Visualizing $\zeta(n \geq 2)$ and Proving Its Irrationality

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1 Introduction

Apery's proof that

$$\zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3}$$

is irrational is difficult [1]. It doesn't generalize to show $\zeta(2n+1)$ is irrational for $n > 2$. Here we develop a much simpler proof that does so generalize.

The proof uses the fact that if a decimal representation of some real number requires an infinite number of digits in all basis, then it must be irrational. Hardy shows that all decimal representations of a rational number a/b in a given base d are finite, repeating, or mixed depending on the relationship between b and d [7]. If all the prime factors of b are those of d, then the decimal representation is finite; if b and d are relatively prime, then the decimal representation is pure repeating; if some prime factors are shared but not all, then the decimal representation is mixed. We observe these three cases with the decimal representations of $1/2$, $1/3$, and $1/6$ in base 10. An irrational number in all bases is an infinite non-repeating decimal. The idea of our proof then is to show

$$\zeta(n) - 1 = z_n = \sum_{k=2}^{\infty} \frac{1}{k^n} \tag{1}$$

can't be represented by a finite decimal in any base.

The current state of affairs with proving z_n, n odd, is irrational is quite limited. It is known that there are infinitely many odd $n > 3$ that are irrational [9] and that at least one of 5, 7, 9, and 11 are irrational [12]. The proofs of these results use combinations of groups and complex analysis. The even case follows easily from the transcendence of π [4, 8] and Bernoulli's famous formula:

$$\zeta(2n) = (-1)^{k+1}\frac{(2\pi)^{2k}B_{2k}}{2(2k)!}.$$

This formula is derived from a trigonometric series expansion [2].

The proof given here uses a simple geometric construction that allows the terms of (1) to be given as sector areas and to be added. There is then some connection with circles on the plane, but our plane is not the complex plane, nor even the Cartesian plane – just concentric circles with sectors designated by a radius. If the radius goes through a point (we call it a dot) on a circle the sector area is given by a finite decimal in a base associated with the circle. The construction allows for a clear visualization of the decimal representations of all terms and partial sums of (1) in all bases k^2, where k is a natural number. We develop this visualization device in Sections 2 and 3. We believe this visualization device makes it highly plausible that all $n \geq 2$ values of z_n are irrational.

The problem of the limit of the partials is addressed with the geometric series in Section 4. A geometric series for our purposes is just an infinite repeating decimal in a base. So $.\overline{1}$ base 4 is such a geometric series. Using our visualization device it is clear that such series can't converge to the circle associated with any term of the series: 1/3, the convergence point is not represented by any finite base 4 decimal or dot on its associated circle. If this is generally provable, then it follows that the convergence point of z_n must not reside as a dot on any of its terms circles, but its terms circles give all finite decimal representations in base k^2, k a natural number greater than 2. This is all rational numbers between 0 and 1, so z_n must be irrational. We need to prove it doesn't go through any such dots.

Finally, Cantor's diagonal process is modified to prove z_n is irrational. It is based on Cantor's classic proof that the real numbers are not countable [10]. A corollary of his proof is that an irrational number exists; he constructs it by modifying a list of decimal numbers, supposed to be all rationals in a fixed base with values between 0 and 1. Each digit down the diagonal of the list is modified. The resulting number is not in the list, so it is irrational, and, further (the point of his proof) the real numbers are proven to be, by contradiction, uncountable. Our modification of Cantor's technique, Section

6, associates (lists) all rational numbers between 0 and 1 using all bases k^n, referencing the visualization sections' circles. It then constructs a number not associated with any rational number using modifications of digits in the base using partial sums of z_n. We show, in a lemma, Section 5, that all such partials are not represented by any of the bases when its term's denominators are used for bases. As in Cantor's original proof, the resultant infinite series is irrational.

