Some Elementary Identities for q-Series and the Generating Functions of the (m,k)-Capsids and (m,r_1,r_2)-Capsids

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August 7, 2017

"Jesus answered and said unto him, What I do thou knowest not now; but thou shalt know hereafter." - John 13:7.

Abstract. We demonstrate some elementary identities for q-series involving the q-Pochhammer symbol, as well as an identity involving the generating functions of the (m, k)-capsids and (m, r_1, r_2)-capsids.

2010 Mathematics Subject Classification. Primary 05A15; Secondary 05A30, 11P81,11P83, 11P84
Key words and phrases. q-series, generating functions of the (m, k) and (m, r_1, r_2)-capsids.

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1. Introduction

The q-series is a topic of Mathematics with several applications: Number Theory, Analysis, Combinatorics, Physics, and Computer Algebra. In this paper, we explore some identities, relating them to the q-Pochhammer symbol. For example, we have shown that

$$\begin{bmatrix} \ell \\ n \end{bmatrix}_q = \frac{(1-q^\ell)}{(1-q^n)(1-q^{\ell-n})} \cdot \frac{(q^n;q)_\infty (q^{\ell-n};q)_\infty}{(q;q)_\infty (q^\ell;q)_\infty},$$

a new version of the q-binomial coefficient, as well as one know elementary identity

$$\begin{bmatrix} \ell \\ n \end{bmatrix}_q = \frac{1-q^\ell}{1-q^n} \cdot \begin{bmatrix} \ell-1 \\ n-1 \end{bmatrix}_q;$$

and a new version of the Cauchy binomial theorem

$$\prod_{k=1}^{\ell} (1+yq^k)$$
$$= 1 + y^\ell q^{\ell(\ell+1)/2} + \frac{(1-q^\ell)(q;q)_\infty}{(q^\ell;q)_\infty} \cdot \sum_{n=1}^{\ell-1} \frac{y^n q^{n(n+1)/2}}{(1-q^n)(1-q^{\ell-n})(q;q)_{n-1}(q;q)_{\ell-n-1}}.$$

In addition, we apply the Lemma 3.1 to the capsids theory of Garvan and Schlosser [8], and find an identity for the (m, k)-capsids and (m, r_1, r_2)-capsids generating functions

$$C_{m,k}(q) = \frac{1-q^m}{1-q^k} \cdot C_{m,m,m}(q^k, q^{-k}, q).$$

We end this article with the following elementary identities:

$$\frac{(a;b)_\infty (c;d)_\infty}{(a;e)_\infty (c;f)_\infty} = \frac{(ab;b)_\infty (cd;d)_\infty}{(ae;e)_\infty (cf;f)_\infty}$$

and

$$\frac{(a;b)_\infty (cd;d)_\infty}{(c;d)_\infty (ab;b)_\infty} = \frac{(a;b/a)_\infty (d;d/c)_\infty}{(b;b/a)_\infty (c;d/c)_\infty}.$$

2. FROM q-BINOMIAL COEFFICIENT TO q-POCHHAMMER SYMBOL

Theorem 2.1. *If $n \leqslant \ell$ and n,ℓ are positive integers, and $0<|q|<1$, then*

$$\begin{bmatrix} \ell \\ n \end{bmatrix}_q = \frac{(1-q^\ell)}{(1-q^n)(1-q^{\ell-n})} \cdot \frac{(q^n;q)_\infty (q^{\ell-n};q)_\infty}{(q;q)_\infty (q^\ell;q)_\infty},$$

where $\begin{bmatrix} \ell \\ n \end{bmatrix}_q$ denotes the q-binomial coefficient and $(a;q)_\infty$ denotes the q-Pochhammer Symbol.

Proof. In [1, p. 3], we prove that

$$\begin{pmatrix} \ell \\ n \end{pmatrix} = \prod_{j=1}^{\infty} \left(\frac{n+j}{j}\right)\left(\frac{j+\ell-n}{j+\ell}\right). \tag{2.1}$$

In [2, p. 85], we encounter

$$\lim_{q\to 1^-} \frac{1-q^v}{1-q} = v \tag{2.2}$$

and [3], we find

$$\lim_{q\to 1^-} \begin{bmatrix} \ell \\ n \end{bmatrix}_q = \begin{pmatrix} \ell \\ n \end{pmatrix}. \tag{2.3}$$

