Relative–Velocity Dependence: a Property of Gravitational Interaction

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Abstract

Herein we propound the property, present its empirical law—a two *Relative-Velocity Dependent* (RVD) terms completion of Newton's law of gravitation—and thereby solve some historic problems as: *Solar cycle*, apparent connection with Jupiter 's revolution, and why the two periods do not quite coincide; about 2.9–4.5×10^{19} joules/yr from earth's rotation slowdown (*Secular retardation*) "missing" in tidal effects and attributed to some "unknown mechanism"; and the "unknown" nature (and magnitude) of the *"driving/propelling force"* in Tectonic plates drift; moreover, the RVD completion of Newton's gravity law predicts that ***tectonic plates drift, globally, to the west***, not randomly, causing earthquakes and volcano eruptions to occur most probable at equinoxes (around March and September). The well-known formula of *Perihelion advance* is also derived. Several experiments are proposed, some feasible now.

Keywords: solar cycle, perihelion advance, secular retardation, continental drift, earthquakes.

Contents

1 Introduction

The objectives of this research are primarily some unsolved problems, as *Secular retardation*, *Tectonic plates drift*, and *Solar cycle*; secondarily, some unexpected solutions to other unsolved or even unknown problems, as the *Areolar speed decrease* (newly reported), and why *particles do not leave the sun at highest energy*; and thirdly, alternative answers to questions encountered on the way, as *Perihelion advance*, *Mountains uplift, Volcanoes, Earthquakes*, etc.

Two obstacles barred physics from embedding the property of RVD[1] of the gravitational interaction (and, in general, of interactions by means of fields): the false appearances of non-existing effects of such a nature, and the real difficulty of finding out the law. One may add, as a third obstacle, since 1916, the absence of the RVD idea from those of GTR[2] otherwise than mass variation with speed, inherited from Special Theory of Relativity in which *speed* is not strictly with respect to an ineracting mass.

The fact that no RVD gravitational effects have been observed does not necessarily mean that they do not exist—this being only one of the three logical possibilities as follows: (i) no RVD gravitational effects exist; (ii) such effects do exist, but too small for having been observed; and (iii) such effects do exist, including some largely observable but wrongly attributed to some hypothetical or, better, to unknown causes. While the today tacitly accepted case is (i), the true case is (iii), and we now enumerate arguments.

1. Interactions by mechanical contact—collision and friction—of two masses are RVD, which suggests that interactions by field (mechanically contactless) might also have somehow this property. Remind that Newton's idea of interaction by field, was not generally accepted at first, including by coryphaei like Leibniz himself who had the conviction of the then generally accepted Descartes's theory that the rotating sun made the planets revolve by means of a rotating flow of ether, i.e., by contact. Now we only bring forth the fact that Newton's kind of interaction is RVD as those by mechanical contact are.

2. Conjecture 3, Section 2, easy to accept, contains implicitly the RVD idea.

3. *Perihelion advance*, an effect well verified, results (Section 4) elegantly from Newton's law RVD completed.

4. Of the 8.5×10^{19} joules/yr from Earth's rotation slowdown (*Secular retardation*), about 2.9 − 4.5×10^{19} joules/yr (analyzed in Section 5) cannot be accounted for by tidal effects, and have been attributed to some unknown cause(s); this is a serious crisis, an antinomy in Geophysics, and nobody has suspected that it has lain in Gravitation.

5. The nature of the "driving/propelling force" (or the "mechanism") in *Plate tectonics* has been unknown, because of which the *Continental drift* theory was repeatedly rejected, at each recrudescence since 1858 until 1960's when it has been accepted as *Sea-floor spread* by H.H. Hess, although the "driving force" (in fact, *braking torque*, to be seen below) has been still unknown just as initially. So, Geophysics has been in an awkward position, having to invoke a force of an unknown nature responsible for its fundamental effect of *Tectonic plates drift* that causes a series of primordial geophysical effects as *Continental drift, Mountains uplift/rise, Volcanoes, Earthquakes*, etc. This is another antinomy, a crisis, in geophysics, as the item 4, and nobody has thought that Gravitation is responsible for them. Now that the RVD is revealed, the antinomies 4 and 5 are complimentary and have one and the same solution: the energetic "surplus" in one problem (Secular retardation) is the "lack" in the other (Continental drift).

6. *Solar cycle*: every planet exerts a RVD (braking) torque upon the sun, causing the solar activity, but that of Jupiter is the largest (causing, in average, 9.46×10^{12} *wats*), while the next, that of Venus (8.97×10^{12} *wats*), contributes to the solar activity, but almost constantly, not cyclically, because of the small

[1] RVD stands for Relative-Velocity Dependence/Dependent (according to context).

[2] GTR stands for General Theory of Relativity.

eccentricity, while the followers, Mercury and Earth together amount to less than Jupiter—which thus rules, approximately, the cyclicity of the solar activity.

7. The recently discovered *Fly-by anomaly*, a gravitational effect [13], is RVD.
8. Seven experiments are proposed (Section 7), of which several are feasible now, to form a peremptory set of arguments.

The fact should be stressed that this is a completion of Newton's law of gravitation, not an alternative theory.

2 RVD completion of Newton gravity law

Researching the law of the RVD of the gravitational interaction is an empirical action, but performed with theoretical tools, so we used some personal conjectures as criteria to follow more or less compulsorily, some of which are mentioned below, each accompanied by a short comment, but they do not convey by themselves to the law, which should ultimately be considered as purely empirical, as we found out it by trying in turn more samples—each temporarily assumed as the law—and abandoned once noted that they either did not predict the *few* available observational data, or predicted inexistent effects, or both, until we came upon that presented below.

Conjecture 1 (on theories) *Any consistent physical theory should not enter into a conflict with any effect whose reality cannot be denied by a priori reasons but it is only considered inexistent on empirical reasons (of having not been seen before, possibly because of smallness).*

Comment In other words, once observed an effect previously considered inexistent only by reason of not having ever been observed, a consistent theory should be open to simply embed the effect, without getting into a conflict with it (and eventually having to either be fundamentally modified or abandoned). Newton's gravity law does comply with this—a fact simplifying our action.

Conjecture 2 (unique speed of fields) *All fields have the same propagation speed in free space, c (of that electromagnetic).*

Comment Neither philosophical, nor empirical reasons contradict this assertion. But one may doubt, however.

Conjecture 3 (of limited accelerating) *A field (irrespective of its nature) cannot accelerate a particle, along its own direction, up to a speed greater than its own propagation speed.*

Comment One can admit this assertion as a matter of common sense. For the electric field one can consider that there exists a laboratory test: Bertozzi's experiment [4]. Note that this *Conjecture* contains in subtext the idea of RVD of interactions by fields: the force law must contain c and $\vec{v}$ to compare. That is, once admitting this conjecture, one admits implicitly the RVD idea.

Conjecture 4 (magnitude) *According to the well-known formula of* Perihelion advance *(9), a magnitude of* $(v/c)^n$, $n \geq 2$ *is to be expected.*

Comment In the beginning of this research we tried even the possibility of a first power law ($n = 1$), but soon abandoned because of predicting non-existing but largely observable effects, and not predicting the ones existing.

Conjecture 5 (Newton's spheric properties) *The Newton law of gravitation RVD completed should comply with the Newton spheric properties (worded in the beginning of section 3).*

Comment This however should not be regarded as compulsory if predicting experimentally unreachable effects now (but we shall see in section 3 that the RVD completion we are going to put forward does comply with this conjecture).

Let M and m be two point masses, and $\vec{r}$ the position vector of m with respect to M, i.e., $\vec{r}$ has its initial point at M and the terminal point at m or, in other words, m lies in the gravitational field of M. Newton's law of gravitation writes $\vec{F}_N = -GMm\vec{r}/r^3 = m\vec{g}_N$. *Newton's gravitational law* (empirically) *RVD completed* is

$$\vec{F} = \vec{F}_N + 3\vec{F}_N\frac{v^2}{c^2} - 6F_N\frac{\vec{v}^3}{c^3} = \vec{F}_N\left(1 + 3\frac{v^2}{c^2}\right) - 6F_N\frac{\vec{v}^3}{c^3}\,, \qquad (1)$$

or using $\vec{g} = \vec{F}/m$ (*force per unit mass*, or *gravitational field strength*, or *gravitational acceleration*),

$$\vec{g} = \vec{g}_N + 3\vec{g}_N\frac{v^2}{c^2} - 6g_N\frac{\vec{v}^3}{c^3} = \vec{g}_N\left(1 + 3\frac{v^2}{c^2}\right) - 6g_N\frac{\vec{v}^3}{c^3}\,, \qquad (1')$$

where $\vec{v}$ is the velocity of m with respect to M, i.e. $\vec{v} \equiv \dot{\vec{r}}$ and, of course, $\vec{v}^3 = v^2\vec{v}$.

Here and a few times below, the sign $\equiv$ stands for "by definition"; we shall also use the notation $\vec{1}_d$ for the unit vector of a given direction d, for instance $\vec{1}_v \equiv \vec{v}/|\vec{v}| = \vec{v}/v$, and $\vec{1}_x$ for the unit vector of the Ox axis of a coordinate system.

Some interesting observations:

1. While the RVD term $3\vec{F}_N v^2/c^2$ (proportional to the 2nd power of v/c) is *even* with respect to $\vec{v}$, the other, $-6F_N v^2\vec{v}/c^3$ (of the 3rd power), is *odd*—a radical property, causing reversed force for reversed relative-velocity, hence a *braking* effect upon rotating bodies (see Figure 1). This may be the cause of the *slow* rotation of Mercury and Venus, the ones closest to the sun.

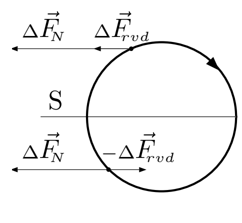

Figure 1: **A rotating mass m (here a ring, clockwise) in a gravitational field is acted by a breaking torque due to the *odd* RVD term.** Any point mass whose projection of $\vec{v}$ on the Newton force $\Delta\vec{F}_N$ points to $-\Delta\vec{F}_N$ acquires an increment $\Delta\vec{F}_{rvd}$ in gravitational force, while the particle mirrored with respect to the line of symmetry S acquires a decrement $-\Delta\vec{F}_{rvd}$.

2. If a particle is moving away from the mass generating $\vec{g}$ (as the sun), that is $\vec{v} = v\vec{1}_r$, then $\vec{g} = \vec{g}_N(1 + 3v^2/c^2 + 6v^3/c^3)$ and, as v tends to c, $\vec{g}$ tends to $10\,\vec{g}_N$, hence particles leave the sun at speeds much smaller than c, than caused by $\vec{g}_N$.
3. If a particle is coming to the mass that generates $\vec{g}$ (as the earth), that is $\vec{v} = -v\vec{1}_r$, then $\vec{g} = \vec{g}_N(1 + 3v^2/c^2 - 6v^3/c^3)$ and, as v increases, $\vec{g}$ becomes $\vec{0}$ (at $v \approx 0.776\,c$)—a conformation to the above *Conjecture 3*—eventually becoming $-2\vec{g}_N$, that is, a repelling force, twice that attractive expected according to Newton's law, making highest energetic particles deviate from their path, tending to avoid collision with the earth. Therefore the synonym "law of universal *attraction*" of the law of gravitation is no longer quite adequate.
4. On hypothesizing that *the lifetime of muons shortens in a gravitational field* of strength like that on the earth, a supplemental or alternative explanation (to that of relativistic time-dilatation) for the fact that muons reach yet the earth's surface is *the weak gravitational field, about zero, they experience at speeds about* $0.776\,c$. See also the related Experiment Proposed 7, section 7.

3 Newton's spheric properties hold

As known, Newton's law of gravitation has the following three properties/theorems concerning the gravitational field of a spheric mass.

1. The gravitational field of a spheric volume mass is equivalent to that of a particle of the same mass lying at the center.

2. The gravitational field at the exterior of a spheric shell mass is equivalent to that of a particle having the same mass situated at the center.

3. The gravitational field at the interior of a spheric shell mass is zero.

The question arises whether these properties hold for the Newton law RVD completed. The answer is affirmative, as follows.

Starting from the fact that for a point/particle mass m, moving or not, in the gravitational field of another point mass M, $\vec{g}_N = -GM\vec{r}/r^3$, one can pass to the case of a particle in the gravitational field of a mass distributed with density $\mu_\bullet$ in a domain $\bullet$: the corresponding expressions for $\vec{g}_{\bullet N}$ and for that RVD completed, $\vec{g}_\bullet$, respectively are (see Figure 2 for notations)

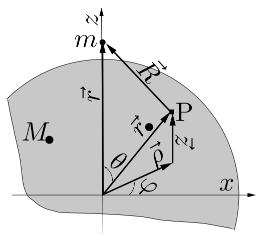

Figure 2: **An m mass particle in the gravitational field of an uniform spheric mass $M_\bullet$, density $\mu_\bullet$, and radius $R_\bullet$.** Every point mass P of $M_\bullet$ acts on m. Evidently, $\vec{r}_\bullet + \vec{R} = \vec{r}$, $\vec{R} = \vec{r} - \vec{r}_\bullet$, $\dot{\vec{R}} = \dot{\vec{r}} \equiv \vec{v}$, and $R^2 = r^2 + r_\bullet^2 - 2rr_\bullet \cos\theta = r^2(1+\zeta^2 - 2\zeta\cos\theta)$, where $\zeta = r_\bullet/r$; $\vec{1}_r \equiv \vec{r}/r = \vec{1}_z$. Analogously for a spheric shelly mass, replacing $\vec{r}_\bullet$ with $\vec{R}_\circ$, but $R_\circ$ constant. $\vec{R}=\vec{r}-\vec{R}_\circ$, $\dot{\vec{R}} = \dot{\vec{r}} \equiv \vec{v}$, $R^2 = r^2 + R_\circ^2 - 2rR_\circ\cos\theta = r^2(1+\zeta^2-2\zeta\cos\theta)$, where $\zeta = R_\circ/r$.

$$\left.\begin{aligned} \vec{g}_{\bullet N} &= -G\int_\bullet \frac{\mu_\bullet \vec{R}}{R^3}\, dV_\bullet\,, \\ \vec{g}_\bullet &= -G\int_\bullet \mu_\bullet \left[\frac{\vec{R}}{R^3}\left(1+3\frac{\dot{\vec{R}}^2}{c^2}\right) - 6\frac{\dot{\vec{R}}^3}{R^2c^3}\right] dV_\bullet\,, \end{aligned}\right\} \tag{2}$$

but $\vec{R} = \vec{r} - \vec{r}_\bullet$, $\dot{\vec{R}} = \dot{\vec{r}} \equiv \vec{v}$, so $\vec{g}_\bullet$ writes

$$\begin{aligned} \vec{g}_\bullet &= -G\int_\bullet \mu_\bullet \left[\frac{\vec{R}}{R^3}\left(1+3\frac{v^2}{c^2}\right) - 6\frac{\vec{v}^3}{R^2c^3}\right] dV_\bullet \\ &= \left(1+3\frac{v^2}{c^2}\right)\vec{g}_{\bullet N} - 6g_{\bullet N}\frac{\vec{v}^3}{c^3}\,, \end{aligned} \tag{3}$$

just the expression for a particle. Q.E.D.

To show that the Newton spheric properties 2 and 3 (concerning a spheric shell $\circ$) hold for the law RVD completed, apply Eq. (2) with notations shown in Figure 2 (of course, here the mass density $\mu_\circ$ is a surface density),

$$\begin{aligned} \vec{g}_\circ &= -G\int_\circ \mu_\circ \left[\frac{\vec{R}}{R^3}\left(1+3\frac{v^2}{c^2}\right) - 6\frac{\vec{v}^3}{R^2c^3}\right] dS_\circ \\ &= \left(1+3\frac{v^2}{c^2}\right)\vec{g}_{\circ N} - 6g_{\circ N}\frac{\vec{v}^3}{c^3}\,, \end{aligned} \tag{4}$$

i.e., the law for a particle at the shell's center, having its mass, $M_\circ$, hence the field strength $\vec{g}_\circ$ is non zero outside, and zero inside, just as Newton's $\vec{g}_{\circ N}$. Q.E.D.

As for all three demonstrations we have used the fact that Newton's law does possess these properties, it is the case to demonstrate them in terms used herein.