2 Term Visualization

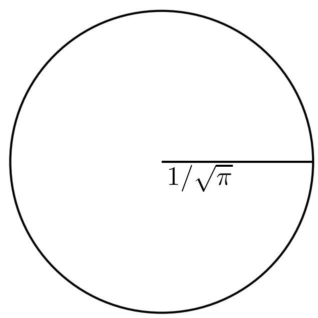

Figure 1: A circle with radius $1/\sqrt{\pi}$ has an area of 1.

We can visualize the first term, 1/4, of this series using a circle. In Figure 1 we have a circle of radius $\sqrt{1/\pi}$. The area of this circle is

$$\pi r^2 = \pi \cdot (\sqrt{1/\pi})^2 = 1.$$

In Figure 2, four equally spaced dots are placed around the circle, giving four equal sector areas. Each area must be 1/4 of the area of the circle or 1/4. Sector areas corresponding to these dots, between 0 and 1, are given by $x/4$, $x = 1, 2, 3$ or a single, non-zero decimal base 4. If a radius on the circle doesn't go through one of the dots, the sector area formed will require more than a single decimal in base 4: Figure 3. We will designate this circle with C_4.

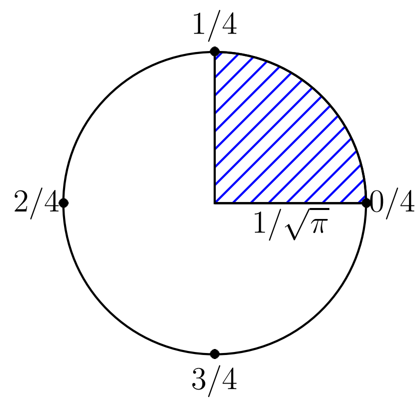

Figure 2: This circle with area 1 is divided up using 2^2. The area of the shaded sector is 1/4.

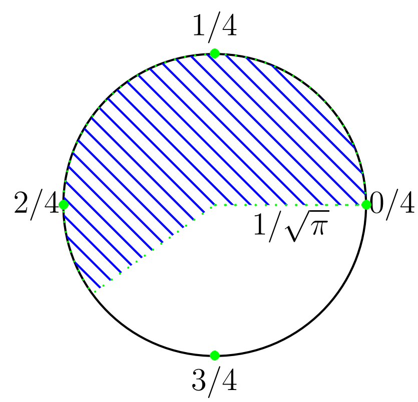

Figure 3: A radius that does not go through any dot generates a sector area that requires more than one decimal, base 4.

The next term is 1/9. The circle in Figure 4 has radius $\sqrt{2/\pi}$ with 9 equally spaced dots around it. Its area is 2:

$$\pi r^2 = \pi \cdot (\sqrt{2/\pi})^2 = 2.$$

We will designate this circle with C_9.

By making C_4 and C_9 concentric circles, Figure 5, the area of the annulus formed is 1: $2 - 1$. If a radius is drawn threw a dot on C_9 it will generate a sector area of $x/9$ on C_4. If a radius misses dots on both circles, then the

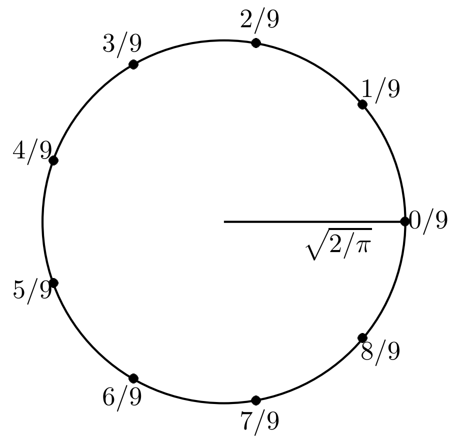

Figure 4: Nine equally spaced dots on a circle of radius $\sqrt{2/\pi}$: C_9.

