Schur-type theorems for k-triangular lattice group-valued set functions with respect to filter convergence

A. Boccuto * X. Dimitriou†

Abstract

We prove some Schur and limit theorems for lattice group-valued k-triangular set functions with respect to filter convergence, by means of sliding hump-type techniques. As consequences, we deduce some Vitali-Hahn-Saks and Nikodým-type theorems.

In the literature there have been many studies about limit theorems for set functions, with values in abstract structures. Here we deal with some versions of these kinds of theorems for k-triangular set functions with values in a lattice group R.

The *M-measures* are monotone set functions m with $m(\emptyset) = 0$, continuous from above and from below and compatible with respect to finite suprema and infima, which are a particular class of 1-triangular set functions and have several applications, for example to intuitionistic fuzzy events and observables. Observe that there are 1-triangular set functions which are not necessarily monotone, for instance the measuroids. We extend earlier results to the setting of k-triangular set functions and filter convergence, by using sliding hump-type techniques. For technical reasons, we use (D)-convergence, since by virtue of the Fremlin Lemma it is possible to replace a sequence of regulators with a single (D)-sequence.

Let Q be a countable set. A *filter* $\mathcal{F}$ of Q is a nonempty collection of subsets of Q with $\emptyset \notin \mathcal{F}$, $A \cap B \in \mathcal{F}$ whenever A, $B \in \mathcal{F}$, and such that $B \in \mathcal{F}$ whenever $A \in \mathcal{F}$ and $B \supset A$.

A filter of Q is *free* iff it contains the filter $\mathcal{F}_{\rm cofin}$ of all cofinite subsets of Q.

Given a free filter $\mathcal{F}$ of Q, we say that a subset of Q is *$\mathcal{F}$-stationary* iff it has nonempty intersection with every element of $\mathcal{F}$. We denote by $\mathcal{F}^*$ the family of all $\mathcal{F}$-stationary subsets of Q.

If $I \in \mathcal{F}^*$, then the *trace* $\mathcal{F}(I)$ of $\mathcal{F}$ on I is the family $\{F \cap I \colon F \in \mathcal{F}\}$. Note that $\mathcal{F}(I)$ is a filter of I.

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*Dipartimento di Matematica e Informatica, University of Perugia, via Vanvitelli 1, I-06123 Perugia, Italy; e-mail: antonio.boccuto@unipg.it, boccuto@yahoo.it

†Department of Mathematics, University of Athens, Panepistimiopolis, Athens 15784, Greece; email: xenofon11@gmail.com, dxenof@math.uoa.gr

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A free filter $\mathcal{F}$ of Q is said to be *diagonal* iff for every sequence $(A_n)_n$ in $\mathcal{F}$ and each $I \in \mathcal{F}^*$ there exists a set $J \subset I$, $J \in \mathcal{F}^*$ such that $J \setminus A_n$ is finite for any n.

Given an infinite set $I \subset Q$, a *blocking* of I is a countable partition $\{D_h : h \in \mathbb{N}\}$ of I into nonempty finite subsets.

A filter $\mathcal{F}$ of Q is *block-respecting* iff for every $I \in \mathcal{F}^*$ and for each blocking $\{D_h : h \in \mathbb{N}\}$ of I there is a set $J \in \mathcal{F}^*$, $J \subset I$ with $\sharp(J \cap D_h) = 1$ for any h, where $\sharp$ denotes the number of elements of the set into brackets

A Dedekind complete lattice group R is *super Dedekind complete* iff for every nonempty set $A \subset R$, bounded from above, there is a finite or countable subset with the same supremum as A.

A bounded double sequence $(a_{t,l})_{t,l}$ in R is a (D)*-sequence* or a *regulator* iff $(a_{t,l})_l$ is a decreasing sequence and $\bigwedge_l a_{t,l} = 0$ for any $t \in \mathbb{N}$.

A lattice group R is *weakly σ-distributive* iff $\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \Big) = 0$ for every (D)-sequence $(a_{t,l})_{t,l}$.

A sequence $(x_n)_n$ in R is (D)*-convergent* to x iff there is a (D)-sequence $(a_{t,l})_{t,l}$ in R such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $n^* \in \mathbb{N}$ with $|x_n - x| \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$ whenever $n \ge n^*$, and we write $(D) \lim_n x_n = x$.

Some examples of super Dedekind complete and weakly σ-distributive lattice groups are the space $\mathbb{N}^{\mathbb{N}}$ of all permutations of $\mathbb{N}$ endowed with the usual componentwise order and the space $L^0(X, \mathcal{B}, \nu)$ of all ν-measurable functions defined on a measure space $(X, \mathcal{B}, \nu)$ with the identification up to ν-null sets, where ν is a positive, σ-additive and σ-finite extended real-valued measure, endowed with almost everywhere convergence.