A solid spheric mass sets up a gravitational field whose Newton strength is

$$\begin{aligned} \vec{g}_{\bullet N} &= -G\int_\bullet \frac{\mu_\bullet \vec{R}}{R^3} dV_\bullet = -G\int_\bullet \frac{\mu_\bullet(\vec{r}-\vec{r}_\bullet)}{R^3} dV_\bullet \\ &= -G\int_0^{R_\bullet}\int_0^{\pi}\int_0^{2\pi} \frac{\mu_\bullet}{R^3}[\vec{r} - r_\bullet(\vec{1}_x \sin\theta\cos\varphi + \vec{1}_y \sin\theta\sin\varphi \\ &\qquad + \vec{1}_z\cos\theta)] r_\bullet^2 \sin\theta\, dr_\bullet\, d\theta\, d\varphi \\ &= -2\pi G\int_0^{R_\bullet}\int_0^{\pi} \frac{\mu_\bullet}{R^3}(\vec{r} - r_\bullet\cos\theta\ \vec{r}/r) r_\bullet^2 \sin\theta\, dr_\bullet d\theta \\ &= -2\pi G\,\vec{r}\int_0^{R_\bullet/r}\int_0^{\pi} \mu_\bullet \frac{(1-\zeta\cos\theta)\,\zeta^2\sin\theta}{(1+\zeta^2-2\zeta\cos\theta)^{3/2}} d\zeta\, d\theta \\ &= 2\pi G\vec{r}\int_0^{R_\bullet/r}\mu_\bullet\left(\frac{2\zeta^2}{1-\zeta^2} - \frac{2\zeta^4}{1-\zeta^2}\right) d\zeta = 4\pi G\,\vec{r}\int_0^{R_\bullet/r}\mu_\bullet\zeta^2 d\zeta \\ &= -\frac{4\pi}{3}G\vec{r}\mu_\bullet\frac{R_\bullet^3}{r^3} = -\frac{GM_\bullet\vec{r}}{r^3}, \quad \zeta \le 1, \end{aligned} \tag{5}$$

where there have used: the notation $\zeta = r_\bullet/r$; integrals (7) and (8); the mass density as $\mu_\bullet(\vec{r}_\bullet) = \mu_\bullet(r_\bullet)$, until the final step when $\mu_\bullet$ was assumed (geometrically and temporally) constant; note the validity of the result for $\lim_{\zeta\to 1}$ (that is, for points on the surface too). Q.E.D.

A spheric shell mass generates a gravitational field whose Newton strength is

$$\begin{aligned} \vec{g}_{\circ N} &= -G\int_\circ \frac{\mu_\circ\vec{R}}{R^3} dS_\circ = -G\int_0^{2\pi}\int_0^{\pi}\mu_\circ\frac{\vec{r}-\vec{R}_\circ}{R^3} R_\circ^2 \sin\theta\, d\theta\, d\varphi \\ &= -GR_\circ^2\int_0^{2\pi}\int_0^{\pi}\frac{\mu_\circ}{R^3}[\vec{r} - R_\circ(\vec{1}_x\sin\theta\cos\varphi + \vec{1}_y\sin\theta\sin\varphi \\ &\qquad + \vec{1}_z\cos\theta)]\sin\theta\, d\theta\, d\varphi \\ &= -2\pi GR_\circ^2\int_0^{\pi}\frac{\mu_\circ(\vec{r}-R_\circ\cos\theta\,\vec{r}/r)}{(r^2+R_\circ^2-2rR_\circ\cos\theta)^{3/2}}\sin\theta\, d\theta \\ &= -2\pi GR_\circ^2\frac{\vec{r}}{r^3}\int_0^{\pi}\mu_\circ\frac{(1-\zeta\cos\theta)\sin\theta}{(1+\zeta^2-2\zeta\cos\theta)^{3/2}} d\theta \\ &= -2\pi\mu_\circ GR_\circ^2\frac{\vec{r}}{r^3}\begin{cases} \left(\frac{2}{1-\zeta^2} - \frac{2\zeta^2}{1-\zeta^2}\right) \texttt{ for } \zeta \le 1, \\ \left[\frac{2}{\zeta(\zeta^2-1)} - \frac{2\zeta}{\zeta^2(\zeta^2-1)}\right] \texttt{ for } \zeta > 1 \end{cases} \\ &= -GM_\circ\frac{\vec{r}}{r^3}\begin{cases} 1 \texttt{ for } \zeta \le 1, \\ 0 \texttt{ for } \zeta > 1, \end{cases} \end{aligned} \tag{6}$$

where $\zeta \le 1$ is drawn since the relationship subsists for $\lim_{\zeta\to 1}$; there have been used: the notation $\zeta = R_\circ/r$; integrals (7) and (8); and the mass density as $\mu_\circ(\vec{r}_\circ) = \mu_\circ(r_\circ)$, until the final step when $\mu_\circ$ was assumed constant. Eq. (6) contains both cases of a spheric shell mass' field: at an arbitrary non-internal point ($\zeta \le 1$), and at an arbitrary internal point ($\zeta > 1$). Q.E.D.

To calculate the integrals (7) and (8) the following temporary notations are used: $\xi = \cos\theta$, $\gamma = 1+\xi^2$, and $\delta = -2\zeta$.

$$\begin{aligned} &\int_0^{\pi}\frac{\sin\theta\, d\theta}{(1+\zeta^2-2\zeta\cos\theta)^{3/2}} = \int_{-1}^{1}\frac{d\xi}{(\gamma+\delta\xi)^{3/2}} = -\frac{2}{\delta}\frac{1}{(\gamma+\delta\xi)^{1/2}}\Bigg|_{\xi=-1}^{1} \\ &= -\frac{2}{\delta}\left[\frac{1}{(\gamma+\delta)^{1/2}} - \frac{1}{(\gamma-\delta)^{1/2}}\right] = \frac{1}{\zeta}\left(\frac{1}{|1-\zeta|} - \frac{1}{1+\zeta}\right) \\ &= \begin{cases} \frac{2}{1-\zeta^2}, \texttt{ if } \zeta < 1, \\ \frac{2}{\zeta(\zeta^2-1)}, \texttt{ if } \zeta > 1. \end{cases} \end{aligned} \tag{7}$$

$$\begin{aligned} &\int_0^{\pi}\frac{\sin\theta\cos\theta\, d\theta}{(1+\zeta^2-2\zeta\cos\theta)^{3/2}} = \int_{-1}^{1}\frac{\xi\, d\xi}{(\gamma+\delta\xi)^{3/2}} \\ &= \frac{2}{\delta^2}\frac{2\gamma+\delta\xi}{(\gamma+\delta\xi)^{1/2}}\Bigg|_{\xi=-1}^{1} = \frac{2}{\delta^2}\left[\frac{2\gamma+\delta}{(\gamma+\delta)^{1/2}} - \frac{2\gamma-\delta}{(\gamma-\delta)^{1/2}}\right] \\ &= \frac{1}{2\zeta^2}\left(\frac{2\gamma+\delta}{|1-\zeta|} - \frac{2\gamma-\delta}{1+\zeta}\right) = \begin{cases} \frac{2\zeta}{1-\zeta^2}, \texttt{ if } \zeta < 1, \\ \frac{2}{\zeta^2(\zeta^2-1)}, \texttt{ if } \zeta > 1. \end{cases} \end{aligned} \tag{8}$$

4 Deviation from Kepler 's first and second laws: *Perihelion advance* & *Areolar speed decrease*

The *Perihelion advance*, an effect of the second power of v/c, is a deviation from Kepler 's first law, while the *Areolar speed decrease* (now newly put forth), of the third power, is a deviation from the second law.

4.1 Perihelion advance

As known, *Perihelion/periastron advance/rotation/ precession/shift* are names of the small remainder of the angular perihelion advance, δ, per revolution, of a planet orbiting the sun—or in general of a body orbiting an *astron—not accounted for by Newton's gravity law.* This is a deviation from Kepler's first law, discovered by Urbain Le Verrier [1] for Mercury, by calculations. The well-known formula of the effect,

$$\delta = \frac{6\pi G M_\odot}{c^2(1-\epsilon^2)a}\,, \tag{9}$$

was found by Paul Gerber [2] from some premises later regarded as inconsistent. A consistent inference was put forward by Albert Einstein, via GTR [3]. Now we deduce it from *Newton's gravity law RVD completed.*

Unlike Gerber, whose reasoning has ultimately been considered both inconsistent and unclear, we either perform or mention all likely useful steps. We do this rather by metamorphic successive equalities than by words.

We have just established (section 3), that the gravitational interaction between the sun and a planet, assumed as spheres, can be treated as an interaction between the respective masses as particles at the respective centers. Newton's law of motion, $M_\oslash\vec a = \vec F$, of a mass $M_\oslash$ (as a planet) in the gravitational field of a mass $M_\odot$ (as the sun) taken as origin, $M_\oslash \ll M_\odot$ so that the center of the masses $M_\odot$ and $M_\oslash$ be approximately at $M_\odot$ (not the case of binary pulsars, for instance), writes

$$\ddot{\vec r} = -\frac{G M_\odot \vec r}{r^3}\left(1+3\frac{v^2}{c^2}\right) - 6\frac{G M_\odot}{r^2}\frac{v^2\vec v}{c^3}\,. \tag{10}$$

Apply $\vec r\times$ to both sides of Eq. (10) and note that $\vec r\times\ddot{\vec r} = d(\vec r\times\dot{\vec r})/dt = \dot{\vec L}/M_\oslash$, where $\vec L$ is the angular moment of $M_\oslash$, obtaining $\dot{\vec L} = -[6\,G M_\odot v^2/(c^3 r^2)]\vec L$, whence

$$\vec L = \vec L_0 \exp\left(-6\frac{G M_\odot}{c^3}\int_0^t \frac{v^2}{r^2}d\tau\right), \tag{11}$$

whence, on multiplying both sides by $\vec r$ and noting that $\vec r\vec L = 0$ (since $\vec L = M_\oslash \vec r\times\vec v$), obtain $\vec r\vec L_0 = 0$, hence $\vec r$ keeps lying in a plain perpendicular to a constant vector $\vec L_0$, i.e., the motion is planar, because of which a plane polar coordinates system (ρ,φ) is convenient, in fact its three-dimensional extension (ρ,φ,z)—cylindrical coordinate system—with the same origin (at $\rho=0$) and the z-axis along $\vec L_0$. However, we continue using the notation $\vec r$ instead of changing to $\vec\rho$, so

$$\left.\begin{aligned} &\vec r = r\vec 1_r\,,\quad \dot{\vec r}\equiv\vec v = \dot r\vec 1_r + r\dot\varphi\vec 1_\varphi\,,\\ &\ddot{\vec r}\equiv\vec a = (\ddot r - r\dot\varphi^2)\vec 1_r + (2\dot r\dot\varphi + r\ddot\varphi)\vec 1_\varphi\,,\end{aligned}\right\} \tag{12}$$

$$\vec L/M_\oslash = \vec r\times\vec v = r^2\dot\varphi\vec 1_z = 2\Omega\vec 1_z = 2\vec\Omega\,, \tag{13}$$

where $\vec\Omega$ is the *areolar velocity* (and Ω the *areolar speed*). Inserting expressions (12) in Eq. (10) and equating the components for each $\vec 1_r$ and $\vec 1_\varphi$, yield two equations,

$$\left.\begin{aligned} &\ddot r - r\dot\varphi^2 = -\tfrac{G M_\odot}{r^2}\left(1+3\tfrac{v^2}{c^2}\right) - 6\tfrac{G M_\odot}{c^3}\tfrac{v^2\dot r}{r^2}\,,\\ &2\dot r\dot\varphi + r\ddot\varphi = -6\tfrac{G M_\odot}{c^3}\tfrac{v^2\dot\varphi}{r}\,.\end{aligned}\right\} \tag{14}$$

Change variable $t\to\varphi$ (so having $d/dt = \dot\varphi\, d/d\varphi$), use primes for derivatives with respect to φ, and note that the second equation in Eqs. (14), whose left side writes $(1/r)d(r^2\dot\varphi)/dt = (1/r)(r^2\dot\varphi)'\dot\varphi$, becomes $(r^2\dot\varphi)' = -6(G M_\odot/c^3)v^2$ or, taking into account Eq. (13), $\Omega' = -3(G M_\odot/c)v^2/c^2$, whence $\Omega-\Omega_0 = -3(G M_\odot/c)\int_0^\varphi(v^2/c^2)d\varphi$, whence

$$\Omega = \Omega_0[1-\chi(\varphi)]\,,\qquad \chi(\varphi)\equiv 3\frac{G M_\odot}{\Omega_0 c}\int_0^\varphi \frac{v^2}{c^2}d\varphi\,. \tag{15}$$

As $d/dt = \dot\varphi\, d/d\varphi$ and $r^2\dot\varphi = 2\Omega$, there follows $\dot r = \dot\varphi r' = (2\Omega/r^2)r'$, $\ddot r = d\dot r/dt = \dot\varphi(\dot\varphi r')' = (2\Omega/r^2)[(2\Omega/r^2)r']' = (2\Omega/r^2)[(2\Omega/r^2)r'' - (4\Omega/r^3)r'^2 + (2\Omega'/r^2)r']$. Insert this expression of $\ddot r$ in the first of Eqs. (14), divide both sides by $(2\Omega/r^2)^2$, note that the term $(\Omega'/\Omega)r'$ is identical with the second term in the right side (hence they cancel out), and take into account the first of Eqs. (15), obtaining

$$r'' - 2\frac{r'^2}{r} - r = -\frac{G M_\odot r^2}{(1-\chi)^2(2\Omega_0)^2}\left(1+3\frac{v^2}{c^2}\right). \tag{16}$$

By function change $r\to u$, as $r=\ell/u$, where ℓ is an arbitrary constant, we have $r' = -\ell u'/u^2$, and $r'' = -\ell u''/u^2 + 2\ell u'^2/u^3$, so the left side of Eq. (14′) becomes $-\ell u''/u^2 - \ell/u = (-\ell/u^2)(u''+u)$; also $v^2 = \dot r^2 + r^2\dot\varphi^2 = \dot\varphi^2 r'^2 + r^2\dot\varphi^2 = (r^2\dot\varphi)^2(r'^2+r^2)/r^4 = (2\Omega)^2(r'^2+r^2)/r^4 = (2\Omega/\ell)^2(u'^2+u^2) = (1-\chi)^2(2\Omega_0/\ell)^2(u'^2+u^2)$; with these preparations Eq. (16) writes

$$u''+u = \frac{G M_\odot \ell}{(1-\chi)^2(2\Omega_0)^2}\left[1+\frac{3}{(1-\chi)^2}\left(\frac{2\Omega_0}{\ell c}\right)^2(u'^2+u^2)\right],$$

which, after setting the arbitrary constant ℓ, and defining a non-dimensional constant κ as

$$\ell = \frac{(2\Omega_0)^2}{G M_\odot}\,,\quad \kappa\equiv\left(\frac{2\Omega_0}{\ell c}\right)^2 = \left(\frac{G M_\odot}{2\Omega_0 c}\right)^2 = \frac{G M_\odot}{\ell c^2}\,, \tag{17}$$

finally writes

$$\left.\begin{aligned} &u''+u = \tfrac{1}{(1-\chi)^2} + 3\kappa\,(u'^2+u^2)\,,\\ &\chi(\varphi) = 6\kappa^{3/2}\textstyle\int_0^\varphi (u'^2+u^2)\,d\varphi\,,\end{aligned}\right\} \tag{18}$$

where the definition (15) of χ has also been used.

The next step is to solve Eq. (18) whose non-linear terms contain in factor the powers 1 and 3/2 of κ—carrying the RVD effect. As κ is small (2.663×10^{-8} for Mercury, decreasing to 2.666×10^{-10} for Pluto, see the table in section 5), we treat the non-linear terms as a small perturbation, solving the equation approximately, by successive approximations, $u_0, u_1, u_2, \ldots$, replacing the non-linear terms in equation with their precedent approximation, and neglecting all terms having in factor κ^ν with $\nu > 3/2$.