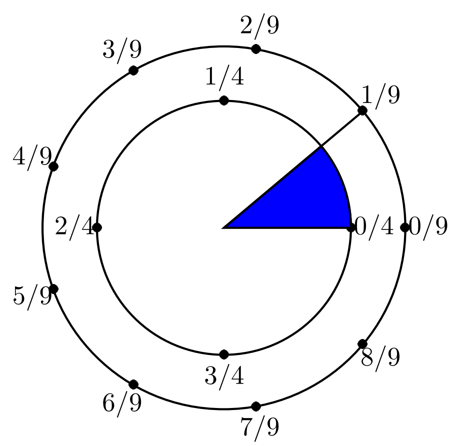

Figure 5: C_4 and C_9 as concentric circles. The area of the blue sector is 1/9.

sector area formed is not equal to a single decimal in either base 4 or base 9. It will require more than a single digit in either of these bases.

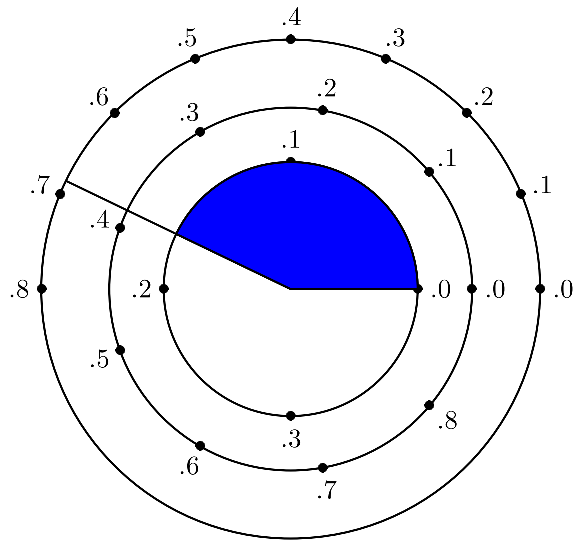

Figure 6: The shaded sector area is not a single decimal in base 4, 9, or 16.

Figure 6 shows the first three terms of z_2 rendered with C_4, C_9, and C_{16}. Clearly, we can continue this process using equally spaced k^2 dots on circles of radius $\sqrt{k/\pi}$. If a given radius misses all dots on all such circles, it seems plausible that it will be irrational. The sector areas generated by radii through a given dot, say x, on the C_{k^2} circle will be given by $.x$ base k^2 and all rational numbers can be so designated.

This is a visualization of the terms of z_2. Next we will visualize adding these terms.

3 Visualization of Partial Sums

Two sector areas can be added.

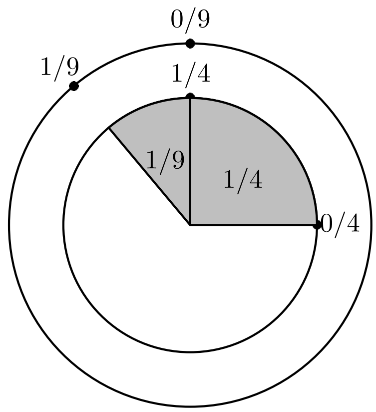

Figure 7: Adding $1/4 + 1/9$ using C_4 and C_9 with C_9 offset.

In Figure 7, 1/4 is added to 1/9 by rotating C_9 in a counter-clockwise direction to line up with the 1/4 dot on C_4. This addition is somewhat analogous to the head to toe (here 1 to 0) method of vector addition. In Figure 8, 1/16 is added to $1/4 + 1/9$ using the same 1 (head) to 0 (toe) method. The resulting radius generates an area on all annuli and C_4's circle that corresponds to $1/4 + 1/9 + 1/16$. Clearly these additions can be used to form such radii for all partial sums of z_2.

