\frac{m}{a}-1\right)} \qquad (16)$$

so that the potential function takes the form

$$V(x)=\frac{m}{2a}m_0 b x^{\frac{2a}{m}} \qquad (17)$$

In this way the Schrödinger eigenvalue problem to be solved may clearly be stated.

4.2.1 Schrödinger eigenvalue problem

a. Problem description

Let us consider a particle described by (4) of mass $M(x)$, moving in the potential $V(x)$. The problem is to find in this context, the Schrödinger wave function $\psi(x)$ and energy eigenvalue E under the conditions that $\psi(x)$ is bounded and of square integrable on the interval $[0,+\infty[$.

b. Schrödinger wave equation

According to [10-12] the Schrödinger eigenvalue problem which has been described previously may be written in the form

$$\psi''(x) - \frac{M'(x)}{M(x)}\psi'(x) + 2M(x)\left[E - V(x)\right]\psi(x) = 0 \tag{18}$$

where prime designates the derivative with respect to x. Substituting the equations (16) and (17), when $\frac{m}{a} = 2,$ into (18) yields the desired Schrödinger equation with position dependent mass

$$\psi''(x) + \frac{1}{x}\psi'(x) + \left[\frac{2E}{x} - 2b\right]\psi(x) = 0 \tag{19}$$

over the interval $0 \le x \prec \infty.$ The Schrödinger wave equation (19) appears to be an interesting case from the physical point of view as well as mathematical viewpoint, since the restriction

$$2E = n + \frac{1}{2} \tag{20}$$

and

$$2b = \frac{1}{4} \tag{21}$$

yields the well known second order ordinary differential equation [20]

$$x\psi''(x) + \psi'(x) + \left(n + \frac{1}{2} - \frac{x}{4}\right)\psi(x) = 0 \tag{22}$$

which is satisfied by the orthonormal Laguerre function

$$\psi_n(x) = e^{-\frac{x}{2}} L_n(x) \tag{23}$$

where $L_n(x)$ designates the Laguerre polynomials [20]. This finding is of the high importance since it will allow testing the efficiency of the Nikiforov-Uvarov

method [21] for solving the Schrödinger wave equation (19). It is interesting to mention that the preceding result may be recovered using the Liouville transformation

$$\psi(x) = x^{-\frac{1}{2}} Z(x) \tag{24}$$

This leads to

$$Z''(x)+\left(\frac{2E}{x}+\frac{1}{4x^2}-2b\right)Z(x)=0 \tag{25}$$

so that for the previous values of E and b, the variable $Z(x)$ may take the form [22]

$$Z(x)=e^{-\frac{x}{2}}x^{\frac{1}{2}}\,L_n(x) \tag{26}$$

such that the wave function $\psi(x)=e^{-\frac{x}{2}}L_n(x)$, is recovered. The equation (19) may be also written in the appropriate form

$$\psi''(x)+\frac{1}{x}\psi'(x)+\left[\frac{2Ex-2bx^2}{x^2}\right]\psi(x)=0 \tag{27}$$

where $b\succ 0$, and $\frac{m}{a}=2$, for application of the Nikiforov-Uvarov (NU) method, on the interval $0\le x\prec\infty$.

4.2.2 Solving of Schrödinger wave equation

The Nikiforov-Uvarov method has been widely used by several authors [11, 12, 23] to solve exactly the Schrödinger wave equation with position-dependent mass. This method is very interesting due to its simplicity and elegance in solving process. The Nikiforov-Uvarov method is usually applied to solve the general second order linear equation [21]

$$\psi''(x)+\frac{\tilde{\tau}(x)}{\sigma(x)}\psi'(x)+\frac{\tilde{\sigma}(x)}{\sigma^2(x)}\psi(x)=0 \tag{28}$$

where $\tilde{\tau}(x)$ is a polynomial at most of first degree while $\sigma(x)$ and $\tilde{\sigma}(x)$ are polynomials at most of second degree. In this regard the wave function $\psi(x)$ is expressed as

$$\psi\ (x)=\phi(x)\,y_n(x) \tag{29}$$

where the function $y_n(x)$ satisfies the hypergeometric type linear differential equation

$$\sigma(x)y_n''(x)+\tau(x)y_n'(x)+\lambda y_n(x)=0 \tag{30}$$

and

$$\frac{\phi'(x)}{\phi(x)} = \frac{\pi(x)}{\sigma(x)} \tag{31}$$

under the condition that $\tau(x)$ shouldbe a polynomial at most of first degree, λ is a constant, and $\pi(x)$ satisfies

$$\pi(x) = (\frac{\sigma'(x) - \tilde{\tau}(x)}{2}) \pm \sqrt{\left(\frac{\sigma'(x) - \tilde{\tau}(x)}{2}\right)^2 - \tilde{\sigma}(x) + k\sigma(x)} \tag{32}$$