A sequence $(x_n)_n$ in R $(D\mathcal{F})$*-converges to* $x \in R$ iff there is a (D)-sequence $(\alpha_{t,l})_{t,l}$ with
$\Big\{ n \in \mathbb{N} : |x_n - x| \le \bigvee_{t=1}^{\infty} \alpha_{t,\varphi(t)} \Big\} \in \mathcal{F}$ for all $\varphi \in \mathbb{N}^{\mathbb{N}}$.

Observe that, when $R = \mathbb{R}$, the $(D\mathcal{F})$-convergence coincides with the usual filter convergence. Moreover, when $\mathcal{F} = \mathcal{F}_{\text{cofin}}$, $(D\mathcal{F})$- and (D)-convergence are equivalent.

We now deal with some basic properties of lattice group-valued set functions. From now on, R is a Dedekind complete lattice group, G is an infinite set, Σ is a σ-algebra of subsets of G, $m : \Sigma \to R$ is a set function and k is a fixed positive integer.

The *semivariation* of m is defined by $v(m)(A) := \bigvee \{|m(B)| : B \in \Sigma, \, B \subset A\}$.

We say that m is k*-triangular* (on Σ) iff

$$m(A) - k\, m(B) \le m(A \cup B) \le m(A) + k\, m(B) \qquad \text{whenever } A, B \in \Sigma, \; A \cap B = \emptyset \tag{1}$$

and $0 = m(\emptyset) \le m(A)$ for each $A \in \Sigma$.

Given a set function $m : \Sigma \to R$ and an algebra $\mathcal{L} \subset \Sigma$, the *semivariation of m with respect to $\mathcal{L}$* is defined by $v_{\mathcal{L}}(m)(A) := \bigvee \{|m(B)| : B \in \mathcal{L}, \, B \subset A\}$. Note that $v(m) = v_{\Sigma}(m)$.

A set function $m : \Sigma \to R$ is *continuous from above at* $\emptyset$ iff for every decreasing sequence $(H_n)_n$ in Σ with $\bigcap_{n=1}^{\infty} H_n = \emptyset$ we get $\bigwedge_n v_{\mathcal{L}}(m)(H_n) = 0$, where $\mathcal{L}$ is the σ-algebra generated by $(H_n)_n$ in H_1.

A topology τ on Σ is a *Fréchet-Nikodým topology* iff the functions $(A,B) \mapsto A\Delta B$ and $(A,B) \mapsto A \cap B$ from $\Sigma \times \Sigma$ (endowed with the product topology) to Σ are continuous, and for any τ-neighborhood V of $\emptyset$ in Σ there exists a τ-neighborhood U of $\emptyset$ in Σ such that, if $E \in \Sigma$ is contained in some suitable element of U, then $E \in V$.

Let τ be a Fréchet-Nikodým topology on Σ. A set function $m : \Sigma \to R$ is *τ-continuous* on Σ iff for each decreasing sequence $(H_n)_n$ in Σ, with $\tau\text{-}\lim_n H_n = \emptyset$, we get $\bigwedge_n v_{\mathcal{L}}(m)(H_n) = 0$, where $\mathcal{L}$ denotes the σ-algebra generated by the sets H_n, $n \in \mathbb{N}$, in H_1.

We say that the set functions $m_j : \Sigma \to R$, $j \in \mathbb{N}$, are *equibounded* iff there is $u \in R$ with $|m_j(A)| \le u$ for all $j \in \mathbb{N}$ and $A \subset \Sigma$.

Lemma 0.1 *Let R be any Dedekind complete lattice group, $(a_{j,n})_{j,n}$ be a double sequence in R and $\mathcal{F}$ be a diagonal filter of $\mathbb{N}$. If $(D\mathcal{F}) \lim_{j\in\mathbb{N}} a_{j,n} = 0$ for each $n \in \mathbb{N}$ with respect to a single regulator $(b_{t,l})_{t,l}$ (independent of n), then there is a (D)-sequence $(c_{t,l})_{t,l}$ (independent of n) such that for any $I \in \mathcal{F}^*$ there is $J \in \mathcal{F}^*$, $J \subset I$, with $(D) \lim_{j\in J} a_{j,n} = 0$ for any $n \in \mathbb{N}$ with respect to $(c_{t,l})_{t,l}$.*

Proposition 0.2 *If $m : \Sigma \to R$ is k-triangular, then $v(m)$ is k-triangular too. Moreover, for any $n \ge 2$ and for every pairwise disjoint sets $E_1, E_2, \ldots, E_n \in \Sigma$ we get*

$$m(E_1) - k\sum_{q=2}^{n} m(E_q) \le m\Big(\bigcup_{q=1}^{n} E_q\Big) \le m(E_1) + k\sum_{q=2}^{n} m(E_q). \tag{2}$$