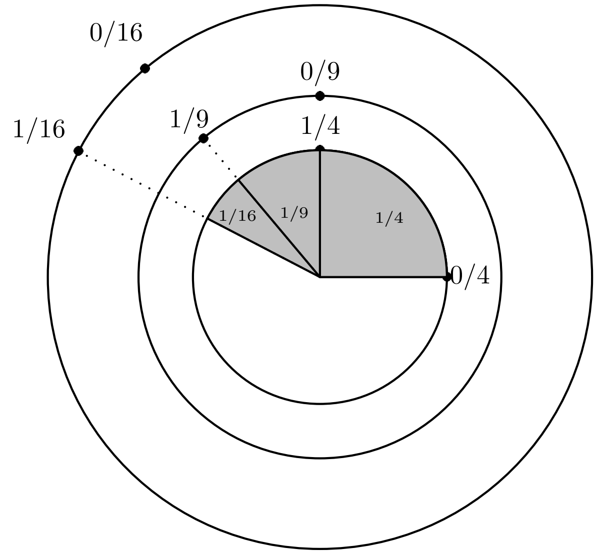

Figure 8: Adding $1/4 + 1/9 + 1/16$ using C_4, C_9, and C_{16} with the offset method. The area of the shaded sector is the sum.

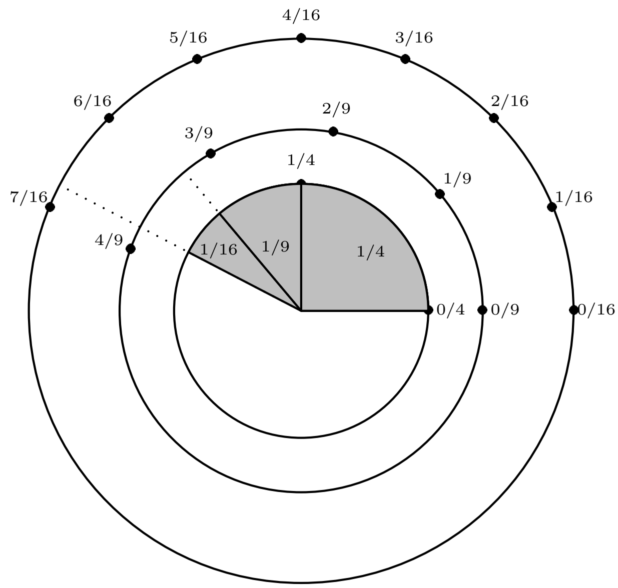

Figure 9: The radius associated with the sum $1/4 + 1/9 + 1/16$ misses all dots on C_4, C_9, and C_{16}.

Figure 7 and Figure 8 show rotations of C_9 and C_{16} to effect fraction additions. Figure 9 shows the resulting radius with the un-rotated versions of these circles. Figure 9 accurately shows that the partial

$$S_3 = \sum_{k=2}^{3} \frac{1}{k^2}$$

generates a radius that does not go through any of its first few term's dotted circles. We can infer that the sum is not expressible as a single decimal digit in base 4, 9, or 16.

The denominators of z_2 are just all decimal bases squared. So if a radius misses all dots on all C_{k^2} circles then its value must require more than one decimal in all bases. How is that possible if the number is rational? Given, p/q, $0 < p/q < 1$, pq/q^2 can be written as a single decimal in base q^2. It corresponds to a dot on C_{q^2}.

We can now visualize the problem of proving z_2 is irrational. We need to show that the limit radius generated by adding the terms of z_2 does not go through any of the dots on any of the circles defined by its terms. The difficulty is that one can converge to a dot on a circle without a radius going through the dot. This is the case for the geometric series which we will analyze next.

4 Geometric series

Infinite repeating decimals are really geometric series. For example, in base 4,

$$.\overline{1} = \sum_{k=1}^{\infty} \frac{1}{4^k}.$$

This geometric series has a convergence point of 1/3. All its terms occur in z_2, so we can use our dotted concentric circles to understand the relationship between the rotated C_{4^k} circles used to construct this sum and C_3, the un-rotated circle having a dot the sum converges to, that is, the radius associated with this infinite sum goes through.