If κ were zero, then Eq. (18) would be just that in the Newton case, $u_0''+u_0 = 1$, whose solution is $u_0 = 1+\epsilon\cos\varphi$, meeting the condition of passing through periastron at $\varphi=0$, ϵ being the eccentricity; taking as the zeroth approximation just the Newton solution u_0 is convenient for a fast convergence. In the established approximation, the function χ in the second of Eqs. (18) writes $\chi\approx 6\kappa^{3/2}[(1+\epsilon^2)\varphi + 2\epsilon\sin\varphi]$, and $(1-\chi)^{-2}\approx 1+2\chi$, so Eqs. (18) become

$$u''+u = 1+3\kappa(u'^2+u^2)+2\chi\,,\ \chi\approx 6\kappa^{3/2}[(1+\epsilon^2)\varphi+2\epsilon\sin\varphi]\,. \tag{19}$$

Corresponding to the sequence of approximations $\{u_n\}$ we have a sequence of *linear* equations,

$$\left.\begin{aligned} &u_1''+u_1 = 1+3\kappa(u_0'^2+u_0^2)+2\chi_0\,,\\ &u_2''+u_2 = 1+3\kappa(u_1'^2+u_1^2)+2\chi_1\,,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}\right\} \tag{20}$$

Using the expression $u_0 = 1+\epsilon\cos\varphi$, and the expression (19) of χ, the first of Eqs. (20) becomes

$$\left.\begin{aligned} u_1''+u_1 = {}&1+3\kappa(1+\epsilon^2+2\epsilon\cos\varphi)\\ &+12\kappa^{3/2}[(1+\epsilon^2)\varphi+2\epsilon\sin\varphi]\,.\end{aligned}\right| \tag{21}$$

Directly verifiable, by differentiation, the general solution of Eq. (21) is

$$\left.\begin{aligned} u_1(\varphi) = {}&c_1\sin\varphi + c_2\cos\varphi\\ &+\sin\varphi\textstyle\int h(\varphi)\cos\varphi\,d\varphi - \cos\varphi\int h(\varphi)\sin\varphi\,d\varphi\,.\end{aligned}\right| \tag{22}$$

Take into account the facts that: sin and cos are linear independent; their coefficients c_1 and c_2 are arbitrary constants; and $h(\varphi)$ is the whole right side of Eq. (21) ; there results

$$h(\varphi) = 1+3\kappa(1+\epsilon^2+2\varepsilon\cos\varphi)+12\kappa^{3/2}[(1+\epsilon^2)\varphi+2\epsilon\sin\varphi]\,, \tag{23}$$

so the explicit form of Eq. (22) is

$$\left.\begin{array}{r} u_1(\varphi) = c_1 \sin\varphi + c_2 \cos\varphi + 1 + 3\kappa(1+\epsilon^2+\epsilon\varphi\sin\varphi) \\ +12\kappa^{3/2}[(1+\epsilon^2)\varphi + \epsilon(\sin\varphi - \varphi\cos\varphi)]\,, \end{array}\right| \tag{24}$$

whence

$$\left.\begin{array}{r} u_1'(\varphi) = c_1\cos\varphi - c_2\sin\varphi + 3\kappa\epsilon(\varphi\cos\varphi + \sin\varphi) \\ +12\kappa^{3/2}(1+\epsilon^2+\varphi\sin\varphi)\,. \end{array}\right| \tag{25}$$

Now determine the constants c_1 and c_2 in expressions (24) and (25) using the initial conditions (the same for all approximations u_n), $u_1|_{\varphi=0} = 1+\epsilon$ and $u_1'|_{\varphi=0} = 0$, finding $c_1 = -12\kappa^{3/2}(1+\epsilon^2)$ and $c_2 = \epsilon - 3\kappa(1+\epsilon^2)$, which we insert in Eq. (25),

$$\left.\begin{array}{r} u_1'(\varphi) = -12\kappa^{3/2}(1+\epsilon^2)\cos\varphi \;-\; [\epsilon - 3\kappa(1+\epsilon^2)]\sin\varphi \\ +3\kappa\epsilon\,(\varphi\cos\varphi + \sin\varphi) + 12\kappa^{3/2}(1+\epsilon^2+\varphi\sin\varphi)\,. \end{array}\right| \tag{26}$$

Note that our sequence of successive approximations $\{u_n\}_{n\in\mathcal{N}}$ — neglecting the terms having in factor κ^ν for $\nu > 3/2$—stops at $n=1$, since the second of Eqs. (20) (for u_2) coincides with the first (for u_1). In other words, u_1 contains the whole RVD effect of periastron shift in our pre-established approximation, $\kappa^\nu \approx 0$ for $\nu > 3/2$. By its definition, perihelion (or periastron) is a point of extreme (minimum distance), hence $u_1' = 0$ at that point. Expecting a periastron shift δ after a revolution means that $u_1' = 0$ at $\varphi = 2\pi + \delta$ (instead of $\varphi = 2\pi$ in the Newton case). Because of the smallness of κ, a small δ is to be expected, so that we approximate $\sin\delta \approx \delta$, $\cos\delta \approx 1$, $\delta^2 \approx 0$, and $\kappa\delta \approx 0$, i.e., neglect δ^ν for $\nu \geq 3/2$. From Eq. (26), using these approximations, we have successively $u_1'(2\pi+\delta) \approx -\epsilon\delta + 6\pi\kappa\epsilon = \epsilon\,(-\delta + 6\pi\kappa)$, whence, if $\epsilon \neq 0$, $u_1' = 0$ means

$$\delta = 6\pi\kappa\,, \qquad (\epsilon \neq 0), \tag{27}$$

hence the perihelion shift δ is positive, i.e., an advance, indeed. Eq. (27) coincides with the well-known formula (9), via the third form of κ in (17), and $\ell = a\,(1-\epsilon^2)$, ℓ being the *semilatus rectum* of an ellipse in polar coordinates, $r = \ell/(1+\epsilon\cos\varphi)$. Q.E.D.

4.2 Areolar speed decrease

While *Perihelion advance* is a deviation from Kepler's first law, originating from the term proportional to v^2/c^2 in law (1), Eq. (15) is the law of a deviation—a decrease—from the second law, originating from $\vec{v}^3/c^3$. Transcribe it in the form of relative variation, using the approximation (19) of χ,

$$\frac{\Omega - \Omega_0}{\Omega_0} = -\chi(\varphi) \approx -6\kappa^{3/2}[(1+\epsilon^2)\varphi + 2\epsilon\sin\varphi]\,. \tag{28}$$

Of the two terms between square brackets, one, $2\epsilon\sin\varphi$, expresses a periodic variation during a revolution—decrease from perihelion to aphelion, and an equal increase over the returning half of revolution—i.e., an overall conservation. On the contrary, the other term, $(1+\epsilon^2)\,\varphi$, is linear, expressing an irreversible decrease—i.e., non conservation—of the areolar speed (or of the angular momentum, Eq. (11)). The relative decrease over a revolution,

$$\frac{\Omega(2\pi) - \Omega(0)}{\Omega(0)} = -12\pi\,(1+\epsilon^2)\,\kappa^{3/2}, \tag{29}$$

is $\approx -3.7\times10^{-11}$ in case of the earth, that is -3.7×10^{-9} % per year (or 3.7 percents in one billion years, the age of the earth being 4.54 billions). See the table in section 5 for all planets.

5 RVD torque exerted by the central body on an orbiting body: *Secular retardation, Tectonic plates drift, Planets thermal emission*

Consider two non point (or non particle) masses $M_\odot$ and $M_\oslash$, distributed with densities $\mu_\odot$ and $\mu_\oslash$ in solids $\odot$ and $\oslash$ respectively. Geometric notations are shown in Figure 3, with spheric masses, as the sun and a planet.

There occur three torques caused by the gravitational force of a body, say the sun, rotating or not, on a *rotating* mass, orbiting or not, say a planet:

$$\left.\begin{array}{l} \vec{\tau}^{\,(r\oslash)}_{\odot\oslash} \equiv \int_\odot \int_\oslash \vec{r}_\oslash \times d\vec{F}_{\odot\oslash}\,, \quad \vec{\tau}^{\,(\rho\oslash)}_{\odot\oslash} \equiv \int_\odot \int_\oslash \vec{\rho}_\oslash \times d\vec{F}_{\odot\oslash}\,, \\ \vec{\tau}^{\,(z\oslash)}_{\odot\oslash} \equiv \int_\odot \int_\oslash \vec{z}_\oslash \times d\vec{F}_{\odot\oslash}\,, \end{array}\right\} \tag{30}$$

where $\vec{F}_{\odot\oslash}$ is the force exerted by $M_\odot$ on $M_\oslash$; evidently, $\vec{r}_\oslash = \vec{\rho}_\oslash + \vec{z}_\oslash$, and $\vec{\tau}^{\,(r\oslash)}_{\odot\oslash} = \vec{\tau}^{\,(\rho\oslash)}_{\odot\oslash} + \vec{\tau}^{\,(z\oslash)}_{\odot\oslash}$. In fact we aim at the torque $\vec{\tau}^{\,(\rho\oslash)}_{\odot\oslash}$, braking the rotation, to calculate the work $W^{(brv)}_{\odot\oslash}$ done by $\odot$ upon $\oslash$ per revolution (the index $_{(brv)}$ stands for "braking per revolution"), but one may calculate $\vec{\tau}^{\,(r\oslash)}_{\odot\oslash}$, which seems more convenient and leads to the same $W^{(brv)}_{\odot\oslash}$. Indeed,

$$\left.\begin{array}{l} W^{(brv)}_{\odot\oslash} \equiv \int_0^{2\pi P_{\oslash rv}/P_{\oslash rt}} \vec{1}_{\omega_\oslash}\vec{\tau}^{\,(\rho\odot)}_{\odot\oslash}\, d\varphi_\oslash = \int_0^{2\pi} \vec{1}_{\omega_\oslash}\vec{\tau}^{\,(\rho\odot)}_{\odot\oslash}\,\frac{\dot\varphi_\oslash}{\dot\varphi}\, d\varphi \\ \approx \frac{\omega_\oslash}{2\Omega_0}\int_0^{2\pi} \vec{1}_{\omega_\oslash}\vec{\tau}^{\,(\rho\odot)}_{\odot\oslash}\, r^2 d\varphi = \frac{\omega_\oslash}{2\Omega_0}\int_0^{2\pi} \vec{1}_{\omega_\oslash}\vec{\tau}^{\,(r\odot)}_{\odot\oslash}\, r^2 d\varphi\,, \end{array}\right| \tag{31}$$

where Ω is approximated by Ω_0 since the difference is insignificant in this problem; $P_{\oslash rt}$ and $P_{\oslash rv}$ are the periods of rotation and of revolution of $\oslash$; and $\vec{1}_{\omega_\oslash}\vec{\tau}^{\,(\rho\oslash)}_{\odot\oslash}$ is replaced by $\vec{1}_{\omega_\oslash}\vec{\tau}^{\,(r\oslash)}_{\odot\oslash}$ since, according to Eqs. (30),

$$\left.\begin{array}{l} \vec{1}_{\omega_\oslash}\vec{\tau}^{\,(\rho\oslash)}_{\odot\oslash} = \vec{1}_{\omega_\oslash}\int_\odot\int_\oslash \vec{\rho}_\oslash \times d\vec{F}_{\odot\oslash} = \int_\odot\int_\oslash (\vec{1}_{\omega_\oslash}\times\vec{\rho}_\oslash)\, d\vec{F}_{\odot\oslash} \\ = \int_\odot\int_\oslash (\vec{1}_{\omega_\oslash}\times\vec{r}_\oslash)\, d\vec{F}_{\odot\oslash} = \vec{1}_{\omega_\oslash}\int_\odot\int_\oslash \vec{r}_\oslash \times d\vec{F}_{\odot\oslash} = \vec{1}_{\omega_\oslash}\vec{\tau}^{\,(r\oslash)}_{\odot\oslash}, \end{array}\right| \tag{32}$$

where we have used the facts that $\vec{1}_{\omega_\oslash}$ is constant in the process of integration, and $\vec{1}_{\omega_\oslash}\times\vec{r}_\oslash = \vec{1}_{\omega_\oslash}\times\vec{\rho}_\oslash$ since $\vec{1}_z = \vec{1}_{\omega_\oslash}$.

In the sequence of steps (34) (a step is either sign $\equiv$, $=$, or $\approx$) equalities (33) are used. Firstly note that $\vec{R} = \vec{r} + \vec{r}_\oslash - \vec{r}_\odot$ is approximated as $\vec{R} \approx \vec{r} + \vec{r}_\oslash$, keeping $\vec{r}_\oslash$ for its role of carrying the effect of rotation subject to study, and that $\dot{\vec{r}}_\oslash = \vec{\omega}_\oslash \times \vec{r}_\oslash$ since $\vec{r}_\oslash$ is constant in magnitude.

$$\left.\begin{array}{l} \vec{R} = \vec{r} + \vec{r}_\oslash - \vec{r}_\odot \approx \vec{r} + \vec{r}_\oslash\,, \;\; \dot{\vec{R}} = \vec{v} + \vec{\omega}_\oslash \times \vec{r}_\oslash\,, \\ \dot{\vec{R}}^2 = v^2 + 2(\vec{v}\times\vec{\omega}_\oslash)\vec{r}_\oslash + (\vec{\omega}_\oslash\times\vec{r}_\oslash)^2, \\ \dot{\vec{R}}^3 = \dot{\vec{R}}^2\dot{\vec{R}} = [v^2 + 2(\vec{v}\times\vec{\omega}_\oslash)\vec{r}_\oslash + \vec{v}^3 + 2[(\vec{v}\times\vec{\omega}_\oslash)\vec{r}_\oslash]\vec{v} \\ +(\vec{\omega}_\oslash\times\vec{r}_\oslash)^2\vec{v} + v^2\vec{\omega}_\oslash\times\vec{r}_\oslash + 2[(\vec{v}\times\vec{\omega}_\oslash)\vec{r}_\oslash)]\vec{\omega}_\oslash\times\vec{r}_\oslash + (\vec{\omega}_\oslash\times\vec{r}_\oslash)^3, \\ \vec{r}_\oslash\times(\vec{\omega}_\oslash\times\vec{r}_\oslash) = \vec{\omega}_\oslash r_\oslash^2 - (\vec{\omega}_\oslash\vec{r}_\oslash)\vec{r}_\oslash\,, \\ \vec{r}_\oslash\times(\vec{\omega}_\oslash\times\vec{r}_\oslash)^3 = (\vec{\omega}_\oslash\times\vec{r}_\oslash)^2\vec{r}_\oslash\times(\vec{\omega}_\oslash\times\vec{r}_\oslash) \\ \quad = [\omega_\oslash^2 r_\oslash^2 - (\vec{\omega}_\oslash\vec{r}_\oslash)^2][\vec{\omega}_\oslash r_\oslash^2 - (\vec{\omega}_\oslash\vec{r}_\oslash)\vec{r}_\oslash] \\ \quad = \vec{\omega}_\oslash^3 r_\oslash^4 - \omega_\oslash^2 r_\oslash^2(\vec{\omega}_\oslash\vec{r}_\oslash)\vec{r}_\oslash - (\vec{\omega}_\oslash\vec{r}_\oslash)^2\vec{\omega}_\oslash r_\oslash^2 + (\vec{\omega}_\oslash\vec{r}_\oslash)^3\vec{r}_\oslash\,. \end{array}\right\} \tag{33}$$

In (34), the first row contains the insertion of the RVD expression of $\vec{g}_{\odot\oslash}$; the second row implements the zero torque produced by the Newton force (as known physically, and Theorem 2, Example 3, shows mathematically why), and approximates $\vec{g}_{\odot\oslash N}$ as having the same magnitude and direction at any point of $\oslash$; the third row transcribes only the odd terms $\dot{\vec{R}}^2$ and $\dot{\vec{R}}^3$ from their expressions (33), since the others produce zero integrals (according to Theorem 1, the second formula (54); the fourth row, only rearrangements; the fifth row applies directly the first formula (55) for the integrals of the first two terms, while for the third term uses the relation in the sixth row of (33) noting that its integral is $v^2[I^{(2)}_\oslash\vec{\omega}_\oslash - (1/3)I^{(2)}_\oslash\vec{\omega}_\oslash] = (2/3)I^{(2)}_\oslash v^2\vec{\omega}_\oslash$, and for the fourth term use the last expression in (33) and the first formula (56) obtaining the integral $I^{(4)}_\oslash\vec{\omega}_\oslash^{\,3} - (1/3)I^{(4)}_\oslash\vec{\omega}_\oslash^{\,3} - (1/3)I^{(4)}_\oslash\vec{\omega}_\oslash^{\,3} + (1/15)I^{(4)}_\oslash 3\vec{\omega}_\oslash^{\,3} = (8/15)I^{(4)}_\oslash\vec{\omega}_\oslash^{\,3}$; next, standard calculations using Eq. (53) for the expressions of $I^{(2)}_{\oslash}$ and $I^{(4)}_{\oslash}$.

$$\begin{aligned}
\vec{\tau}^{\,(r_\oslash)}_{\odot\oslash} &\equiv \int_\oslash \vec{r}_\oslash \times d\vec{F}_{\odot\oslash} = \int_\oslash \vec{r}_\oslash \times \vec{g}_{\odot\oslash} dM_\oslash = \int_\oslash \mu_\oslash(r_\oslash)\, \vec{r}_\oslash \times \vec{g}_{\odot\oslash} dV_\oslash \\
&= \int_\oslash \mu_\oslash(r_\oslash)\, \vec{r}_\oslash \times \Big[\vec{g}_{\odot\oslash N}\Big(1+3\tfrac{\dot{\vec{R}}^2}{c^2}\Big) - 6g_{\odot\oslash N}\tfrac{\dot{\vec{R}}^3}{c^3}\Big] dV_\oslash \\
&= \int_\oslash \mu_\oslash(r_\oslash)\, \vec{r}_\oslash \times \Big(3\vec{g}_{\odot\oslash N}\tfrac{\dot{\vec{R}}^2}{c^2} - 6g_{\odot\oslash N}\tfrac{\dot{\vec{R}}^3}{c^3}\Big) dV_\oslash \\
&\approx -\tfrac{6g_{\odot\oslash N}}{c^3}\int_\oslash \mu_\oslash(r_\oslash)\, \vec{r}_\oslash \times \Big(\tfrac{c}{2}\vec{1}_r \dot{\vec{R}}^2 + \dot{\vec{R}}^3\Big) dV_\oslash \\
&= -\tfrac{6g_{\odot\oslash N}}{c^3}\int_\oslash \mu_\oslash(r_\oslash)\, \vec{r}_\oslash \times \{c\,[(\vec{v}\times\vec{\omega}_\oslash)\vec{r}_\oslash]\vec{1}_r + 2[(\vec{v}\times\vec{\omega}_\oslash)\vec{r}_\oslash]\vec{v} \\
&\qquad + v^2\vec{\omega}_\oslash\times\vec{r}_\oslash + (\vec{\omega}_\oslash\times\vec{r}_\oslash)^3\}\, dV_\oslash \\
&= -\tfrac{6g_{\odot\oslash N}}{c^3}\int_\oslash \mu_\oslash(r_\oslash)\{c\,[(\vec{v}\times\vec{\omega}_\oslash)\vec{r}_\oslash]\, \vec{r}_\oslash\times\vec{1}_r + 2[(\vec{v}\times\vec{\omega}_\oslash)\vec{r}_\oslash]\, \vec{r}_\oslash\times\vec{v} \\
&\qquad + v^2\vec{r}_\oslash\times(\vec{\omega}_\oslash\times\vec{r}_\oslash) + (\vec{\omega}_\oslash\times\vec{r}_\oslash)^2\vec{r}_\oslash\times(\vec{\omega}_\oslash\times\vec{r}_\oslash)\}dV_\oslash \\
&= -\tfrac{6g_{\odot\oslash N}}{c^3}\Big[\tfrac{c}{3}I^{(2)}_\oslash(\vec{v}\times\vec{\omega}_\oslash)\times\vec{1}_r + \tfrac{2}{3}I^{(2)}_\oslash(\vec{v}\times\vec{\omega}_\oslash)\times\vec{v} \\
&\qquad + \tfrac{2}{3}v^2 I^{(2)}_\oslash\vec{\omega}_\oslash + \tfrac{8}{15}I^{(4)}_\oslash\vec{\omega}^3_\oslash\Big] \\
&= -\tfrac{4g_{\odot\oslash N}}{c^3}I^{(2)}_\oslash\Big[\tfrac{c}{2}(\vec{v}\times\vec{\omega}_\oslash)\times\vec{1}_r + (\vec{v}\times\vec{\omega}_\oslash)\times\vec{v} + v^2\vec{\omega}_\oslash \\
&\qquad + \tfrac{4}{5}\tfrac{I^{(4)}_\oslash}{I^{(2)}_\oslash}\vec{\omega}^3_\oslash\Big] = \\
&= -\tfrac{12}{5}\tfrac{GM_\odot M_\oslash R^2_\oslash\omega_\oslash}{c^3 r^2}\Big[\tfrac{c}{2}(\vec{v}\times\vec{1}_{\omega_\oslash})\times\vec{1}_r + (\vec{v}\times\vec{1}_{\omega_\oslash})\times\vec{v} \\
&\qquad + v^2\vec{1}_{\omega_\oslash} + \tfrac{4}{7}\omega^2_\oslash R^2_\oslash\vec{1}_{\omega_\oslash}\Big].
\end{aligned} \tag{34}$$