Lemma 0.3 *Let R be any Dedekind complete (ℓ)-group and $(a^{(n)}_{t,l})_{t,l}$, $n \in \mathbb{N}$, be a sequence of regulators in R. Then for every $u \in R$, $u \ge 0$ there is a (D)-sequence $(a_{t,l})_{t,l}$ in R with*

$$u \wedge \Big(\sum_{n=1}^{q}\Big(\bigvee_{t=1}^{\infty} a^{(n)}_{t,\varphi(t+n)}\Big)\Big) \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \quad \textit{for every } q \in \mathbb{N} \textit{ and } \varphi \in \mathbb{N}^{\mathbb{N}}.$$

Proposition 0.4 *Let R be a Dedekind complete lattice group, $m : \Sigma \to R$ be a set function, $(H_n)_n$ be any decreasing sequence in Σ, and set $C_n = H_n \setminus H_{n+1}$, $n \in \mathbb{N}$. For every $A \subset \mathbb{N}$ put $\nu(A) = m\Big(\bigcup_{n\in A} C_n\Big)$. Let $\mathcal{L}$ be the σ-algebra generated by the H_n's in H_1 and assume that $\bigwedge_n v_{\mathcal{L}}(m)(H_n) = 0$. Then ν is continuous from above at $\emptyset$, and for every $n \in \mathbb{N}$ it is*

$$\begin{aligned} v(\nu)([n,+\infty[) &= \bigvee\Big(|\nu(B)| : B \subset [n,+\infty[\Big) \le \\ &\le \bigvee\Big(|m(C)| : C \in \mathcal{L} \textit{ with } C \subset H_n\Big) = v_{\mathcal{L}}(m)(H_n). \end{aligned} \tag{3}$$

Theorem 0.5 *Let R be any Dedekind complete (ℓ)-group, $\mathcal{F}$ be a block-respecting filter of $\mathbb{N}$, $m_j : \mathcal{P}(\mathbb{N}) \to R$, $j \in \mathbb{N}$, be a sequence of continuous from above at $\emptyset$ and equibounded k-triangular set functions, and set $\beta_{A,j} := m_j(A)$, $A \in \mathcal{P}(\mathbb{N})$, $j \in \mathbb{N}$. Suppose that:*

0.5.1) *$(D) \lim_j m_j(\{n\}) = 0$ for any $n \in \mathbb{N}$;*

0.5.2) *the family* $(\beta_{A,j})_{A\in\mathcal{P}(\mathbb{N}),j\in\mathbb{N}}$ $(RD\mathcal{F})$*-converges to* 0.

Then we get:

0.5.3) $(D\mathcal{F})\lim_j v(m_j)(\mathbb{N})=0$.

0.5.4) *If* $\mathcal{F}$ *is also diagonal and* R *is super Dedekind complete and weakly* σ*-distributive, then* 0.5.2) *implies* 0.5.3).

The following Vitali-Hahn-Saks-type theorem is a consequence of Theorem 0.5.

Theorem 0.6 *Let* R *be a super Dedekind complete and weakly* σ*-distributive* (ℓ)*-group,* $\mathcal{F}$ *be a diagonal and block-respecting filter of* $\mathbb{N}$*,* τ *be a Fréchet-Nikodým topology on* Σ*,* $m_j:\Sigma\to R$*,* $j\in\mathbb{N}$*, be a sequence of equibounded,* τ*-continuous and* k*-triangular set functions. Assume that the family* $m_j(A)$*,* $A\in\Sigma$*,* $j\in\mathbb{N}$*,* $(RD\mathcal{F})$*-converges to* 0*. Then for each decreasing sequence* $(H_n)_n$ *in* Σ *with* $\tau\text{-}\lim_n H_n=\emptyset$ *and for every* $\mathcal{F}$*-stationary set* $I\subset\mathbb{N}$ *there exists an* $\mathcal{F}$*-stationary set* $J\subset I$*, with*

$$\bigwedge_n\Big[\bigvee_{j\in J}v_{\mathcal{L}}(m_j)(H_n)\Big]=0, \tag{4}$$

where $\mathcal{L}$ *is the* σ*-algebra generated by the* H_n*'s in* H_1*.*

We now give the following Nikodým convergence-type theorem.

Theorem 0.7 *Let* R*,* $\mathcal{F}$*,* Σ*,* $\mathcal{L}$ *be as in Theorem* 0.6*,* $m_j:\Sigma\to R$*,* $j\in\mathbb{N}$*, be a sequence of equibounded* k*-triangular set functions, continuous from above at* $\emptyset$*. Suppose that the family* $m_j(A)$*,* $A\in\Sigma$*,* $j\in\mathbb{N}$*,* $(RD\mathcal{F})$*-converges to* 0*. Then for each decreasing sequence* $(H_n)_n$ *in* Σ *with* $\bigcap_{n=1}^{\infty}H_n=\emptyset$ *and for every* $\mathcal{F}$*-stationary set* $I\subset\mathbb{N}$ *there exists an* $\mathcal{F}$*-stationary set* $J\subset I$*, with* $\bigwedge_n\Big[\bigvee_{j\in J}v_{\mathcal{L}}(m_j)(H_n)\Big]=0$.