Notice that all single decimals for all rational numbers in base k^2 have radii associated with them on an un-rotated circle. As k^2 gives the square of all possible bases, natural numbers greater than 1, all rational numbers representable with a single digit between 0 and 1 are included in the union of these radii. This implies that we can read from a system of dotted circles the decimal expansion in a given base, like base 4. Also all convergent infinite series with terms of the form $1/a_k$ with a_k strictly increasing have partial radii that rotate counter-clockwise around the circle and go through points farther and farther from the center. This forces series that converge to a rational number to have their convergence point given by a radius going through an un-rotated "earlier" dot. We can see these patterns in Figure 10.

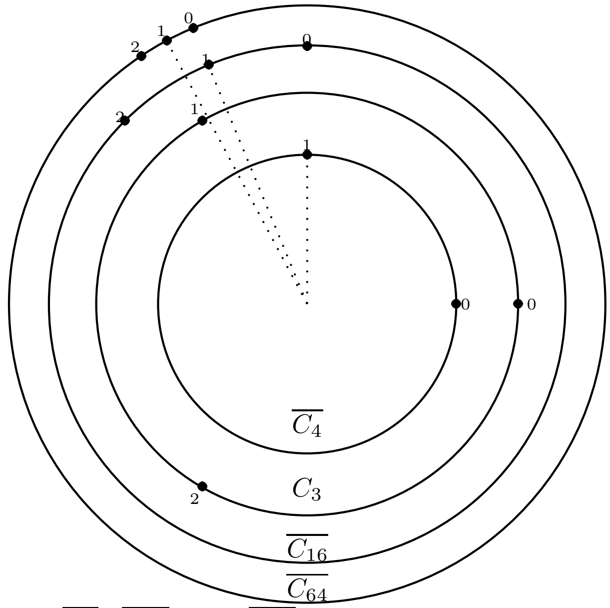

Figure 10: Circles $\overline{C_4}$, $\overline{C_{16}}$ and $\overline{C_{64}}$ are rotated (indicated with overline) to generate the radius associated with .111 base 4. C_3, unrotated, has the convergence point for $.\overline{1}$: $1/3$.

Draw all rational radii and note that the rules for pure, mixed, and finite decimals will be observed. The only radii remaining to be drawn are the ones that go through no rational dots on the circles and hence are irrational. Placing an irrational sector area in circle C_4 and extending its radius outward through all circles, it must be that the radius will go through no dots on any circle – otherwise the irrationality of the sector area would be contradicted. There is only one radius for every area, rational and irrational. Unlike decimal representations where $.4\overline{9} = .5$, there is no ambiguity with fractions and areas. We can read the decimals from our figure and as the radius never goes through a point in all bases we never terminate. If the number were rational this would imply that its denominator has prime factors that are not shared by any natural number, it has some prime factors that are relatively prime to all natural numbers: a contradiction.

In the next two sections we attempt to build to a proof that z_2 is irrational.

5 Lemma

Our aim is to show that the reduced fractions that give the partial sums of z_2 require a denominator greater than that of the last term defining the partial sum. Restated this says that partial sums can't be expressed as a finite decimal using for a base the denominators of any of the partial sum's terms. As it is no more difficult, we will argue the general case, i.e. for $z_n = \zeta(n) - 1$, n an integer greater than 1. We just need this result for $\zeta(2)$.

The first lemma is typically given as an exercise in courses in number theory; for example, in [2], it's an exercise.

Lemma 1. *The reduced fraction, r/s giving*

$$s_k^m = \sum_{j=2}^{k} \frac{1}{j^m} = \frac{r}{s} \tag{2}$$

is such that 2^m divides s.