Apply the left hand side of (2.2) and (2.3) into (2.1), and obtain

$$\begin{aligned}
\lim_{q\to 1^-} \begin{bmatrix} \ell \\ n \end{bmatrix}_q &= \lim_{q\to 1^-} \prod_{j=1}^{\infty} \left(\frac{1-q^{j+n}}{1-q^j}\right)\left(\frac{1-q^{j+\ell-n}}{1-q^{j+\ell}}\right) \\
&= \lim_{q\to 1^-} \left[\frac{(q^\ell-1)q^n}{(q^n-1)(q^n-q^\ell)} \cdot \frac{(q^n;q)_\infty (q^{\ell-n};q)_\infty}{(q;q)_\infty (q^\ell;q)_\infty}\right] \\
&= \lim_{q\to 1^-} \left[\frac{(q^\ell-1)q^n}{(q^n-1)q^n(1-q^{\ell-n})} \cdot \frac{(q^n;q)_\infty (q^{\ell-n};q)_\infty}{(q;q)_\infty (q^\ell;q)_\infty}\right] \\
&= \lim_{q\to 1^-} \left[\frac{(1-q^\ell)}{(1-q^n)(1-q^{\ell-n})} \cdot \frac{(q^n;q)_\infty (q^{\ell-n};q)_\infty}{(q;q)_\infty (q^\ell;q)_\infty}\right].
\end{aligned}$$

By quantization process, we eliminate the limit formula in previous equation and get

$$\begin{bmatrix} \ell \\ n \end{bmatrix}_q = \frac{(1-q^\ell)}{(1-q^n)(1-q^{\ell-n})} \cdot \frac{(q^n;q)_\infty (q^{\ell-n};q)_\infty}{(q;q)_\infty (q^\ell;q)_\infty},$$

which is the desired result. □

Example 2.2. Set $\ell=4$ and $n=2$ in previous Theorem

$$\begin{aligned}
\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q &= \frac{(1-q^4)}{(1-q^2)^2} \cdot \frac{(q^2;q)_\infty^2}{(q;q)_\infty (q^4;q)_\infty} \\
\Rightarrow (1+q^2)(1+q+q^2) &= \frac{(1-q^4)}{(1-q^2)^2} \cdot \frac{(q^2;q)_\infty^2}{(q;q)_\infty (q^4;q)_\infty} \\
\Leftrightarrow (1+q^2)(1+q+q^2) &= \frac{(1+q^2)}{(1-q^2)} \cdot \frac{(q^2;q)_\infty^2}{(q;q)_\infty (q^4;q)_\infty} \\
\Leftrightarrow \frac{(q^2;q)_\infty^2}{(q;q)_\infty (q^4;q)_\infty} &= (1-q^2)(1+q+q^2).
\end{aligned}$$

Theorem 2.3. *If $0<y,q<1$ and $\ell \in \mathbb{N}^+$, then*

$$\begin{aligned}
&\prod_{k=1}^{\ell} (1+yq^k) \\
=&1+y^\ell q^{\ell(\ell+1)/2} + \frac{(1-q^\ell)(q;q)_\infty}{(q^\ell;q)_\infty} \cdot \sum_{n=1}^{\ell-1} \frac{y^n q^{n(n+1)/2}}{(1-q^n)(1-q^{\ell-n})(q;q)_{n-1}(q;q)_{\ell-n-1}},
\end{aligned}$$

where $(a;q)_\infty$ denotes the q-Pochhammer Symbol.

Proof. Multiply the equation of the Theorem 2.1 by $y^n q^{n(n+1)/2}$ and sum from 1 at $\ell-1$ with respect to n, and encounter

$$\sum_{n=1}^{\ell-1} y^n q^{n(n+1)/2}\begin{bmatrix} \ell \\ n \end{bmatrix}_q = \frac{(1-q^\ell)}{(q;q)_\infty (q^\ell;q)_\infty}\cdot \sum_{n=1}^{\ell-1} y^n q^{n(n+1)/2} \frac{(q^n;q)_\infty (q^{\ell-n};q)_\infty}{(1-q^n)(1-q^{\ell-n})}. \tag{2.4}$$

On the other hand, we know that [4, p. 300]

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \Rightarrow (aq^\alpha;q)_\infty = \frac{(a;q)_\infty}{(a;q)_\alpha}. \tag{2.5}$$