The polynomial $\pi(x)$ is required to be at most of first degree with the conditions that

$$\tau(x) = \tilde{\tau}(x) + 2\pi(x) \tag{33}$$

$$k = \lambda - \pi'(x) \tag{34}$$

and

$$\lambda = \lambda_n = -n\tau'(x) - \frac{n(n-1)}{2}\sigma''(x) \ , \ n = 0,1,2,3,... \tag{35}$$

As such the hypergeometric type function $y_n(x)$ is considered as a polynomial of degree n with an expression given by the Rodrigues formula

$$y_n(x) = \frac{A_n}{\rho(x)} \frac{d^n}{dx^n}\left[\sigma(x)^n \rho(x)\right] \tag{36}$$

in the sense that the weight function $\rho(x)$ satisfies

$$\frac{d}{dx}\left[\sigma(x)\rho(x)\right] = \tau(x)\rho(x) \tag{37}$$

and A_n denotes the normalization constant. In the context where

$$\sigma(x) = x, \ \tilde{\tau}(x) = 1, \ \tilde{\sigma}(x) = 2Ex - 2bx^2$$

the polynomial $\pi(x)$ may take the expression

$$\pi(x) = \pm\sqrt{2bx^2 + (k - 2E)x} \tag{38}$$

which becomes under the requirement that the derivative of $\tau(x)$ must be negative

$$\pi(x) = -x\sqrt{2b} \tag{39}$$

where

$$k = 2E \tag{40}$$

and

$$\tau(x) = \tilde{\tau}(x) + 2\pi(x) \tag{41}$$

reduces to

$$\tau(x) = 1 - 2x\sqrt{2b} \tag{42}$$

In this regard one may easily compute the exact discrete bound state energy spectrum

a. Discrete bound state energy eigenvalues

The substitution of $\tau(x)$ into (35) leads to

$$\lambda_n = 2n\sqrt{2b} \tag{43}$$

In this fashion the application of (34) yields

$$\lambda_n = 2E - \sqrt{2b} \tag{44}$$

so that the comparison of (43) with (44) provides the desired exact discrete energy eigenvalues

$$E_n = \sqrt{2b}\left(n + \frac{1}{2}\right) \tag{45}$$

that is

$$E_n = \left(n + \frac{1}{2}\right)\sqrt{\frac{4\omega_0{}^2 - \lambda^2}{2a}m} \tag{46}$$

In this perspective the discrete bound state wave functions may be exactly calculated.

b. Discrete bound state wave functions

According to the above, the function $\phi(x)$ defined by (31) may be computed under the form

$$\phi(x) = C_0 e^{-x\sqrt{2b}} \tag{47}$$

so that the hypergeometric type function $y_n(x)$ may take the expression

$$\psi_n(x) = C_n e^{-x\sqrt{2b}} e^{2x\sqrt{2b}} \frac{d^n}{dx^n}\left[x^n e^{-2x\sqrt{2b}}\right] \tag{48}$$

which may be rewritten in the form

$$\psi_n(X) = C_n e^{-\frac{X}{2}} e^X \frac{d}{dX}\left(X^n e^{-X}\right)$$

that is

$$\psi_n(X) = C_n e^{-\frac{X}{2}} L_n(X) \tag{49}$$

where $L_n(X)$ denotes the Laguerre polynomials and $X = 2\sqrt{2b}\,x$. The normalization condition

$$\int_0^\infty C_n{}^2 e^{-X} [L_n(X)]^2 dX = 1 \tag{50}$$

requires that [20]

$$C_n{}^2 = 1 \tag{51}$$

that is

$$C_n = 1 \tag{52}$$

such that the wave function $\psi_n(x)$ may be, in definitive, written as

$$\psi_n(X) = e^{-\frac{X}{2}} L_n(X) \tag{53}$$

That being so, the ability of the developed generalized Painlevé-Gambier XVII equation to model the stress relaxation process in material system exhibiting viscoelasticity behavior may be investigated.

4.3 Analysis of stress relaxation curves

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