Proof. The set $\{2, 3, \ldots, k\}$ will have a greatest power of 2 in it, a; the set $\{2^m, 3^m, \ldots, k^m\}$ will have a greatest power of 2, ma. Also $k!$ will have a powers of 2 divisor with exponent b; and $(k!)^m$ will have a greatest power of 2 exponent of mb. Consider

$$\frac{(k!)^m}{(k!)^m} \sum_{j=2}^{k} \frac{1}{j^m} = \frac{(k!)^m/2^m + (k!)^m/3^m + \cdots + (k!)^m/k^m}{(k!)^m}. \tag{3}$$

The term $(k!)^m/2^{ma}$ will pull out the most 2 powers of any term, leaving a term with an exponent of $mb - ma$ for 2. As all other terms but this term will have more than an exponent of 2^{mb-ma} in their prime factorization, we have the numerator of (3) has the form

$$2^{mb-ma}(2A + B),$$

where $2 \nmid B$ and A is some positive integer. This follows as all the terms in the factored numerator have powers of 2 in them except the factored term $(k!)^m/2^{ma}$. The denominator, meanwhile, has the factored form

$$2^{mb}C,$$

where $2 \nmid C$. This leaves 2^{ma} as a factor in the denominator with no powers of 2 in the numerator, as needed. $\square$

Lemma 2. *If p is a prime such that $k > p > k/2$, then p^m divides s in* (2)*.*

Proof. First note that $(k, p) = 1$. If $p|k$ then there would have to exist r such that $rp = k$, but by $k > p > k/2$, $2p > k$ making the existence of a natural number $r > 1$ impossible.

The reasoning is much the same as in Lemma 1. Consider

$$\frac{(k!)^m}{(k!)^m}\sum_{j=2}^{k}\frac{1}{j^m} = \frac{(k!)^m/2^m + \cdots + (k!)^m/p^m + \cdots + (k!)^m/k^m}{(k!)^m}. \qquad (4)$$

As $(k, p) = 1$, only the term $(k!)^m/p^m$ will not have p in it. The sum of all such terms will not be divisible by p, otherwise p would divide $(k!)^m/p^m$. As $p < k$, p^m divides $(k!)^m$, the denominator of r/s, as needed. □

Theorem 1. *If*

$$s_k^m = \frac{1}{2^m} + \frac{1}{3^m} + \cdots + \frac{1}{k^m} = \frac{r}{s}, \qquad (5)$$

with r/s reduced, then $s > k^m$.

Proof. Bertrands postulate states that for any $k \geq 2$, there exists a prime p such that $k < p < 2k$ [7]. If k of (5) is even we are assured that there exists a prime p such that $k > p > k/2$. If k is odd $k-1$ is even and we are assured of the existence of prime p such that $k-1 > p > (k-1)/2$. As $k-1$ is even, $p \neq k-1$ and $p > (k-1)/2$ assures us that $2p > k$, as $2p = k$ implies k is even, a contradiction.

For both odd and even k, using Bertrand's postulate, we have assurance of the existence of a p that satisfies Lemma 2. Using Lemmas 1 and 2, we have $2^m p^m$ divides the denominator of (5) and as $2^m p^m > k^m$, the proof is completed. □

So, for z_2, we have the following.

Definition 1.

$$D_{k^2} = \{0, 1/k^2, \ldots, (k^2-1)/k^2\} = \{0, .1, \ldots, .(k^2-1)\} \textit{ base } k^2$$

Corollary 1.

$$s_n^2 \notin \bigcup_{k=2}^{n} D_{k^2}$$

Proof. Immediate. □

6 Cantor

Cantor's diagonal method is a rear view mirror technique that reduces the question of convergence to finite cases that are systematically eliminated as they build an infinite convergence point – involving an infinite procedure.

Here's an example of its use to show the existence of an irrational number. List all rational numbers between 0 and 1. They are countable, so this can be done. Use base 10.

$$.d_{11}d_{12}d_{13}\ldots \tag{6}$$
$$.d_{21}d_{22}d_{23}\ldots \tag{7}$$
$$.d_{31}d_{32}d_{33}\ldots \tag{8}$$
$$.d_{41}d_{42}d_{43}\ldots \tag{9}$$
$$.d_{51}d_{52}d_{53}\ldots \tag{10}$$
$$\vdots \tag{11}$$