As $W^{(brv)}_{\odot\oslash}$ and $W^{(brv)}_{\oslash\odot}$ are proportional to $(v/c)^3$, we neglect deviations from Kepler 's first and second laws, as not being interested in greater precision, so we use

$$\left.\begin{aligned}
&r = \ell/(1+\epsilon\cos\varphi)\,, \\
&\dot{r} = \dot{\varphi}\, dr/d\varphi = (r^2\dot{\varphi}/r^2)dr/d\varphi = c\sqrt{\kappa}\epsilon\sin\varphi\,, \\
&v^2 = \dot{r}^2 + r^2\dot{\varphi}^2 = c^2\kappa(1+\epsilon^2+2\epsilon\cos\varphi)\,;
\end{aligned}\right\} \tag{35}$$

$$\left.\begin{aligned}
&\vec{1}_{\omega_\oslash} = \vec{1}_x\sin\iota_\oslash + \vec{1}_z\cos\iota_\oslash = (\vec{1}_r\cos\varphi - \vec{1}_\varphi\sin\varphi)\sin\iota_\oslash \\
&\qquad + \vec{1}_z\cos\iota_\oslash\,, \\
&\vec{v}\,\vec{1}_{\omega_\oslash} = (\dot{r}\vec{1}_r + r\dot{\varphi}\vec{1}_\varphi)[(\vec{1}_r\cos\varphi - \vec{1}_\varphi\sin\varphi)\sin\iota_\oslash + \vec{1}_z\cos\iota_\oslash] \\
&\quad = \dot{r}\sin\iota_\oslash\cos\varphi - r\dot{\varphi}\sin\iota_\oslash\sin\varphi = c\sqrt{\kappa}\epsilon\sin\iota_\oslash\sin\varphi\cos\varphi \\
&\qquad - c\sqrt{\kappa}\sin\iota_\oslash\sin\varphi(1+\epsilon\cos\varphi) = -c\sqrt{\kappa}\sin\iota_\oslash\sin\varphi\,, \\
&[(\vec{v}\times\vec{1}_{\omega_\oslash})\times\vec{1}_r]\vec{1}_{\omega_\oslash} = [\dot{r}\vec{1}_{\omega_\oslash} - (\vec{1}_r\vec{1}_{\omega_\oslash})\vec{v}]\vec{1}_{\omega_\oslash} \\
&\quad = \dot{r} - (\vec{1}_r\vec{1}_{\omega_\oslash})(\vec{v}\,\vec{1}_{\omega_\oslash}) = c\sqrt{\kappa}\epsilon\sin\varphi + c\sqrt{\kappa}\sin^2\iota_\oslash\sin\varphi\cos\varphi \\
&\qquad = c\sqrt{\kappa}\sin\varphi\,(\epsilon + \sin^2\iota_\oslash\cos\varphi)\,, \\
&[(\vec{v}\times\vec{1}_{\omega_\oslash})\times\vec{v}]\vec{1}_{\omega_\oslash} = [v^2\vec{1}_{\omega_\oslash} - (\vec{v}\,\vec{1}_{\omega_\oslash})\vec{v}]\vec{1}_{\omega_\oslash} = v^2 - (\vec{v}\,\vec{1}_{\omega_\oslash})^2 \\
&\qquad = c^2\kappa\,(1+\epsilon^2+2\epsilon\cos\varphi - \sin^2\iota_\oslash\sin\varphi)\,.
\end{aligned}\right\} \tag{36}$$

The sequence of steps (37) starts by inserting the final expression (34) of $\vec{\tau}^{\,(r)}_{\odot\oslash}$ in the final expression (31) of $W^{(brv)}_{\odot\oslash}$; step 2 uses the second expression (17) of κ, and expressions from Eqs (33); next, standard calculations.

$$\begin{aligned}
W^{(brv)}_{\odot\oslash} &= -\tfrac{12}{5}\tfrac{GM_\odot M_\oslash R^2_\oslash\omega_\oslash}{2\Omega_0 c^3}\int_0^{2\pi}\Big\{\tfrac{c}{2}[(\vec{v}\times\vec{\omega}_\oslash)\times\vec{1}_r]\vec{1}_{\omega_\oslash} \\
&\qquad + [(\vec{v}\times\vec{\omega}_\oslash)\times\vec{v}]\vec{1}_{\omega_\oslash} + v^2 + \tfrac{4}{7}\omega^2_\oslash R^2_\oslash\Big\}\, d\varphi \\
&= -\tfrac{12}{5}\sqrt{\kappa}M_\oslash\tfrac{\omega^2_\oslash R^2_\oslash}{c^2}\int_0^{2\pi}\Big[\tfrac{c^2}{2}\sqrt{\kappa}\,(\epsilon+\sin^2\iota_\oslash\cos\varphi)\sin\varphi \\
&\quad + c^2\kappa\,(1+\epsilon^2+2\epsilon\cos\varphi) \\
&\quad + c^2\kappa(1+\epsilon^2+2\epsilon\cos\varphi - \sin^2\iota_\oslash\sin^2\varphi) + \tfrac{4}{7}(\omega_\oslash R_\oslash)^2\Big]d\varphi \\
&= -\tfrac{48\pi}{5}\kappa^{3/2}M_\oslash\omega^2_\oslash R^2_\oslash\Big[1+\epsilon^2 - \tfrac{1}{4}\sin^2\iota_\oslash + \tfrac{2}{7\kappa}\Big(\tfrac{\omega_\oslash R_\oslash}{c}\Big)^2\Big].
\end{aligned} \tag{37}$$

Seeing Table 1, notice that the 1.162×10^{12} watts resulted from *Secular retardation* caused by the RVD (braking) torque are the equivalent of 100 power stations of 11.62 gigawatts each working permanently for *Secular retardation* hence for *Tectonic plates drift, Continental drift, Mountains uplift, Volcanoes, Earthquakes, etc.*

Secular Retardation—essential support of the odd RVD term in law (1)—is the earth rotation slowdown caused, about one half, by tidal effects, and the remainder by RVD torque. Here are some significant quotations from Melchior [7]. *"The secular retardation is a phenomenon which has existed fore more than a billion years as demonstrated by paleontological discoveries." "... The total dissipation corresponding to astronomical and paleontological evidence is* 8.5×10^{26} *erg yr*$^{-1}$*." "... estimation made by Miller (1966) gave the dissipation in shallow seas as* $4.4{\times}10^{26}$ *erg yr*$^{-1}$*, while Cox and Sandstrom (1962) estimate that the scattering into internal modes in open ocean is responsible for* $1.2{\times}10^{26}$ *erg yr*$^{-1}$*. Miller also finds that in fact the flux from the deep seas is far too small to be considered. Kaula (1975) postulates that the dissipation takes place in shallow and shelf seas through processes unidentified. (R.R.) Newton (1973) noted that the site of at least half the tidal dissipation has not been identified in the oceans ...".*

From these data one can reckon the remaining energy, in 10^{26} *erg yr*$^{-1}$ as unit, to be accounted for by the RVD torque, namely 8.5 - (4.4 + 1.2 + 0) = 2.9, hence the result is 2.9×10^{26} *erg yr*$^{-1}$ = 2.9×10^{19} *joules yr*$^{-1}$. However, beyond the numerical data, the above Newton's 1973 assertion implies a magnitude up to about 4.5×10^{19} *joules yr*$^{-1}$. So, a value in the interval (2.9 − 4.5)$\times10^{19}$ *joules yr*$^{-1}$ is to be expected from the RVD braking torque, in agreement with 3.66×10^{19} *joules yr*$^{-1}$ in Table 1 (the fifth column).

Resume quotations from Melchior [7] with paleontologic evidences. *"... paleontologists discovered that fossil corals can be considered as fossil clocks. These animals develop themselves by secreting one ring every day, its width being a function of the quantity of light they have received. One can therefore expect and observe an annual modulation in their structure which allows to be counted more or less exactly the number of days (rings) contained in one year (wavelength). An obvious check is that one finds 365 rings for the presently living corals. The fossil corals have been submitted to precise and delicate measurements conducted by Wells, Pannella and MacClintock. ... The duration of the day at the Devonian epoch was about 7238 s which represents in 380 million years a loss of 7238 s ... that is 1.9 s in 100000 years, in very close agreement with the astronomical results for the last 3000 years.".*

Geophysicists have been able to identify in their science two antinomies that finally proved to be complementary, both having one and the same gravitational solution: the surplus in one problem (energy in *Secular retardation*), was that lacked in the other ("driving/propelling force" in *Continental drift*).

Finally, also quotations from Melchior [7], this time with astronomical evidences. *"The secular retardation of the Earth's rotation is a classical astronomical phenomenon deduced from the recorded longitude of the places of observations of the total eclipses in Antiquity." "Besides the eclipses, more recent observations of the Sun (declinations since 1760 and right ascensions since 1835), Venus and Mercury transits on the Sun's disk reconfirm the size of this phenomenon."*

Zefir is the gentle evening wind to the west, an evidence of the RVD action on the earth's atmosphere. It is not perceived when superposed with stronger winds, of thermodynamic causes.

Evening Tide is for sea waters what Zephyr is for air.

Pororoca is the Amazon's reverse flow, occurring in the evening, twice a year, near equinoxes—when the the RVD torque is maximum. Perhaps it is at the same time an effect of resonance of the solar RVD and lunar tidal actions on ocean waters.

Tectonic Plates Drift takes most of the energy from *Secular retardation* consuming it in *Continental drift* (of course, to the west *globally*, not *totally* because of the earth's axis tilt), causing Rifts, Mountains uplift, Volcanoes, Earthquakes, Geysers, and other more or less significant effects.

Continental Drift Abraham Ortelius is credited as the first to notice (1596), on geometric coincidences (complementary forms), the possibility of continental drift. Antonio Snider-Pellegrini proposed (1858) a comprehensive theory of an initially unique continent, bringing forward as evidence, matching plant fossils. Frank Bursley Taylor re-proposed (1908) the theory, improved by his studies on mountain ranges as the Andes , Rockies, Alps, and Himalayas. Three years later, independent of Taylor, Alfred Wegener re-reproposed the theory (1912) much extended, with a more comprehensive scenario and more evidences, and searched for further evi-

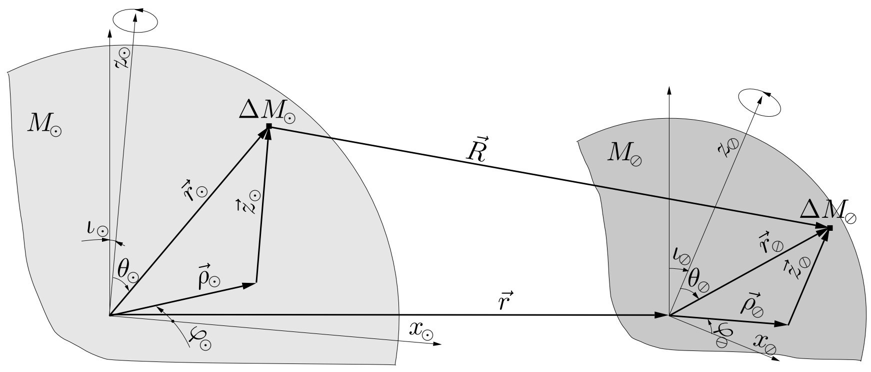

Figure 3: **Gravitational interaction between the central and the orbiting spheric bodies—geometric notations** Every particle/point mass $\Delta M_\odot$ of a *rotating* central body $\odot$ acts on every point mass $\Delta M_\oslash$ of a rotating and orbiting body $\oslash$. The position vector of $\Delta M_\oslash$ with respect to $\Delta M_\odot$ is $\vec{R}$, hence $\vec{r}_\odot + \vec{R} = \vec{r} + \vec{r}_\oslash$. Rotation axes are tilted with $\iota_\odot$ and $\iota_\oslash$ angles. As origins there are taken the spheres' centers, and as z-axes just the rotating axes, i.e., their unit vectors coincide, $\mathbf{1}_z = \mathbf{1}_\omega$. Coordinate systems: Cartesian, (x, y, z); spheric, (r, φ, θ); and cylindric, (ρ, φ, z). y-axes are not shown.

Planets	κ $\times 10^{-10}$	δ $\times 10^{-2}$	$\Delta\Omega/\Omega$ $\times 10^{-13}$	$-W^{(brv)}_{\odot\oslash}$ $\times 10^{20}$	$-W^{(brv)}_{\odot\oslash}/P_{rv}$ $\times 10^{12}$	$-W^{(brv)}_{\oslash\odot}/P_{rv}$ $\times 10^{10}$	Rank
Mercury	266.3	4311	1708	4.1×10^{-6}	5.4×10^{-5}	441.5	3
Venus	136.5	865.1	601.3	7.7×10^{-6}	4.0×10^{-5}	896.7	2
Earth	98.77	385.1	370.1	**0.366**	**1.162**	400.4	4
Mars	65.37	135.5	201.0	0.006	9.6×10^{-3}	12.36	6
Jupiter	19.02	6.252	31.35	8976	2404	**946.0**	**1**
Saturn	10.38	1.374	12.65	82.64	8.914	43.82	5
Uranus	5.157	0.239	4.425	1.607	0.061	0.654	7
Neptune	3.284	0.078	2.244	1.320	0.025	0.250	8
Pluto	0.666	0.042	1.743	2.6×10^{-9}	3.3×10^{-11}	1.5×10^{-5}	9

Table 1: **Solar System Table, Metric, Four Digits** κ is defined in Eqs. (17), hence $\kappa = GM_\odot/[a\,(1-\epsilon^2)\,c^2]$; δ, given by formula (27), is expressed in arc seconds per century; column $\Delta\Omega/\Omega$ is generated by Eq. (29); $W^{(brv)}_{\odot\oslash}$ is given by formula (37), and $W^{(brv)}_{\oslash\odot}$ by (41); Ω is computed using $\Omega = \pi a^2\sqrt{1-\epsilon^2}/P_{rv}$ (orbit's area over orbital period).

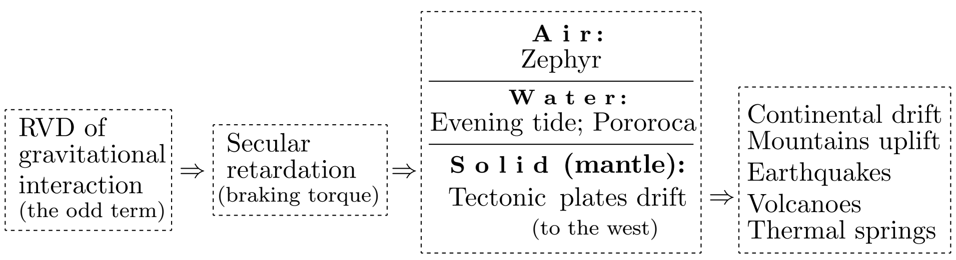

Figure 4: **Causal chain of RVD geophysical effects**

dences (until his death at work, in Greenland). Therefore the fair name of the continental drift is "Ortelius-Pellegrini-Taylor-Wegener theory". The theory was sustained ardently by its authors, by far most notably Wegener, but categorically rejected because of lack of the "driving force", until 1960's when it was generally accepted as *Sea-floor spread* by Hess, though the driving force lacked just as initially. Nobody suspected that the fault lain in gravitation (hence between 1858 and 2017 geophysics was ahead of gravitation!...) A conclusive quotation [8]: "*... the lack of an acceptable mechanism was ... a strong reason to reject continental drift. Ironically, ... plate tectonics was accepted ...*".