Replacing q by a and $n-1$ by α in (2.5), find

$$(q^n;q)_\infty = \frac{(q;q)_\infty}{(q;q)_{n-1}}. \tag{2.6}$$

Replacing q by a and $\ell-n-1$ by α in (2.5), get

$$(q^{\ell-n};q)_\infty = \frac{(q;q)_\infty}{(q;q)_{\ell-n-1}}. \tag{2.7}$$

From (2.4), (2.6) and (2.7), it follows that

$$\sum_{n=1}^{\ell-1} y^n q^{n(n+1)/2}\begin{bmatrix} \ell \\ n \end{bmatrix}_q = \frac{(1-q^\ell)(q;q)_\infty}{(q^\ell;q)_\infty}\cdot \sum_{n=1}^{\ell-1} \frac{y^n q^{n(n+1)/2}}{(1-q^n)(1-q^{\ell-n})(q;q)_{n-1}(q;q)_{\ell-n-1}}. \tag{2.8}$$

Sum $1+y^\ell q^{\ell(\ell+1)/2}$ in both members of (2.8) and encounter

$$\begin{aligned} &\sum_{n=0}^{\ell} y^n q^{n(n+1)/2}\begin{bmatrix} \ell \\ n \end{bmatrix}_q \\ =&1+y^\ell q^{\ell(\ell+1)/2}+\frac{(1-q^\ell)(q;q)_\infty}{(q^\ell;q)_\infty}\cdot \sum_{n=1}^{\ell-1} \frac{y^n q^{n(n+1)/2}}{(1-q^n)(1-q^{\ell-n})(q;q)_{n-1}(q;q)_{\ell-n-1}} \end{aligned} \tag{2.9}$$

The Cauchy Binomial Theorem [5] assures us that

$$\sum_{n=0}^{\ell} y^n q^{n(n+1)/2}\begin{bmatrix} \ell \\ n \end{bmatrix}_q = \prod_{k=1}^{\ell} (1+yq^k). \tag{2.10}$$

From (2.9) and (2.10), we obtain

$$\begin{aligned} &\prod_{k=1}^{\ell} (1+yq^k) \\ =&1+y^\ell q^{\ell(\ell+1)/2}+\frac{(1-q^\ell)(q;q)_\infty}{(q^\ell;q)_\infty}\cdot \sum_{n=1}^{\ell-1} \frac{y^n q^{n(n+1)/2}}{(1-q^n)(1-q^{\ell-n})(q;q)_{n-1}(q;q)_{\ell-n-1}}, \end{aligned}$$

which is the desired result. □

3. Some Theorems

Lemma 3.1. *If* $0<z \,\mathrm{or}\, q \leqslant 1$ *and* $z,q\in\mathbb{C}$*, then*

$$\frac{1}{1-zq} = \frac{(zq^2;q)_\infty}{(zq;q)_\infty},$$

where $(a;q)_\infty$ *denotes the q-Pochhammer Symbol.*

Proof. In previous paper [6, p. 2], we proved that

$$\begin{aligned} \left(1-\frac{a}{q}\right)\frac{(b/q;q)_\infty}{(b;q)_\infty} &= \left(1-\frac{b}{q}\right)\frac{(a/q;q)_\infty}{(a;q)_\infty} \\ \Rightarrow \frac{1-\frac{a}{q}}{1-\frac{b}{q}} &= \frac{(a/q;q)_\infty (b;q)_\infty}{(a;q)_\infty (b/q;q)_\infty}. \end{aligned} \tag{3.1}$$

Replacing q^2 by a and zq^2 by b in (3.1) and encounter

$$\frac{1-q}{1-zq}=\frac{(q;q)_\infty(zq^2;q)_\infty}{(q^2;q)_\infty(zq;q)_\infty}\Rightarrow\frac{1}{1-zq}=\frac{(q;q)_\infty(zq^2;q)_\infty}{(1-q)(q^2;q)_\infty(zq;q)_\infty}$$
$$\Leftrightarrow\frac{1}{1-zq}=\frac{(zq^2;q)_\infty}{(zq;q)_\infty},$$

which is the desired result. □

Theorem 3.2. *If $0<|q|<1$, $q\in\mathbb{C}$, and n is a positive integer, then*

$$\prod_{k=2}^{5}\frac{(q^{2n+k};q^n)_\infty}{(q^{n+k};q^n)_\infty}=\frac{(q^2;q)_n}{(1-q^2)(1-q^3)(1-q^4)(1-q^5)(q^6;q)_n}.$$