Go down the diagonal and change the value of the decimal to 3, if it is not 3 and 7, if it is: Table 1. Record the changes following a decimal point.

row	new	original
1	$.c_1d_{12}d_{13}\ldots$	$.d_{11}d_{12}\ldots$
2	$.d_{21}c_2d_{23}\ldots$	$.d_{21}d_{22}\ldots$
2	$.d_{31}d_{32}c_3\ldots$	$.d_{31}d_{32}\ldots$
2	$.d_{41}d_{42}d_{43}c_4\ldots$	$.d_{41}d_{42}\ldots$

Table 1: Cantor's diagonal method building an irrational number: $.c_1c_2\ldots$.

We notice that $.c_1$ of row 1 is different $.d_{11}d_{12}\ldots$ and $.c_1c_2$ is different than $.d_{21}d_{22}\ldots$ of row 2, as well as $.d_{11}d_{12}\ldots$ of row 1. We can actually get a bound for the difference with these numbers.

We have reduced the infinite construction of $.c_1c_2\ldots$ to finite considerations and we can conclude that the infinite decimal $.c_1c_2c_3\ldots$ is not in the list. As it is also between 0 and 1, it must be irrational. We are looking in the rear view mirror as we go down the diagonal, forward. As we see a new number ahead of us, we will change it. Think of a space ship trajectory given by the radius of earlier sections. We our building our trajectory by small increments avoiding the dots ahead. The result $.c_1 c_2 \dots$ is a sum of discrete steering wheel corrections. It avoids the limit radius of all rational trajectories – some of which are infinite sums.

We have *constructed* an irrational number. This is different than proof by contradiction or induction.

The application of Cantor's diagonal method just given changes the numerators of sums of fractions. We will change the numerators and denominators. List all the rational numbers between 0 and 1 using D_{k^2}. These are arranged down a diagonal in Table 2. Our mission is to create a number that isn't in the first row, then isn't the first or second row, and then repeat this process infinitely many times.

D_4							
	D_9						
		D_{16}					
			D_{25}				
				D_{49}			
					D_{64}		
						D_{81}	
							$\ddots$

Table 2: A list of all rational numbers between 0 and 1.

Notice this is something like a hydra list. If you cut out a row all the numbers will continue to exist (grow back) later. For example, removing D_{25} doesn't change the list of numbers because any number that is a multiple of 25 will have fractions that when reduced give the same values, D_{100} for example ($4/100 = 1/25$). Also notice, unlike Cantor's one value at a time changes, we are going to give a value not in a set with several values. We need to construct a value not in D_4, then not in D_4 and D_9, then not in D_4, D_9, and D_{16}. If this process never ends, the number constructed will not be in any D_{k^2} and so it must be irrational.

The diagonal arrangement of Table 2 is just a contrivance to make the program visually more comprehensible. One could write all the numbers in each set one after the other and then do the procedure with the same effect. Let's get to the procedure.

The modification of Cantor is really simple; we add to make the change to rational numbers we encounter. Recall Cantor executes a swap based on a criterion. There is no real difference, Cantor could say if the decimal digit encountered is a 3 add .000004 to it where the zeros give the right position to yield the swap of 7 for 3. The important net is that the number is changed and the way it is changed can be recorded and builds a number not in the list. We add partials of z_2 to cause the number changes in our list. We are using fractions instead of decimals, but these are just representations of the same thing. We change the number using the same fraction repeatedly four times say for D_4, if the list was not in the diagonal arrangement of Table 1. Table 2 gives the program.

1/4							
1/9	1/4	1/4	1/4	1/4			
D_4	1/9	1/9	1/9	1/9			
	D_9	1/16	1/16	1/16			
		D_{16}	1/25	1/25			
			D_{25}	1/36			
				D_{36}			
					D_{49}		
						D_{64}	
							$\ddots$

Table 3: A list of all rational numbers between 0 and 1 modified to exclude them all via partial sums of z_2.