Earthquakes As the *RVD (braking) torque is maximum at equator, at equinoxes (March and September)*, it is this where and when earthquakes and volcano eruptions occur most probably. However, a motion of a tectonic plate can cause stress on its neighbors at large distances, so the place of an earthquake is difficult to predict. As to month, a tectonic plate may have little up to release by an earthquake, before equinox (as March 9, the Andreanof Islands, Alaska, 9.1 magnitude), or may end this process afterwards (as March 28, 1964, Prince William Sound, Alaska, 9.2 magnitude). As a good plead for March, we quote "*March is Earthquake Month, and Other Shaky 'Facts'. A glance at geologic statistics might lead one to believe March is earthquake month. After all, the two strongest recorded earthquakes in U.S. history occurred in this month"* [14]. As a plead for September (as well as for March) we can see USGS statistics, of which the most suitable (graphing number of earthquakes per month) is that of Kansas [15] : note the more numerous earthquakes around March and September of each year 2013–2016, a fine confirmation. An even better coincidence is to be expected if one takes into consideration $\vec{\tau}_{\odot\oslash}^{(z\oslash)}$ (defined in Eqs. (30) but not calculated), which although does not brake Earth's rotation, does exert some tension on tectonic plates, being thus involved in determining earthquakes.

Mountain Rise/Uplift takes place when the drift of a tectonic plate is blocked by another, possible blocked by an other, with such a force that there is no other possibility to resolve the mechanical tension than rising some portion, or a volcano, or both.

Volcano is an eruption of melt rock (lava) occurring when tectonic plates are so strongly blocked that the simplest way of resolving the mechanical tension is a release to the surface and expelling material (gases, lava, and rock fragments).

Thermal Spring is analogous to a volcano, involving hot water instead of magma, at smaller scale.

5.1 Planetary Thermal Emission

After exploring the multitude of effects entailed by *Secular retardation* on the earth, one can think of such effects on other planets, primarily on Jupiter whose rotation speed $\omega_\oslash R_\oslash$ is much greater hence, as known, the "Jovian physics" (analogy to geophysics) is infernal. As seen in Table I, the average breaking power over a revolution $W_{\odot\oslash}^{(brv)}/P_{rv} = 2.40{\times}10^{15}$ *wats* upon Jupiter is much smaller than the thermal (infrared) power it emits, 1.38×10^{18} *wats* as resulting from table 1 of Ingersol [10]. This comparison is not quite relevant because $W_{\odot\oslash}^{(brv)}/P_{rv}$ is an average value, while the value at equinoxes is greater, especially in case of Saturn (at "Saturnian equinoxes") whose rotation axis inclination is greater, and extremely in case of Uranus whose inclination is the greatest. One can expect the braking torque to play a role in maintaining the thermal emission of the outer planets, mainly Jupiter and Saturn [9], [10].

6 RVD torque exerted by an orbiting body on the central body: *Solar cycle*

Calculating the torque $\vec{\tau}_{\oslash\odot}^{(r\odot)}$ and work $W_{\oslash\odot}^{(brv)}$ is analogous to that of $\vec{\tau}_{\odot\oslash}^{(r\oslash)}$ and $W_{\odot\oslash}^{(brv)}$. The same Figure 2 is used as a scheme for notations, but with reversed direction of $\vec{r}$ and $\vec{R}$, keeping the rest unchanged. So $\vec{r}_\oslash + \vec{R} = \vec{r} + \vec{r}_\odot$, and approximate $\vec{R} = \vec{r} + \vec{r}_\odot - \vec{r}_\oslash \approx \vec{r} + \vec{r}_\odot$, keeping $\vec{r}_\odot$ for its essential role in the effect of rotation we are studying. Relations (38) are used in the sequence of steps (39), just as Eqs. (33) in the sequence (34).

$$\left.\begin{aligned}
&\vec{R} = \vec{r} + \vec{r}_\odot - \vec{r}_\oslash \approx \vec{r} + \vec{r}_\odot\,,\ \dot{\vec{R}} = \vec{v} + \vec{\omega}_\odot\times\vec{r}_\odot\,,\\
&\dot{\vec{R}}^2 = v^2 + 2(\vec{v}\times\vec{\omega}_\odot)\vec{r}_\odot + (\vec{\omega}_\odot\times\vec{r}_\odot)^2,\\
&\dot{\vec{R}}^3 = \dot{\vec{R}}^2\dot{\vec{R}} = [v^2 + 2(\vec{v}\times\vec{\omega}_\odot)\vec{r}_\odot + (\vec{\omega}_\odot\times\vec{r}_\odot)^2](\vec{v} + \vec{\omega}_\odot\times\vec{r}_\odot)\\
&\quad = \vec{v}^{\,3} + 2[(\vec{v}\times\vec{\omega}_\odot)\vec{r}_\odot]\vec{v} + (\vec{\omega}_\odot\times\vec{r}_\odot)^2\vec{v} + v^2\vec{\omega}_\odot\times\vec{r}_\odot\\
&\qquad +2[(\vec{v}\times\vec{\omega}_\odot)\vec{r}_\odot)]\vec{\omega}_\odot\times\vec{r}_\odot + (\vec{\omega}_\odot\times\vec{r}_\odot)^3,\\
&\vec{r}_\odot\times(\vec{\omega}_\odot\times\vec{r}_\odot) = \vec{\omega}_\odot r_\odot^2 - (\vec{\omega}_\odot\vec{r}_\odot)\vec{r}_\odot\,,\\
&\vec{r}_\odot\times(\vec{\omega}_\odot\times\vec{r}_\odot)^3 = (\vec{\omega}_\odot\times\vec{r}_\odot)^2\vec{r}_\odot\times(\vec{\omega}_\odot\times\vec{r}_\odot)\\
&\quad = [\omega_\odot^2 r_\odot^2 - (\vec{\omega}_\odot\vec{r}_\odot)^2][\vec{\omega}_\odot r_\odot^2 - (\vec{\omega}_\odot\vec{r}_\odot)\vec{r}_\odot]\\
&\quad = \vec{\omega}_\odot^{\,3} r_\odot^4 - \omega_\odot^2 r_\odot^2(\vec{\omega}_\odot\vec{r}_\odot)\vec{r}_\odot - (\vec{\omega}_\odot\vec{r}_\odot)^2\vec{\omega}_\odot r_\odot^2 + (\vec{\omega}_\odot\vec{r}_\odot)^3\vec{r}_\odot\,.
\end{aligned}\right\} \tag{38}$$

In the sequence of steps (39) for $\vec{\tau}_{\oslash\odot}^{(r\odot)}$ the first row passes the stages analogous to (31) and (32) for $\vec{\tau}_{\odot\oslash}^{(r\oslash)}$; the second row inserts the RVD expression of $\vec{g}_{\oslash\odot}$; the third row takes into account the zero torque involved by the Newton gravitational force (Theorem 2, Example 3), and approximates $g_{\oslash\odot N}$ as having the same magnitude and direction at any point of $\odot$; the fourth row transcribes only odd terms of $\dot{\vec{R}}^2$ and $\dot{\vec{R}}^3$ from their expressions (38), since the others give null integrals according to the second formula (54); the fifth row, only rearrangements; the sixth row writes the values of the integrals of the first two terms applying the first formula (55), while for the third term use its (38) expression obtaining the integral as $v^2[I_\odot^{(2)}\vec{\omega}_\odot - (1/3)I_\odot^{(2)}\vec{\omega}_\odot] = (2/3)I_\odot^{(2)}v^2\vec{\omega}_\odot$, and the fourth (last) term uses its (38) expression and obtains the integral as $I_\odot^{(4)}\vec{\omega}_\odot^{\,3} - (1/3)I_\odot^{(4)}\vec{\omega}_\odot^{\,3} - (1/3)I_\odot^{(4)}\vec{\omega}_\odot^{\,3} + (1/15)I_\odot^{(4)}3\vec{\omega}_\odot^{\,3} = (8/15)I_\odot^{(4)}\vec{\omega}_\odot$; the seventh and eighth rows, obvious.

$$\left.\begin{aligned}
\vec{\tau}_{\oslash\odot}^{(r\odot)} &\equiv \int_\odot \vec{r}_\odot\times d\vec{F}_{\oslash\odot} = \int_\odot \vec{r}_\odot\times\vec{g}_{\oslash\odot}dM_\odot = \int_\odot \mu(r_\odot)\,\vec{r}_\odot\times\vec{g}_{\oslash\odot}dV_\odot\\
&= \int_\odot \mu(r_\odot)\,\vec{r}_\odot\times\Big[\vec{g}_{\oslash\odot N}\Big(1 + 3\frac{\dot{\vec{R}}^2}{c^2}\Big) - 6g_{\odot\oslash N}\frac{\dot{\vec{R}}^3}{c^3}\Big]dV_\odot\\
&= \int_\odot \mu(r_\odot)\,\vec{r}_\odot\times\Big(3\vec{g}_{\oslash\odot N}\frac{\dot{\vec{R}}^2}{c^2} - 6g_{\oslash\odot N}\frac{\dot{\vec{R}}^3}{c^3}\Big)\,dV_\odot\\
&\approx -\frac{6g_{\oslash\odot N}}{c^3}\int_\odot \mu(r_\odot)\,\vec{r}_\odot\times\Big(\frac{c}{2}\dot{\vec{R}}^2\vec{1}_r + \dot{\vec{R}}^3\Big)\,dV_\odot\\
= -\frac{6g_{\oslash\odot N}}{c^3}&\int_\odot \mu(r_\odot)\vec{r}_\odot\times\Big\{c\vec{1}_r[(\vec{v}\times\vec{\omega}_\odot)\vec{r}_\odot] + 2[(\vec{v}\times\vec{\omega}_\odot)\,\vec{r}_\odot]\vec{v}\\
&\qquad + v^2\vec{\omega}_\odot\times\vec{r}_\odot + (\vec{\omega}_\odot\times\vec{r}_\odot)^3\Big\}\,dV_\odot\\
= -\frac{6g_{\oslash\odot N}}{c^3}&\int_\odot \mu(r_\odot)\Big\{c\,[(\vec{v}\times\vec{\omega}_\odot)\vec{r}_\odot]\vec{r}_\odot\times\vec{1}_r + 2[(\vec{v}\times\vec{\omega}_\odot)\vec{r}_\odot]\vec{r}_\odot\times\vec{v}\\
&\qquad + v^2\vec{r}_\odot\times(\vec{\omega}_\odot\times\vec{r}_\odot) + \vec{r}_\odot\times(\vec{v}\times\vec{\omega}_\odot)^3\Big\}dV_\odot\\
= -\frac{6g_{\oslash\odot N}}{c^3}&\Big[\frac{c}{3}I_\odot^{(2)}(\vec{v}\times\vec{\omega}_\odot)\times\vec{1}_r + \frac{2}{3}I_\odot^{(2)}(\vec{v}\times\vec{\omega}_\odot)\times\vec{v} + \frac{2}{3}I_\odot^{(2)}v^2\vec{\omega}_\odot\\
&\qquad + \frac{8}{15}I_\odot^{(4)}\vec{\omega}_\odot^{\,3}\Big] =\\
= -\frac{4g_{\oslash\odot N}I_\odot^{(2)}}{c^3}&\Big[\frac{c}{2}(\vec{v}\times\vec{\omega}_\odot)\times\vec{1}_r + (\vec{v}\times\vec{\omega}_\odot)\times\vec{v} + v^2\vec{\omega}_\odot + \frac{4}{5}\frac{I_\odot^{(4)}}{I_\odot^{(2)}}\vec{\omega}_\odot^{\,3}\Big]\\
= -\frac{12}{5}\frac{GM_\oslash M_\odot R_\odot^2\omega_\odot}{c^3r^2}&\Big[\frac{c}{2}(\vec{v}\times\vec{1}_{\omega\odot})\times\vec{1}_r + (\vec{v}\times\vec{1}_{\omega\odot})\times\vec{v} + v^2\vec{1}_{\omega\odot}\\
&\qquad + \frac{4}{7}\omega_\odot^2R_\odot^2\vec{1}_{\omega\odot}\Big].
\end{aligned}\right| \tag{39}$$

Note the full analogy between the expression (39) of $\vec{\tau}_{\oslash\odot}^{(r\odot)}$, and (34) of $\vec{\tau}_{\odot\oslash}^{(r\oslash)}$: indexes $_\oslash$ and $_\odot$ are interchanged.

Eqs. (40) and (35) are used in the sequence of steps (41).

$$\left.\begin{aligned}
\vec{1}_{\omega\odot} &= \vec{1}_x\sin\iota_\odot + \vec{1}_z\cos\iota_\odot\\
&= (\vec{1}_r\cos\varphi - \vec{1}_\varphi\sin\varphi)\sin\iota_\odot + \vec{1}_z\cos\iota_\odot\,,\\
\vec{v}\,\vec{1}_{\omega\odot} &= (\dot{r}\vec{1}_r + r\dot{\varphi}\vec{1}_\varphi)[(\vec{1}_r\cos\varphi - \vec{1}_\varphi)\sin\iota_\odot + \vec{1}_z\cos\iota_\odot]\\
&= \dot{r}\sin\iota_\odot\cos\varphi - r\dot{\varphi}\sin\iota_\odot\sin\varphi + \sqrt{\kappa}\,\epsilon\sin\iota_\odot\sin\varphi\cos\varphi\\
&\quad -c\sqrt{\kappa}\sin\iota_\odot\sin\varphi\,(1 + \epsilon\cos\varphi) = -\,c\sqrt{\kappa}\sin\iota_\odot\sin\varphi\,,\\
[(\vec{v}\times\vec{1}_{\omega\odot})\times\vec{1}_r]\vec{1}_{\omega\odot} &= [\dot{r}\vec{1}_{\omega\odot} - (\vec{1}_r\vec{1}_{\omega\odot})\,\vec{v}]\vec{1}_{\omega\odot} = \dot{r} - (\vec{1}_r\vec{1}_{\omega\odot})(\vec{v}\,\vec{1}_{\omega\odot})\\
&= c\sqrt{\kappa}\epsilon\sin\varphi + c\sqrt{\kappa}\sin^2\iota_\odot\sin\varphi\cos\varphi\\
&= c\sqrt{\kappa}\sin\varphi\,(\epsilon + \sin^2\iota_\odot\cos\varphi)\,,\\
[(\vec{v}\times\vec{1}_{\omega\odot})\times\vec{v}]\vec{1}_{\omega\odot} &= \{v^2\vec{1}_{\omega\odot} - (\vec{v}\,\vec{1}_{\omega\odot})\vec{v}\}\vec{1}_{\omega\odot} = v^2 - (\vec{v}\,\vec{1}_{\omega\odot})^2\\
&= c^2\kappa\,(1 + \epsilon^2 + 2\epsilon\cos\varphi - \sin^2\iota_\odot\sin^2\varphi)\,.
\end{aligned}\right\} \tag{40}$$

In the following sequence of steps the first row passes the stages similar to (31) and (32); the second row inserts the last expression (39) of $\vec{\tau}_{\oslash\odot}^{(r\odot)}$; the third and fourth rows use expressions from (40); and the fifth row, usual calculations.

$$\begin{aligned} W_{\oslash\odot}^{(brv)} &\equiv \int_0^{2\pi P_{rv}/P_{rt}} \vec{\tau}_{\oslash\odot}^{(r\odot)} \vec{1}_{\omega_\odot} d\varphi_\odot = \int_0^{2\pi} \vec{\tau}_{\oslash\odot}^{(r\odot)} \vec{1}_{\omega_\odot} \frac{\dot\varphi_\odot}{\dot\varphi}\, d\varphi \\ &\approx \frac{\omega_\odot}{2\Omega_0} \int_0^{2\pi} \vec{\tau}_{\oslash\odot}^{(r\odot)} \vec{1}_{\omega_\odot} r^2 d\varphi \\ = -\frac{12}{5} &\frac{GM_\oslash M_\odot(\omega_\odot R_\odot)^2}{2\Omega_0 c^3} \int_0^{2\pi} \Big\{ \tfrac{c}{2} [(\vec{v}\times\vec{1}_{\omega_\odot})\times\vec{1}_r]\vec{1}_{\omega_\odot} \\ &+ [(\vec{v}\times\vec{1}_{\omega_\odot})\times\vec{v}]\vec{1}_{\omega_\odot} + v^2 + \tfrac{4}{7}\omega_\odot^2 R_\odot^2 \Big\}\, d\varphi \\ = -\frac{12}{5} &\sqrt{\kappa} M_\oslash \frac{(\omega_\odot R_\odot)^2}{c^2} \int_0^{2\pi} \Big[\tfrac{c^2}{2}\sqrt{\kappa}\,(\epsilon + \sin^2\iota_\odot \cos\varphi)\sin\varphi + \\ c^2\kappa(1+\epsilon^2+2\epsilon\cos\varphi) &+ c^2\kappa(1+\epsilon^2+2\epsilon\cos\varphi - \sin^2\iota_\odot \sin^2\varphi) \\ &+ \tfrac{4}{7}\omega_\odot^2 R_\odot^2 \Big]\, d\varphi \\ = -\frac{48\pi}{5} &\kappa^{3/2} M_\oslash (\omega_\odot R_\odot)^2 \left[1 + \epsilon^2 - \tfrac{1}{4}\sin^2\iota_\odot + \tfrac{2}{7\kappa}\left(\tfrac{\omega_\odot R_\odot}{c}\right)^2 \right]. \end{aligned} \tag{41}$$

Solar cycle Though sunspots were observed by Galileo Galilei (1610) and contemporaries, their cyclicity was discovered by S.H. Schwabe (1843) after 17 years observations. The rough fit of the solar cycle with Jupiter's revolution has for long been a temptation to consider them in a causal correlation, but just the lack of the cause/mechanism opposed stubbornly, somehow like, but not so dramatic, as in case of *Continental drift*. On seeing Table 1 (the last two columns), now it is clear that *Jupiter rules approximately the* Solar cycle with its average power of 9.46×10^{12} *watts* exerted upon the sun, wile the next, Venus, exerts an almost constant work upon the sun, not cyclic, because of the small eccentricity; the other significant contributors to the solar activity, Mercury and Earth, together amount less than Jupiter.