Proof. We define

$$g(q):=\frac{(q^2;q)_\infty}{(q^6;q)_\infty}=\lim_{n\to\infty}\frac{(q^2;q)_n}{(q^6;q)_n}, \tag{3.2}$$

hence,

$$g(q)=\lim_{n\to\infty}\frac{(1-q^2)(1-q^3)(1-q^4)(1-q^5)}{(1-q^{n+2})(1-q^{n+3})(1-q^{n+4})(1-q^{n+5})}. \tag{3.3}$$

On the other hand, replacing q by q^n in Lemma 3.1, we encounter

$$\frac{1}{1-zq^n}=\frac{(zq^{2n};q^n)_\infty}{(zq^n;q^n)_\infty}. \tag{3.4}$$

Setting $z=q^2$, $z=q^3$, $z=q^4$ and $z=q^5$, respectively, in (3.4) and multiplying each other, we get

$$\frac{1}{(1-q^{n+2})(1-q^{n+3})(1-q^{n+4})(1-q^{n+5})}=\frac{(q^{2n+2};q^n)_\infty(q^{2n+3};q^n)_\infty(q^{2n+4};q^n)_\infty(q^{2n+5};q^n)_\infty}{(q^{n+2};q^n)_\infty(q^{n+3};q^n)_\infty(q^{n+4};q^n)_\infty(q^{n+5};q^n)_\infty} \tag{3.5}$$

From (3.2), (3.3) and (3.5), it follows that

$$\prod_{k=2}^{5}\frac{(q^{2n+k};q^n)_\infty}{(q^{n+k};q^n)_\infty}=\frac{(q^2;q)_n}{(1-q^2)(1-q^3)(1-q^4)(1-q^5)(q^6;q)_n},$$

which is the desired result. □

Theorem 3.3. *If $0<|q|<1$, $q\in\mathbb{C}$, and n is a positive integer, then*

$$\frac{(q^{4n+1};q^{2n})_\infty}{(q^{2n+1};q^{2n})_\infty}=\frac{(q;q^2)_n}{(1-q)(q^3;q^2)_n}.$$

Proof. We define

$$g(q):=\frac{(q^2;q^2)_\infty}{(q^3;q^2)_\infty}=\lim_{n\to\infty}\frac{(q^2;q^2)_n}{(q^3;q^2)_n}, \tag{3.6}$$

hence,

$$g(q)=\lim_{n\to\infty}\frac{(1-q)(q^2;q^2)_n}{(1-q^{2n+1})(q;q^2)_n}. \tag{3.7}$$

On the other hand, replacing q by q^{2n} in Lemma 3.1, we encounter

$$\frac{1}{1-zq^{2n}}=\frac{(zq^{4n};q^{2n})_\infty}{(zq^{2n};q^{2n})_\infty}. \tag{3.8}$$

Setting $z=q$ in (3.8), we get

$$\frac{1}{1-q^{2n+1}}=\frac{(q^{4n+1};q^{2n})_\infty}{(q^{2n+1};q^{2n})_\infty}. \tag{3.9}$$

From (3.6), (3.7) and (3.9), it follows that

$$\frac{(q^{4n+1};q^{2n})_\infty}{(q^{2n+1};q^{2n})_\infty}=\frac{(q;q^2)_n}{(1-q)(q^3;q^2)_n},$$

which is the desired result. □

4. Some Applications for old Results

Theorem 4.1. *If* $0<|q|<1$, $q\in\mathbb{C}$, *and* n,ℓ *are a positive integers, such that* $n<\ell$, *then*

$$\left[\begin{array}{c}\ell\\ n\end{array}\right]_q=\frac{1-q^{\ell}}{1-q^n}\cdot\left[\begin{array}{c}\ell-1\\ n-1\end{array}\right]_q,$$

where $\left[\begin{array}{c}\ell\\ n\end{array}\right]_q$ *denotes the q-binomial coefficient.*

Proof. We proved above that (see Theorem 2.1)

$$\left[\begin{array}{c}\ell\\ n\end{array}\right]_q=\frac{(1-q^{\ell})}{(1-q^n)(1-q^{\ell-n})}\cdot\frac{(q^n;q)_\infty(q^{\ell-n};q)_\infty}{(q;q)_\infty(q^{\ell};q)_\infty}. \tag{4.1}$$