The procedure is to add the numbers above each D_{k^2}. The result is not in D_{k^2}. This is Corollary 1. So, for example, $1/4+1/9$ is not in D_4, $1/4+1/9$ is not in D_4 or D_9, $1/4+1/9+1/16$ is not in D_4, D_9, or D_{16}, etc.. Just like Cantor allows us to conclude a number we construct is not in a list, we can conclude the number we construct, z_2 is not in the list. As our list consists of all rational numbers between 0 and 1, z_2 must be irrational. Note: because of repetition of numbers in D_{k^2}, one could chop off the first columns, but the tail would still be proven to be irrational. The slight asymmetry in the first $1/4+1/9$ is placed for aesthetic reasons in the table. A missing partial will be rational. The tail only will make the number irrational.

It is worth noting that Cantor needs to be careful with his *if 3, 7 else*

3 program. If he replaced everything with 9's or 0's then an ambiguity of $.1\overline{9} = .2$ might arise; he might not have assurance that a number is excluded from the list. Working with fractions (or all bases), as we are, and not a single number base, this problem does not arise.

7 Other series

The telescoping series

$$\sum_{k=2}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = 1/2 - 1/3 + 1/3 - 1/4 + \cdots = 1/2$$

or

$$\sum_{k=2}^{\infty}\frac{1}{n(n+1)} = \frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots = 1/2$$

shows the necessity of partials escaping terms. For example, the sum of the first three terms is 3/10 which can be expressed with 6/20 in D_{20}. Partial sums backtrack to earlier denominators, thus preventing Cantor's diagonal process from being valid. The geometric series has partials that sum to fractions with denominators from the last term of the partial, but the term's denominators don't cover all pertinent rational numbers.

For both examples, placing them in Cantor's diagonal of Table 3 shows the necessity of partials escaping their terms and the terms covering the rationals.

8 Conclusion

Do the ideas given here give a proof that $\zeta(n \geq 2)$ is irrational, all natural number n? As all bases k^n have the same prime factors as k, the answer is yes: Table 3, in conjunction with Section 5, works when these other series are used.

With the assumption of Lemmas 1 and 2 and Theorem 1 of Section 5, does the proof distill to a geometric proof? Note that the denominator of the partial sums of a z_k series with upper bound n will be much larger than n^k, more like $(n!)^k$. Also simple number theory proofs show that $(n-1, n) = (n, n+1) = 1$, that the natural numbers are consecutively relatively prime. So one suspects such partial sums will have denominators that have increasing prime factors. This points to the central intuition about this series; the fractions added have denominators growing by one (with a power) and this marks how the series differs from the "neatness" of the geometric and telescoping series. If one grants Theorem 1 as intuitively plausible is Figure 9 of Section 3 enough: the nudging of a trajectory by the terms (the addition of terms) of any $\zeta(k)$ builds a trajectory that never "hits" a rational dot; all rational sector areas are perpetually offset yielding a sector area that must be irrational – is that enough?

The radii used are unique to every number, rational and irrational. Irrational areas, unlike irrational numbers represented by decimals, have no infinite procedure associated with them. The circles represent a binary system: through a dot or not through a dot (a single decimal is used only). If a radius does not go through a dot, it is not representing a rational sector area. In this system, rational numbers have a first circle that its radius goes through. Make a radius that doesn't have a first circle. One such will be $\zeta(k)$.

Finally, the mission is to show z_{2n+1}, $n \geq 2$ are irrational, others are known to the irrational. So we can assume that z_2 is irrational and hence that its limit radius never goes through a dot on the circles associated with its terms, but then this property is per the prime factors of these circles which are shared for z_{2n+1}, $n > 2$. We, then, can dispense with the results dependent on Bertrand's postulate (Section 5) and Cantor's diagonal method, Section 6; if the result is true for one $\zeta(n \geq 2)$, it's true for them all: the 1/4 sector is divided into 4, the 1/9 into 9 and the fractal the pattern for z_2 is repeated for z_3 and all others.

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