Grandpierre (1996) [11], situated at the final end of the sequence Edmonds (1882) → K.D. Wood (1972) → Curie (1973) → R.M. Wood (1975) → Gribbin,Plagemann (1977) → Verma (1986) → Seymour,Willmott,Turner (1992) → Desmoulins (1995), and not only, took over the ideas of the planetary individual tides and that of co-alignment of Jupiter, Earth, and Venus, and, in addition, hypothesized a role of "the local *magnetic field in the solar core" supposed (*artificially*) by chance roughly fitting Jupiter's period*; but he inserted three valuable assertions, as follows: "Hantzsche (1978) ... argues against a purely tidal planetary theory" and "K.D. Wood (1972) replied *the best physical explanation is ... some nontidal factor contributing to the formation of sunspots*"; "*the appropriate physical mechanism is not known*" (Novotny, 1983).

Niroma's cogent analysis [12] on Jupiter's primordial role in solar cycle is convincing, the remaining step being the cause/mechanism: "My study is a pure statistical theory and it shows interesting patterns. I leave to the physicists the arena to think any explanations...".

Summarize the above preceding studies of the *Solar cycle* as follows: (i) rightly, all admitted some planetary cause, including co-alignments; (ii) rightly, all pointed out the inner planets, plus Jupiter; (iii) wrongly, most of them (except Hantzsche, Novotny, and eventually K.D. Wood) upheld the tidal cause; (iv) and only one, Niroma, determined Jupiter's prevailing role.

Institutions on Solar Physics—as Zurich Observatory (daily, from 1849), and NASA—watch the sun's picture as to the sunspots number and their evolution, but now, as the cause/mechanism is known, a graph of the instantaneous torque $\vec{\tau}_{\odot\oslash}^{(r)}\vec{1}_{\omega_\oslash}$, not its integral (work) as we have calculated above, and keeping comparison with observations would be fully expressive.

7 Experiments proposed

The following first two experiments are outdoor, feasible now, and concern the *odd* RVD term—the most revolutionary.

Experiment Proposed 1 ***The maximum (westward) drift speed of tectonic plates is at equator at equinoxes*** The experiment consists in a high precision monitoring a topographic distance, and concerns the *odd* RVD term in Newton's law of gravitation (1). The westward drift is maximum, as to place, at the equator and vicinity (greatest relative-velocity), and, as to time, at equinoxes—when it is exactly to the west (greatest projection of velocity, as the earth's equatorial plane contains the sun's center).

A west–east distance between two well marked points (W and E in Figure 5*) on opposite sides of a drift trench or a drift valley should be measured from time to time, for instance daily, if not continuously, and a graph recorded.*

Experiment Proposed 2 ***Westward drift of tectonic plates*** The experiment consists in a high precision topographic measurement, repeated at least once (but not monitoring as in Experiment Proposed 1), and concerns the *odd* RVD term in the gravity law (1).

As already mentioned at Experiment Proposed 1, the effect is maximum at the equator and vicinity, at equinoxes—when it is exactly to the west—but it is present and detectable at any time. The main difficulty is the lack of a fixed (not drifting) reference frame, in fact a fixed point. Therefore the problem has to be solved using relative distances and their comparison.

One should survey a relief map to note how Ecuador advances to the west as if pulling westward the continent, rising gradually Galapagos and making them boil by thermal springs, as well as the entire Andes are in a march to the west, pushing Peru, and Chile toward Pacific. Somehow analogously, Himalayan heights push Nepal and Bhutan to the west, causing earthquakes. Measurements are highly required to confirm this westward trend, as it is shockingly obvious.

As tectonic plates have different geometric forms and dimensions, their speeds are improbable not to differ. Therefore two cases are to be expected at a north-south juncture of two plates: the western plate's speed is (i) greater, causing a rift (and likely volcanoes), or (ii) smaller, causing a collision with its neighbor (involving mountains uplift/rise, volcanoes, and earthquakes). The case (i) is well exemplified at The Great African Rift (juncture of African and Arabic plates), at the rift valley joining the Mid-Atlantic Ridge from Iceland to the Romanche Trench, while the case (ii) at Himalayas (the Indian and Eurasian plates juncture), at Andes (South America and Nazca plates juncture).

In Figure 5 (a), W and E are some arbitrary points in a west-east line, on the banks of a south-north rift valley or a rift tranche, and A, B, C, ... are other several marked points, within the rift valley/tranche, situated in the vertical plane containing the line WE. It is known that the distance between W and E increases in time from its initial value $\overline{WE}$ to a value $\overline{WE}+\varepsilon$. The purpose of this experiment is to see whether the increment ε is produced by different westward speeds of both banks, not by motion apart from each other (as now thought by the scientific community).

According to the presently accepted Hess' *Sea-floor spread* theory, the increment ε is the ***sum*** of the displacements of the two banks: that western to the west, and that eastern to the east, Figure 5 (b), the inner points A, B, C, ... remaining roughly unmoved, provided that no dilatation is involved.

According to the RVD prediction, ε is the ***difference*** between the two displacements, both westward, Figure 5 (c), the inner points A, B, C, ... are pushed westward by the eastern bank to the new positions A', B', C', ... but distances between these points keep approximately constant, provided no contractions happen.

The rift valley/tranche is supposed not to be bound up with any of the two plates, otherwise the scheme is not valid and these measurements should be performed at several places and the conclusion should be drawn statistically.

The westward drift, as well as the object of Experiment Proposed 1, has no competing theory to account for, since it was never observed, hence it is not necessary an accuracy like the one in the case of having to discriminate between theories predicting westward drift differently sized.

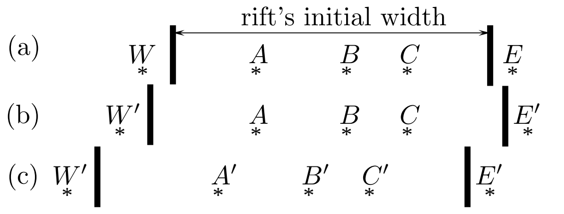

Figure 5: **Topographic distances high precision measurement, repeated**. (a) initial status of a rift's banks; (b) final status of the rift's banks according to H.H. Hess' *Sea-floor spread* theory; (c) final status of the rift's banks according to the RVD of gravitational force herein propounded.

Experiment Proposed 3 ***Weight increase with temperature*** The experiment concerns the *even* RVD term in the law of gravitation. Apparently the weight G (unlike G, the constant) of a mass m in a volume $\square$ *at rest* on Earth is given by the Newton strength of the (earth's) gravitational field, as mg_N, but since at temperatures different from $0\,K$ the micro-particles—atoms or molecules—are in motion, it is the Newton RVD completed strength ($1'$) that applies.

Recall (section 3) that the gravitational field of a spheric mass, as the earth, is equivalent to that of a point mass located at the center, and notate $\langle Q\rangle$ the mean value of a microscopic quantity, thus having

$$\vec G=\int_\square \vec g\, dm_\square=\int_\square \vec g_N\left[\left(1+3\frac{v^2}{c^2}\right)-6g_N\frac{\vec v^3}{c^3}\right]dm_\square=m\vec g_N\left(1+3\frac{\langle v^2\rangle}{c^2}\right), \tag{42}$$

since $\langle\vec v^3\rangle=\vec 0$ because the quantity is odd in $\vec v$ and, statistically, to any particle having $\vec v$ there exists a particle having $-\vec v$, so canceling out. In fact a scale in equilibrium at a temperature is no longer in equilibrium at another temperature of one side.

If the mas m is of an ideal gas whose micro-particles have mass m_μ,

$$N_A\frac{m_\mu\langle v^2\rangle}{2}=\frac{3}{2}k_BT\,,$$

where N_A is the Avogadro number, k_B is the Bolzmann constant, and T is the Kelvin temperature.

An experimenter should be open to five expectancies: Newton weight, mg_N; Newton RVD completed weight, $mg_N(1+3\langle v^2\rangle/c^2)$; relativistic weight (involving an increased mass), $mg_N[1+(1/2)\langle v^2\rangle/c^2]$, since $(1-v^2/c^2)^{-1/2}\approx 1+(1/2)v^2/c^2$; superposition (sum) of the RVD and of the relativistic effects, $mg_N[1+(7/2)\langle v^2\rangle/c^2]$; and a wholly unpredicted result. The experiment should be accurate enough to discriminate between these cases.

We now describe some laboratory experiments, using a ring, disc, or sphere, spinning about the axis of symmetry. Three effects take place:

- increase in weight, caused by the RVD *even* term (since the diametrically opposite contributions of the *odd* term cancel out);
- braking torque, caused by the RVD odd term;
- weight center shift, caused by the RVD odd term.

Corresponding to the three effects, three experiments proposed follow. Horizontal orientation of the spinning axis seams most convenient for all three effects. Major difficulties: extremely weak effect; vacuum; great $\omega R_\ominus$; and security (because of spinning speed near the material resistance limit).

Experiment Proposed 4 ***Weight increase of a spinning ring /disc/sphere*** This experiment is intended as a test for the *even* RVD term in the gravitation law, i.e., a laboratory confirmation of *Perihelion advance*. The weight/gravity denoted G (unlike G, the constant) of a ring, disc, or sphere, denoted $\ominus$, having mass $m_\ominus$ and radius $R_\ominus$, is

$$\vec G=\int_\ominus \vec g\, dm_\ominus=\int_\ominus \vec g_N\left[\left(1+3\frac{v^2}{c^2}\right)-6g_N\frac{\vec v^3}{c^3}\right]dm_\ominus\ , \tag{43}$$

but, as $\vec g_N$ is the same at any point of $\ominus$, and $\vec v^3$ gives a zero integral (as an odd function in a domain having as axis of spinning just its axis of symmetry), one can write scalarly

$$\begin{aligned} G&=m_\ominus g_N+\tfrac{3g_N}{c^2}\textstyle\int_\ominus v^2dm_\ominus=m_\ominus g_N+3g_N\tfrac{\omega^2}{c^2}\textstyle\int_\ominus\rho^2dm_\ominus\\ &=m_\ominus g_N+3g_N J^{(2)}_\ominus\tfrac{\omega^2}{c^2}\,, \end{aligned} \tag{44}$$

where $J^{(2)}_\ominus$ is the moment of inertia of $\ominus$ with respect to its spinning axis, namely: $m_\ominus R^2_\ominus$ for ring; $(1/2)m_\ominus R^2_\ominus$ for disc; and $(2/5)m_\ominus R^2_\ominus$ for sphere. Thus from (44) the increase in weight, ΔG, is

$$\Delta G\equiv G-m_\ominus g_N=3km_\ominus g_N\left(\frac{\omega R_\ominus}{c}\right)^2, \tag{45}$$

where k is 1 for ring, 1/2 for disc, and 2/5 for sphere.

Performance of this kind of experiment can be expected from the superconductors domain. It would be a reference laboratory experiment for the RVD even term, $3v^2/c^2$, as now there exists one evidence only, the *Perihelion advance*, while for the odd term, $-6\vec v^3/c^3$, there are many (but not laboratory) evidences.

Experiment Proposed 5 ***Braking torque on a spinning ring /disc/sphere*** In a gravitational field, an initial ω_0 decreases in time, regardless of the spinning axis orientation. The braking torque exerted by the gravitational field upon a rotating body on the earth is

$$\begin{aligned} \vec\tau_\ominus&=\textstyle\int_\ominus d\vec\tau_\ominus=\int_\ominus\vec\rho_\ominus\times d\vec F=\int_\ominus\vec\rho_\ominus\times\vec g\,dm_\ominus\\ &=\textstyle\int_\ominus\vec\rho_\ominus\times\left[\vec g_N\left(1+3\frac{v^2}{c^2}\right)-6g_N\frac{\vec v^3}{c^3}\right]dm_\ominus\\ &=\vec 0\,-\,6\tfrac{g_N}{c^3}\textstyle\int_\ominus\vec\rho_\ominus\times\vec v^3dm_\ominus=-6\frac{g_N}{c^3}\int_\ominus v^2\vec\rho_\ominus\times\vec v\,dm_\ominus\\ &=-6\tfrac{g_N}{c^3}\textstyle\int_\ominus(\omega\rho_\ominus)^2\vec\rho_\ominus\times(\vec\omega\times\vec\rho_\ominus)\,dm_\ominus\\ &=-6g_N\tfrac{\vec\omega^3}{c^3}\textstyle\int_\ominus\rho^4_\ominus\,dm_\ominus=-6g_N\frac{\vec\omega^3}{c^3}J^{(4)}_\ominus, \end{aligned} \tag{46}$$

where $\vec 0$ is the value of the torque produced by the Newton and the RVD *even* terms; assuming the mass uniformly distributed, $J^{(4)}_\ominus=k_4m_\ominus R^4_\ominus$ where k_4 is 1 for ring, 1/3 for disc, and 8/35 for sphere.

To see how ω decreases in time, use the dynamic equation (for rotation), assuming no friction, $J^{(2)}_\ominus\dot{\vec\omega}=\vec\tau_\ominus$, where $J^{(2)}_\ominus=k_2m_\ominus R^2_\ominus$, where k_2 is 1 for ring, 1/ 2 for disc, and 2 /5 for sphere, hence

$$\begin{aligned} &\dot{\vec\omega}=\tfrac{\vec\tau_\ominus}{J^{(2)}_\ominus}=-6kg_N\tfrac{\vec\omega^3R^2_\ominus}{c^3},\\ &k=k_4/k_2=1;\ 2/3;\ 4/7\ \texttt{(ring;disc;sphere)}, \end{aligned} \tag{47}$$

whence (as $\vec 1_\omega$ is constant) we have scalarly $\dot\omega/\omega^3-6kg_NR^2_\ominus/c^3$, $-1/(2\omega^2)=-(6kg_NR^2_\ominus/c^3)\,t-1/(2\omega_0^2)$,

$$\frac{\omega}{\omega_0}=\frac{\nu}{\nu_0}=\left[1+12kg_N\frac{\omega_0^2R^2_\ominus}{c^3}t\right]^{-1/2},\ \ \frac{\Delta\nu}{\nu_0}\approx-6kg_N\frac{\omega_0^2R^2_\ominus}{c^3}\,t\,, \tag{47′}$$

where ν is the number of rotations per unit of time (frequency of rotation, $\omega=2\pi\nu$), and $\Delta\nu=\nu-\nu_0$. Numeric exemplifications: (i) a disc having $R\ominus=1/4$ meters and $\nu_0=120000$ rotations/minute slows down $7.23\times10^{-7}\,\%$ in $t=1$ year; and (ii) a disc of $R\ominus=1$ meter and $\nu_0=60000$ rotations/minute slows down 1% in $t=1382387.6$ years. Hence the gravitational slow down of rotation is far from being reachable by laboratory experiments now and in a near future. We have described it to form an idea only. But the effect is fully detectable at planetary scale—the above-discussed *Secular retardation* and consequences.

Experiment Proposed 6 ***Gravity center shift of a horizontal axis spinning disc*** The experiment concerns the *odd* RVD term in the gravitation law. A disc $\ominus$ of mass $m_\ominus$ and radius $R_\ominus$ is hanged either with a thread, by means of an axle, from a fixed point, or by a light framework from a pivot parallel to the disc's axis. In both cases the assembly is allowed of positioning with the center of gravity in the vertical plane containing the above point or pivot. The velocity of a point mass of the disc is $\vec{v} = \vec{\omega}\times\vec{\rho}$ in usual notations. The odd RVD gravitational forces point upward on one half of the disc, and downward on the other half. Calculate the magnitude $\mathcal{M}$ of this torque (moment of forces): integrate the product of the arm (horizontal projection of $\vec{\rho}$, that is, $\vec{\rho}\,\vec{1}_x = \rho\cos\varphi$), and the vertical projection of the forces, using the equalities $(-6g_N\vec{v}^{\,3}/c^3)\vec{1}_y = -6(g_N/c^3)v^2\vec{v}\,\vec{1}_y$, and $\vec{v}\,\vec{1}_y = (\vec{\omega}\times\vec{\rho})\vec{1}_y = (\vec{1}_y\times\vec{\omega})\vec{\rho} = \omega\vec{1}_x\,\vec{\rho} = \omega\rho\cos\varphi$:

$$\left.\begin{aligned} \mathcal{M} &\equiv \textstyle\int_\ominus(\vec{1}_x\,\vec{\rho})(\vec{1}_y\,d\vec{F}) = \int_\ominus(\vec{1}_x\,\vec{\rho})[\vec{1}_y(-6g_N\frac{\vec{v}^{\,3}}{c^3}dm_\ominus)] \\ &= -6\tfrac{g_N}{c^3}\textstyle\int_\ominus(\vec{1}_x\,\vec{\rho})[v^2\vec{1}_y\,\vec{v}\,)\,dm_\ominus \\ &= -6g_N\tfrac{\omega^3}{c^3}\textstyle\int_\ominus\rho^4\cos^2\varphi\,dm_\ominus = -m_\ominus g_N\left(\frac{\omega R_\ominus}{c}\right)^3 R_\ominus\,. \end{aligned}\right| \tag{48}$$

The gravity center of the whole device shifts from its initial position C to C', a horizontal distance ε (see Figure 5) such that to balance the torque $\mathcal{M}$, that is, $m_{device}g\varepsilon = \mathcal{M}$, where m_{device} is the mass of the whole suspended device including the spinning disc, whence

$$\varepsilon = \frac{m_\ominus}{m_{device}}\frac{g_N}{g}\left(\frac{\omega R_\ominus}{c}\right)^3 R_\ominus \approx \frac{m_\ominus}{m_{device}}\left(\frac{\omega R_\ominus}{c}\right)^3 R_\ominus\,. \tag{49}$$

Note the increase of the effect with ω^3 and $R_\circ^4$. The size of the effect to obtain is confined to disc's material's resistance to the centrifugal force.

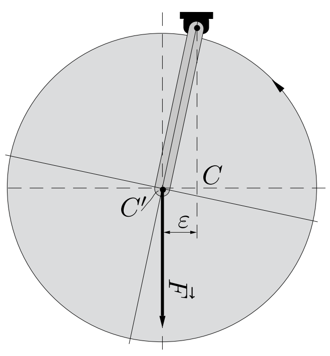

Figure 6: **Horizontal axis disc, spinning and free to pivot in the plane of rotation** The RVD odd gravitational forces point downward on the right half of the disc, and upward on the left half, thus generating a torque (moment of force) $\vec{\mathcal{M}}$ counteracted by an ε shift of the gravity center (from C to C'), to be measured directly by interferometry. $\vec{F}$ is the weight of the suspended device including the disc. Suspending such a device with a long thread, and measuring the angle, may also be a version.

Since there is no competing theory to account for this effect, any performance of the experiment in which the effect exceeds the experimental errors, is relevant; and even if the effect is systematically repetitive, irrespective of accuracy; for the latter case, one might first try working without vacuum. This experiment would meant a laboratory confirmation of tectonic plates westward drift and subsequent effects (mountain rise, earth quakes, volcanoes, etc).

Experiment Proposed 7 ***Muons' anisotropic vertical travel*** On comparing the numbers of muons recorded by two detectors, one above and one beneath an isotropic source, a much smaller number is expected to detect the above detector, because they encounter a stronger gravitational field, according to the fourth observation in the end of section 2.

Discussions

Observations and comments are scattered throughout the article, especially in the end of sections.

Unlike Experiments Proposed 1 and 2—outdoor, and dealing with planetary effects—the other are at or beyond the edge of today technology, challenging the most advanced experimenters and laboratories.

The odd term in the gravitational law RVD completed has no contribution to perihelion advance, and the even term has no contribution to the work done per revolution—though it does produce a torque, but it is zero on the average (zero integral over a revolution).

The only evidence for the *even* term in the RVD completion of Newton's law of gravitation is *Perihelion advance*, but this does not require just the term $3v^2/c^2$, since $3\vec{v}_\perp^{\,2}/c^2$, where $\vec{v}_\perp$ is the component of $\vec{v}$ perpendicular to the field's direction, leads to the same formula (9), but does not posses the Newton spheric properties (the subject of section 3).

Unlike the case of the *even* RVD term, for which we benefited from the exact formula of *Perihelion advance* to choose the coefficient 3, the $2.9 - 4.5\times10^{19}$ *joules/yr* energetic interval from secular retardation that guided us in choosing the coefficient 6 for the *odd* RVD term in Eq. (1) allows of the choice 7, as well (keeping choice within the integers set), i.e., there is an incertitude to be removed by measurement.

It is quite impressive how geologists, considering waters and sands, have been able to discriminate between the amount of *Secular retardation* energy dissipated by tidal effects and the one, a *surplus* of the same order of magnitude, dissipated by an "unknown mechanism"—unaccounted for by any geophysical processes, in reality, as we now know, the process of *Tectonic plates drift*— thus contributing to advance the gravitation science. And it is equally impressive the dramatic fight between the obvious *Continental drift*, with its hero, Wegener, and the lack of the necessary "driving force". Now we have finally shown that the *surplus* in one problem (Secular retardation) is just the *deficit* in the other (Continental drift).

An interesting question is how GTR can be adapted to incorporate the property of RVD, especially the *odd* term—proportional to $(\vec{v}/c)^3$. In constructing GTR—a geometric view of gravitation—Einstein started from Newton's law of gravity and, briefly, now it should be "reset" and "restarted" from Newton's law RVD completed.

Appendix

Passing from orthogonal Cartesian coordinates (x, y) with $(\vec{1}_x,\vec{1}_y)$ base of (unit) vectors to polar coordinates $(\,\rho,\varphi)$ with base $(\vec{1}_\rho,\vec{1}_\varphi)$, we have $\vec{1}_\rho = \vec{1}_x\cos\varphi + \vec{1}_y\sin\varphi$, $\quad\vec{1}_\varphi = -\vec{1}_x\sin\varphi + \vec{1}_y\cos\varphi$, $\quad\dot{\vec{1}}_\rho = \vec{1}_\varphi\,\dot\varphi$, $\quad\dot{\vec{1}}_\varphi = -\vec{1}_\rho\dot\varphi$, $\quad\ddot{\vec{1}}_\rho = -\vec{1}_\rho\dot\varphi^2 + \vec{1}_\varphi\ddot\varphi$, $\quad\ddot{\vec{1}}_\varphi = -\vec{1}_\rho\ddot\varphi - \vec{1}_\varphi\dot\varphi^2$, and

$$\left.\begin{aligned} &\vec{\rho} = \vec{1}_\rho\,\rho\,, \quad \vec{v} \equiv \dot{\vec{\rho}} = \vec{1}_\rho\,\dot\rho + \vec{1}_\varphi\,\rho\,\dot\varphi\,, \\ &\vec{a} \equiv \ddot{\vec{\rho}} = \vec{1}_\rho\,(\,\ddot\rho - \rho\,\dot\varphi^2) + \vec{1}_\varphi\,(\,2\dot\rho\,\dot\varphi + \rho\,\ddot\varphi)\,. \end{aligned}\right\} \tag{50}$$

Let $\circ$ be a spheric solid of radius $R_\circ$, containing a mass $M_\circ$ distributed with density $\mu_\circ(\vec{r}_\circ) = \mu_\circ(r_\circ)$. Concerning polar and axial moments of inertia $I_\circ^{(n)}$ and $J_\circ^{(n)}$ respectively, defined as

$$I_\circ^{(n)} \equiv \int_\circ \mu_\circ(r_\circ)r^n dV_\circ\,,\ J_\circ^{(n)} \equiv \int_\circ \mu_\circ(r_\circ)\rho^n dV_\circ\,,\quad (n{=}0,1,2,...), \tag{51}$$

note successively: $I_\circ^{(0)}{=}J_\circ^{(0)}{=}M$; $I_\circ^{(2)}$ and $J_\circ^{(2)}$ are the usual (second order) polar and axial moments of inertia;

$$\left.\begin{aligned} I_\circ^{(n)} &= \textstyle\int_0^{R_\circ}\mu_\circ(r_\circ)\,r_\circ^{\,n+2}dr_\circ\int_0^\pi\sin\theta\,d\theta\int_0^{2\pi}d\varphi \\ &= 4\pi\textstyle\int_0^{R_\circ}\mu_\circ(r_\circ)\,r_\circ^{\,n+2}dr_\circ\,, \qquad (n = 0,1,2,...)\,, \\ J_\circ^{(n)} &= 2\pi\textstyle\int_0^{R_\circ}\mu_\circ(r_\circ)\,r_\circ^{\,n+2}dr_\circ\int_0^\pi\sin^{n+1}\theta\,d\theta\,, \\ J_\circ^{(2n)} &= \tfrac{2^n n!}{(2n+1)!!}I_\circ^{(2n)}, \quad J_\circ^{(2n+1)} = \tfrac{(2n+1)!!}{2^{n+2}(n+1)!}\pi I_\circ^{(2n+1)}, \end{aligned}\right\} \tag{52}$$

and, when the mass is homogeneously distributed in $\circ$,

$$I_\circ^{(n)} = \frac{3}{n+3}M_\circ R_\circ^n, \qquad \texttt{if}\ \ \mu_\circ(r_\circ) = \mu_\circ = \texttt{constant}\,. \tag{53}$$

Theorem 1 *If a function $\mu_\circ(r_\circ)$, specifically a mass density, is spherically even, that is, $\mu_\circ(\vec{r}_\circ) = \mu(-\vec{r}_\circ)$, in particular $\mu_\circ(\vec{r}_\circ) = \mu_\circ(r_\circ)$, in a spheric solid $\circ$ of radius $R_\circ$ and total mass $M_\circ$, and an either vector or scalar function $f(\vec{r}_\circ)$ is spherically odd, that is,*

$f(-\vec{r}_\circ) = -f(\vec{r}_\bullet)$, and the vectors $\vec{a}_1, \vec{a}_2, \cdots$ and $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$, are independent on $\vec{r}_\circ$, then:

$$\left.\begin{aligned} &\int_\circ f(\vec{r}_\circ)\,dV_\circ = 0\,,\ \texttt{in particular}\\ &\int_\circ \mu(r_\circ)(\vec{a}_1\vec{r}_\circ)(\vec{a}_2\vec{r}_\circ)\cdots(\vec{a}_{2n}\vec{r}_\circ)\vec{r}_\circ\,dV_\circ = \vec{0}\,,\\ &\int_\circ \mu(r_\circ)(\vec{a}_1\vec{r}_\circ)(\vec{a}_2\vec{r}_\circ)\cdots(\vec{a}_{2n}\vec{r}_\circ)(\vec{a}_{2n+1}\vec{r}_\circ)\,dV_\circ = 0\,; \end{aligned}\right\} \tag{54}$$

$$\int_\circ \mu(r_\circ)(\vec{a}\,\vec{r}_\circ)\vec{r}_\circ\,dV_\circ = \frac{1}{3}I_\circ^{(2)}\vec{a}\,,\quad \int_\circ \mu(r_\circ)(\vec{a}\,\vec{r}_\circ)(\vec{b}\,\vec{r}_\circ)\,dV_\circ = \frac{1}{3}I_\circ^{(2)}\vec{a}\,\vec{b}\,; \tag{55}$$

$$\left.\begin{aligned} &\int_\circ \mu(r_\circ)(\vec{a}\,\vec{r}_\circ)(\vec{b}\,\vec{r}_\circ)(\vec{c}\,\vec{r}_\circ)\vec{r}_\circ\,dV_\circ\\ &\qquad = \tfrac{1}{15}I_\circ^{(4)}[\vec{a}\,(\vec{b}\,\vec{c}) + \vec{b}\,(\vec{a}\,\vec{c}) + \vec{c}\,(\vec{a}\,\vec{b})]\,,\\ &\int_\circ \mu(r_\circ)(\vec{a}\,\vec{r}_\circ)(\vec{b}\,\vec{r}_\circ)(\vec{c}\,\vec{r}_\circ)(\vec{d}\,\vec{r}_\circ)dV_\circ\\ &\qquad = \tfrac{1}{15}I_\circ^{(4)}[(\vec{a}\,\vec{b})(\vec{c}\,\vec{d}) + (\vec{a}\,\vec{c})(\vec{b}\,\vec{d}) + (\vec{a}\,\vec{d})(\vec{b}\,\vec{c})]; \end{aligned}\right\} \tag{56}$$

$I_\circ^{(n)}$ are defined and expressed by Eqs. (51) and (53).

Remark 1 One can find (rather tediously) the generalizations for any $n \ge 1$,

$$\left.\begin{aligned} &\int_\circ \mu_\circ(r_\circ)(\vec{a}_1\vec{r}_\circ)(\vec{a}_2\vec{r}_\circ)...(\vec{a}_{2n-1}\vec{r}_\circ)\vec{r}_\circ dV_\circ\\ &= \tfrac{I_\circ^{(2n)}}{(2n+1)!!}\sum_{\substack{\imath,\,p,\,q=1\\ \jmath_q\ne \jmath_p\ne\imath}}^{2n-1}\vec{a}_\imath\prod_{\jmath=1}^{2n-3}(\vec{a}_\jmath\,\vec{a}_{\jmath+1})\int_\circ \mu_\circ(r_\circ)(\vec{a}_1\vec{r}_\circ)(\vec{a}_2\vec{r}_\circ)...(\vec{a}_{2n}\vec{r}_\circ)dV_\circ\\ &\qquad\qquad = \tfrac{I_\circ^{(2n)}}{(2n+1)!!}\sum_{\substack{p,\,q=1\\ \jmath_q\ne \jmath_p}}^{2n-1}\prod_{\jmath=1}^{2n-1}(\vec{a}_\jmath\,\vec{a}_{\jmath+1})\,, \end{aligned}\right| \tag{57}$$

but these formulas become inoperative for $n>3$ because of generating too numerous terms, $(2n–1)!!$ (that is, 105 for $n=4$).
Remark 2 Obviously, if the arbitrary $\mu_\circ(r_\circ)$ is replaced by $\mu_\circ(r_\circ)r_\circ^{\,p}$, then $I_\circ^{(2n)}/(2n+1)!!$ in Eqs. (57) is replaced by $I_\circ^{(2n+p)}/(2n+1)!!$ (this case is frequently encountered in the above); and if $\mu_\circ(r_\circ)$ is replaced by $\mu_\circ(r_\circ)\,\varrho(r_\circ)$ where ϱ is an arbitrary function of one variable (a case not encountered herein), then $I_\circ^{(2n)}/(2n+1)!!$ in Eqs. (57) is multiplied by the ratio defined as

$$\mathcal{R}_\circ(\,\varrho\,;2n) \equiv \int_0^{R_\circ}\mu_\circ(r_\circ)\,\varrho(r_\circ)\,r_\circ^{\,2n+2}dr_\circ \Big/ \int_0^{R_\circ}\mu_\circ(r_\circ)\,r_\circ^{\,2n+2}dr_\circ\,. \tag{58}$$

Proof of Theorem 1 The first formula (54) is a simple notice: for every value $f(\vec{r}\,)$ there exists a value $f(-\vec{r}\,) = -f(\vec{r}\,)$, so that they cancel out in the process of integration. Of the formulas (55) we prove the second one only, since the proof for the first is practically identical, one having only to replace the expression $\vec{b}\vec{r} = a_x x + a_y y + a_z z$ with the analogous expression $\vec{r} = \vec{1}_x x + \vec{1}_y y + \vec{1}_z z$, formally replace a with $\vec{1}$ (this is an advantage of the notation $(\vec{1}_x, \vec{1}_y, \vec{1}_z)$ for vector base, instead of $(\vec{\imath}, \vec{\jmath}, \vec{k})$). The observation is also valid for the two formulas (56) because of which we shall demonstrate the first one only. Note that in the following succession of steps (equalities), the third is made by omitting the terms whose integrals with respect to φ and θ are zero, using the following filter: there are nonzero only $\int_0^{2\pi}\sin^{2m}\varphi\cos^{2n}\varphi\,d\varphi$ and $\int_0^{\pi}\sin^{m}\theta\cos^{2n}\theta\,d\theta$, with m and n arbitrary integers. All integrals with respect to φ and θ occurring in the proof are gathered at Eqs. (59).

$$\left.\begin{aligned} &\int_\circ \mu_\circ(r)(\vec{a}\,\vec{r}\,)(\vec{b}\,\vec{r}\,)\,dV_\circ\\ &= \int_\circ \mu_\circ(r)(a_x x + a_y y + a_z z)(c_x x + c_y y + c_z z)\,dV_\circ\\ &= \int_0^{R_o} r^4 dr\int_0^{2\pi}\!\!\int_0^{\pi}(a_x\sin\theta\cos\varphi + a_y\sin\theta\sin\varphi + a_z\cos\theta)\\ &\qquad(b_x\sin\theta\cos\varphi + b_y\sin\theta\sin\varphi + b_z\cos\theta)\sin\theta\,d\theta\,d\varphi\\ &= I_\circ^{(2)}\int_0^{2\pi}\!\!\int_0^{\pi}(a_x b_x\sin^3\theta\cos^2\varphi + a_y b_y\sin^3\theta\sin^2\varphi\\ &\qquad\qquad + a_z b_z\sin\theta\cos^2\theta)\,d\theta\,d\varphi =\\ &= (4\pi/3)\,I_\circ^{(2)}(a_x b_x + a_y b_y + a_z b_z) \;=\; (4\pi/3)\,I_\circ^{(2)}\vec{a}\vec{b}. \end{aligned}\right|$$

Q.E.D.

Now we prove the first formula (56) by a sequence of eight equalities, as steps, (60); each sign of equality is marked with its order number, as $\stackrel{n}{=}$, to refer to and find them easily. Between $\stackrel{1}{=}$ and $\stackrel{2}{=}$, the quantities $\vec{a}\,\vec{r}$, $\vec{b}\,\vec{r}$, $\vec{c}\,\vec{r}$ and $\vec{r}$ are expressed by their components in orthogonal Cartesian coordinates (x, y, z), while between $\stackrel{2}{=}$ and $\stackrel{3}{=}$ the components of $\vec{r}$ are expressed in spherical coordinates (r, φ, θ), convenient for integration over a spherical domain; the integral with respect to r is denoted as in (51). Between $\stackrel{3}{=}$ and $\stackrel{4}{=}$ the product of the first two pair of parentheses is worked out, resulting nine terms in a single pair of parentheses, and likewise with the other two pairs of parentheses (also obtaining nine terms), and the new two pairs of parentheses are worked out between $\stackrel{4}{=}$ and $\stackrel{5}{=}$, but not all the 81 resulting terms are transcribed, but only 27, whose integrals with respect to φ and θ are nonzero, using the filter already specified and used; the integrals of the three underlined terms are equal to $4\pi/5$, while the integrals of all the other terms are $4\pi/15$—which we take as common factor for all terms between $\stackrel{5}{=}$ and $\stackrel{6}{=}$. Grouping separately the simple, double, and triple underlined terms, then grouping the results, one obtains $\vec{c}\,(\vec{a}\vec{b}\,)$ written between $\stackrel{6}{=}$ and $\stackrel{7}{=}$ together with the remaining terms. Here again group alike underlined terms obtaining $\vec{a}\,(\vec{b}\,\vec{c}\,)$ written between $\stackrel{7}{=}$ and $\stackrel{8}{=}$ where grouping the simple, double, and not underlined terms, the proof ends. Q.E.D.
Note Theorem 1 does not contain the generalization of formulas (56) for an arbitrary even number of products in the integrand, because the occurring integrals with respect to φ and θ are available as recurrence rules only. In fact, they would not be very useful, because the calculations of RVD forces and torques become so tedious that it becomes more convenient to calculate exactly, working with elliptic integrals, instead of approximating by expending the denominators in series thus appearing such products.

Trigonometric integrals frequently used in (60):

$$\left.\begin{aligned} &\int_0^{2\pi}\sin^{2n}\varphi\,d\varphi = \int_0^{2\pi}\cos^{2n}\varphi\,d\varphi = \tfrac{(2n-1)!!}{2^n n!}2\pi\,,\ (n=1,2,3,...)\,;\\ &\int_0^{2\pi}\sin^{2n+1}\varphi\,d\varphi = \int_0^{2\pi}\cos^{2n+1}\varphi\,d\varphi = 0\,,\ \int_0^{\pi}\cos^{2n+1}\theta\,d\theta = 0\,,\\ &\int_0^{\pi}\sin^{2n}\theta\,d\theta = \int_0^{\pi}\cos^{2n}\theta\,d\theta = \tfrac{(2n-1)!!}{2^n n!}\pi\,,\ \int_0^{\pi}\sin^{2n+1}\theta\,d\theta = \tfrac{2^{n+1}n!}{(2n+1)!!}\,,\\ &(n=0,1,2,3,...)\,;\\ &\int_0^{2\pi}\sin^2\varphi\cos^2\varphi\,d\varphi = \tfrac{\pi}{4},\ \int_0^{\pi}\sin\theta\cos^2\theta\,d\theta = \tfrac{2}{3},\ \int_0^{\pi}\sin^3\theta\cos^2\theta\,d\theta = \tfrac{4}{15}. \end{aligned}\right\} \tag{59}$$

$$
\begin{aligned}
&\int_\circ \mu_\circ(r)(\vec{a}\,\vec{r}\,)(\vec{b}\,\vec{r}\,)(\vec{c}\,\vec{r}\,)\,\vec{r}\,dV_\circ \\
&\overset{1}{=} \int_\circ \mu_\circ(r)(a_x x + a_y y + a_z z)(b_x x + b_y y + b_z z)(c_x x + c_y y + c_z z)(\vec{1}_x x + \vec{1}_y y + \vec{1}_z z)dV_\circ \\
&\overset{2}{=} \int_0^{R_\circ}\mu_\circ(r)r^6 dr \int_0^\pi\int_0^{2\pi}(a_x\sin\theta\cos\varphi + a_y\sin\theta\sin\varphi + a_z\cos\theta)(b_x\sin\theta\cos\varphi + b_y\sin\theta\sin\varphi + b_z\cos\theta) \\
&\qquad (c_x\sin\theta\cos\varphi + c_y\sin\theta\sin\varphi + c_z\cos\theta)(\vec{1}_x\sin\theta\cos\varphi + \vec{1}_y\sin\theta\sin\varphi + \vec{1}_z\cos\theta)\sin\theta\,d\theta\,d\varphi \\
&\overset{3}{=} I_\circ^{(4)}\int_0^\pi\int_0^{2\pi}(a_xb_x\sin^2\theta\cos^2\varphi + a_xb_y\sin^2\theta\sin\varphi\cos\varphi + a_xb_z\sin\theta\cos\theta\cos\varphi + a_zb_x\sin\theta\cos\theta\cos\varphi + a_zb_y\sin\theta\cos\theta\sin\varphi + a_zb_z\cos^2\theta) \\
&\qquad (\vec{1}_xc_x\sin^2\theta\cos^2\varphi + \vec{1}_xc_y\sin^2\theta\sin\varphi\cos\varphi + \vec{1}_xc_z\sin\theta\cos\theta\cos\varphi + \vec{1}_yc_x\sin^2\theta\sin\varphi\cos\varphi + \vec{1}_yc_y\sin^2\theta\sin^2\varphi \\
&\qquad + \vec{1}_yc_z\sin\theta\cos\theta\sin\varphi + \vec{1}_zc_x\sin\theta\cos\theta\cos\varphi + \vec{1}_zc_y\sin\theta\cos\theta\sin\varphi + \vec{1}_zc_z\cos^2\theta)\sin\theta\,d\theta\,d\varphi \\
&\overset{4}{=} I_\circ^{(4)}\int_0^\pi\int_0^{2\pi}(\vec{1}_xa_xb_xc_x\underline{\sin^5\theta\cos^4\varphi} + \vec{1}_ya_xb_xc_y\sin^5\theta\sin^2\varphi\cos^2\varphi + \vec{1}_za_xb_xc_z\sin^3\theta\cos^2\theta\cos^2\varphi \\
&\qquad + \vec{1}_xa_xb_yc_y\sin^5\theta\sin^2\varphi\cos^2\varphi + \vec{1}_ya_xb_yc_x\sin^5\theta\sin^2\varphi\cos^2\varphi + \vec{1}_xa_xb_zc_z\sin^3\theta\cos^2\theta\cos^2\varphi + \vec{1}_za_xb_zc_xz\sin^3\theta\cos^2\theta\cos^2\varphi \\
&\qquad + \vec{1}_xa_yb_xc_y\sin^5\theta\sin^2\varphi\cos^2\varphi + \vec{1}_ya_yb_xc_x\sin^5\theta\sin^2\varphi\cos^2\varphi + \vec{1}_xa_yb_yc_x\sin^5\theta\sin^2\varphi\cos^2\varphi + \vec{1}_ya_yb_yc_y\underline{\sin^5\theta\sin^4\varphi} \\
&\qquad + \vec{1}_za_yb_yc_z\sin^3\theta\cos^2\theta\sin^2\varphi + \vec{1}_ya_yb_zc_z\sin^3\theta\cos^2\theta\sin^2\varphi + \vec{1}_za_yb_zc_y\sin^3\theta\cos^2\theta\sin^2\varphi \\
&\qquad + \vec{1}_xa_zb_xc_z\sin^3\theta\cos^2\theta\cos^2\varphi + \vec{1}_za_zb_xc_x\sin^3\theta\cos^2\theta\cos^2\varphi + \vec{1}_ya_zb_yc_z\sin^3\theta\cos^2\theta\sin^2\varphi + \vec{1}_za_zb_yc_y\sin^3\theta\cos^2\theta\sin^2\varphi \\
&\qquad + \vec{1}_xa_zb_zc_x\sin^3\theta\cos^2\theta\cos^2\varphi + \vec{1}_ya_zb_zc_y\sin^3\theta\cos^2\theta\sin^2\varphi + \vec{1}_za_zb_zc_z\underline{\sin\theta\cos^4\theta})\,d\theta\,d\varphi \\
&\overset{5}{=} I_\circ^{(4)}(4\pi/15)(3\underline{\vec{1}_xa_xb_xc_x} + \underline{\vec{1}_ya_xb_xc_y} + \underline{\vec{1}_za_xb_xc_z} + \vec{1}_xa_xb_yc_y + \vec{1}_ya_xb_yc_x + \vec{1}_xa_xb_zc_z + \vec{1}_za_xb_zc_x + \vec{1}_xa_yb_xc_y + \vec{1}_ya_yb_xc_x + \underline{\underline{\vec{1}_xa_yb_yc_x}} \\
&\qquad + 3\underline{\underline{\vec{1}_ya_yb_yc_y}} + \underline{\underline{\vec{1}_za_yb_yc_z}} + \vec{1}_ya_yb_zc_z + \vec{1}_za_yb_zc_y + \vec{1}_xa_zb_xc_z + \vec{1}_za_zb_xc_x + \vec{1}_ya_zb_yc_z + \vec{1}_za_zb_yc_y + \underline{\underline{\underline{\vec{1}_xa_zb_zc_x}}} + \underline{\underline{\underline{\vec{1}_ya_zb_zc_y}}} + 3\underline{\underline{\underline{\vec{1}_za_zb_zc_z}}}) \\
&\overset{6}{=} I_\circ^{(4)}(4\pi/15)[\vec{c}(\vec{a}\vec{b}) + 2\underline{\vec{1}_xa_xb_xc_x} + \underline{\vec{1}_xa_xb_yc_y} + \vec{1}_ya_xb_yc_x + \underline{\vec{1}_xa_xb_zc_z} \\
&\qquad + \vec{1}_za_xb_zc_x + \vec{1}_xa_yb_xc_y + \underline{\underline{\vec{1}_ya_yb_xc_x}} + 2\underline{\underline{\vec{1}_ya_yb_yc_y}} + \underline{\underline{\vec{1}_ya_yb_zc_z}} + \vec{1}_za_yb_zc_y + \vec{1}_xa_zb_xc_z + \underline{\underline{\underline{\vec{1}_za_zb_xc_x}}} + \vec{1}_ya_zb_yc_z + \underline{\underline{\underline{\vec{1}_za_zb_yc_y}}} + 2\underline{\underline{\underline{\vec{1}_za_zb_zc_z}}}] \\
&\overset{7}{=} I_\circ^{(4)}(4\pi/15)[\vec{c}(\vec{a}\vec{b}) + \vec{a}(\vec{b}\vec{c}) + \underline{\vec{1}_xa_xb_xc_x} + \underline{\vec{1}_ya_xb_yc_x} + \underline{\vec{1}_za_xb_zc_x} + \underline{\underline{\vec{1}_xa_yb_xc_y}} + \underline{\underline{\vec{1}_ya_yb_yc_y}} + \underline{\underline{\vec{1}_za_yb_zc_y}} + \vec{1}_xa_zb_xc_z + \vec{1}_ya_zb_yc_z + \vec{1}_za_zb_zc_z] \\
&\overset{8}{=} I_\circ^{(4)}(4\pi/15)[\vec{c}(\vec{a}\vec{b}) + \vec{a}(\vec{b}\vec{c}) + \vec{b}(\vec{a}\vec{c})]\,.
\end{aligned}
\tag{60}
$$

Q.E.D.

Theorem 2 *Given a scalar function of one vector variable, $f(\vec{r}_\circ)$, in a spheric solid domain $\circ$ whose center is the initial point of $\vec{r}_\circ$, if the only odd dependency of f on $\vec{r}_\circ$ is of the form $\vec{k}\vec{r}_\circ$, where $\vec{k}$ is a vector independent on $\vec{r}_\circ$, then*

$$\int_\circ f(\vec{r}_\circ)\vec{r}_\circ dV_\circ = K\vec{k}\,, \tag{61}$$

where K is a scalar independent on $\vec{r}_\circ$.

Proof In words, the integral (61) is a vector along $\vec{k}$. For proof, bring $\vec{k}$ with its terminal point at the center of $\circ$, like the variable vector $\vec{r}_\circ$, and consider the equatorial plane of $\circ$ having $\vec{k}$ as the normal, thus separating $\circ$ in two halves (calottes). The product $\vec{k}\vec{r}_\circ$ is positive in one calotte, and negative in the other, and the values in each calotte are symmetric with respect to $\vec{k}$, and so is the function $f(\vec{r}_\circ)$, thus the integral (the resultant) over each calotte and over the entire $\circ$.

Example 1 One of the simplest sample of function f is $f(\vec{r}_\circ) = \mu_\circ(r_\circ)\vec{k}\vec{r}_\circ$, where $\mu_\circ$ may be the density of mass distributed in $\circ$, for which the integral in the theorem is

$$\int_\circ \mu(r_\circ)(\vec{k}\vec{r}_\circ)\vec{r}_\circ dV_\circ = \frac{1}{3}I_\circ^{(2)}\vec{k}\,,$$

according to Theorem 1, first formula (55). Hence the scalar K in the theorem is $(1/3)I_\circ^{(2)}$.

Example 2 Function $f(\vec{r}_\circ) = \mu(r_\circ)r_\circ^2\vec{k}\vec{r}_\circ$ is another simple sample, encountered in the above. Its integral corresponding to the one in Theorem 2 is

$$\int_\circ \mu(r_\circ)r_\circ^2(\vec{k}\vec{r}_\circ)\vec{r}_\circ dV_\circ = \frac{1}{3}I_\circ^{(4)}\vec{k}\,,$$

by Theorem 1, Remark 2. The scalar K is $(1/3)I_\circ^{(4)}$.

Example 3 Demonstrate that the Newton gravitational force of a homogeneous spheric mass, as the sun, produces a zero torque $\vec{\tau}_{\odot\oslash}^{(r_\oslash)}$ on a homogeneous spheric mass, as a planet. This banal physical fact, used above in mathematical contexts, is not quite banal mathematically. In small steps, using notations shown in Figure 3, one can write

$$
\begin{aligned}
\vec{\tau}_{\odot\oslash N}^{(r_\oslash)} &\equiv \int_\oslash \vec{r}_\oslash\times d\vec{F}_{\odot\oslash N} = \int_\oslash \vec{r}_\oslash\times\vec{g}_{\odot\oslash N}\,dM_\oslash = \int_\oslash \mu_\oslash(r_\oslash)\vec{r}_\oslash\times\vec{g}_{\odot\oslash N}\,dV_\oslash = \int_\oslash \mu_\oslash(r_\oslash)\vec{r}_\oslash\times\Big[-G\int_\odot \tfrac{\mu_\odot(r_\odot)\,\vec{R}}{R^3}dV_\odot\Big]dV_\oslash \\
&= \int_\oslash\int_\odot \mu_\oslash(r_\oslash)\,\mu_\odot(r_\odot)\tfrac{\vec{r}_\oslash\times\vec{R}}{R^3}dV_\oslash\,dV_\odot = \int_\oslash\int_\odot \mu_\oslash(r_\oslash)\,\mu_\odot(r_\odot)\tfrac{\vec{r}_\oslash\times(\vec{r}-\vec{r}_\odot)}{[r^2+r_\oslash^2+r_\odot^2-2\vec{r}\vec{r}_\odot+2(\vec{r}-\vec{r}_\odot)\vec{r}_\oslash]^{3/2}}dV_\oslash\,dV_\odot \\
&= -\int_\odot \mu_\odot(r_\odot)(\vec{r}-\vec{r}_\odot)\times\int_\oslash \tfrac{\mu_\oslash(r_\oslash)\,\vec{r}_\oslash}{[r^2+r_\oslash^2+r_\odot^2-2\vec{r}\vec{r}_\odot+2(\vec{r}-\vec{r}_\odot)\vec{r}_\oslash]^{3/2}}dV_\oslash\,dV_\odot = -\int_\odot \mu_\odot(r_\odot)(\vec{r}-\vec{r}_\odot)\times[2K(\vec{r}-\vec{r}_\odot)] = \vec{0}\,,
\end{aligned}
\tag{62}
$$

where K is a scalar independent on $\vec{r}_\oslash$, according to Theorem 2 of which $\vec{k}$ is $2(\vec{r}-\vec{r}_\odot)$. Q.E.D.

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References

[1] Le Verrier, U. (1859) Comptes rendus hebdomadaires des séances de l'Académie des Sciences, **49**, 379–383.

[2] Gerber, P. (1898), Zeitschrift für Mathematik und Physik, **43**, 93—104.

[3] Einstein, A. (1916) Annalen der Physik, **49**, 769—822.

[4] Bertozzi, W. (1964) AmJPh **32** (7), 551–555.

[5] Wegener, A.L. (1912) Petermanns Geographische Mitteilungen **63**, 185–195, 253–256, and 305–309.

[6] Hess, H.H. (1962) Petrologic Studies: a Volume to Honor A.F. Buddington, 599–620.

[7] Melchior, P. (1983) The Tides of the Planet Earth, 438–441, Pergamon Press, Oxford.

[8] Frankel, H. (2012) *The Continental Drift Controversy: Wegener and the Early Debate*, p. xxi. Cambridge University Press.

[9] Hanel, R.A., Conrath, B.J., Kunde, V.G., Pearl, J.C., and Pirraglia J.A. (1983) Icarus **53**, 262–285.

[10] Ingersol, A.P. (1990) Science **248**, 308–315.

[11] Grandpierre, A. (1996) Astrophys. Space Sci. **243**(2), 393–400.

[12] Niroma, T. (1998), http://personal.inet.fi/tiede/tilmari/sunspots.html .

[13] Anderson, J.D, Campbell, J.K., Ekelund, J.E., Ellis, J., and Jordan, J.F. (2008) Phys. Rev. Lett, 100, 091102-1–091102-4.

[14] Schirber, M. (2015), https://www.livescience.com/6887-march-earthquake-month-shaky-facts.html .

[15] Kansas Geological Survey (2017), http://kmuw.org/post/qa-geophysicist-kansas-earthquakes .