In [7], we encounter the definition of q-gamma function

$$\Gamma_q(x):=\frac{(1-q)^{1-x}(q;q)_\infty}{(q^x;q)_\infty}\Rightarrow(q^x;q)_\infty=\frac{(1-q)^{1-x}(q;q)_\infty}{\Gamma_q(x)}. \tag{4.2}$$

Substitute the right hand side of (4.2) into the right hand side of (4.1), replacing x by n, by $\ell-n$ and by ℓ, respectively, and find

$$\begin{aligned}\left[\begin{array}{c}\ell\\ n\end{array}\right]_q&=\frac{(1-q^{\ell})(1-q)^{1-n}(1-q)^{1-\ell+n}}{(1-q^n)(1-q^{\ell-n})(1-q)^{1-\ell}}\cdot\frac{\Gamma_q(\ell)}{\Gamma_q(n)\Gamma_q(\ell-n)}\\
&=\frac{(1-q^{\ell})(1-q)^{1-n}(1-q)^{1-\ell+n}}{(1-q^n)(1-q^{\ell-n})(1-q)^{1-\ell}}\cdot\frac{[\ell-1]_q!}{[n-1]_q![\ell-n-1]_q!}\\
&=\frac{(1-q^{\ell})(1-q)^{1-n}(1-q)^{1-\ell+n}[\ell-n]_q!}{(1-q^n)(1-q^{\ell-n})(1-q)^{1-\ell}[\ell-n-1]_q!}\cdot\frac{[\ell-1]_q!}{[n-1]_q![\ell-n]_q!}\\
&=\frac{(1-q^{\ell})(1-q)^{1-n}(1-q)^{1-\ell+n}(q;q)_{\ell-n}(1-q)^{\ell-n-1}}{(1-q^n)(1-q^{\ell-n})(1-q)^{1-\ell}(q;q)_{\ell-n-1}(1-q)^{\ell-n}}\cdot\left[\begin{array}{c}\ell-1\\ n-1\end{array}\right]_q\\
&=\frac{(1-q^{\ell})(1-q)^{1-n}(q;q)_{\ell-n}}{(1-q^n)(1-q^{\ell-n})(1-q)^{1-\ell}(q;q)_{\ell-n-1}(1-q)^{\ell-n}}\cdot\left[\begin{array}{c}\ell-1\\ n-1\end{array}\right]_q\\
&=\frac{1-q^{\ell}}{1-q^n}\cdot\left[\begin{array}{c}\ell-1\\ n-1\end{array}\right]_q,\end{aligned} \tag{4.3}$$

and this complete the proof. □

Corollary 4.2. *If* $0<|q|<1$, $q\in\mathbb{C}$, *and* n,ℓ *are positive integers, such that* $n<\ell$, *then*

$$\frac{(q;q)_\ell}{(q;q)_n}=\frac{1-q^{\ell}}{1-q^n}\cdot\frac{(q;q)_{\ell-1}}{(q;q)_{n-1}}.$$

Proof. Using the definition of q-binomial coefficient [8], we obtain

$$\left[\begin{array}{c}\ell\\ n\end{array}\right]_q:=\frac{(q;q)_\ell}{(q;q)_n(q;q)_{\ell-n}} \tag{4.4}$$

and

$$\left[\begin{array}{c}\ell-1\\ n-1\end{array}\right]_q=\frac{(q;q)_{\ell-1}}{(q;q)_{n-1}(q;q)_{\ell-n}}. \tag{4.5}$$

From Theorem 4.1, (4.4) and (4.5), we conclude that

$$\begin{aligned}&\frac{(q;q)_\ell}{(q;q)_n(q;q)_{\ell-n}}=\frac{1-q^{\ell}}{1-q^n}\cdot\frac{(q;q)_{\ell-1}}{(q;q)_{n-1}(q;q)_{\ell-n}}\\
&\Rightarrow\frac{(q;q)_\ell}{(q;q)_n}=\frac{1-q^{\ell}}{1-q^n}\cdot\frac{(q;q)_{\ell-1}}{(q;q)_{n-1}},\end{aligned}$$

which is the desired result. □

Corollary 4.3. *If* $0<|q|<1$, $q\in\mathbb{C}$, *and* n,ℓ *are positive integers, such that* $n<\ell$, *then*

$$\frac{(q;q)_{n-1}(q;q)_\ell}{(q;q)_n(q;q)_{\ell-1}}=\frac{(q^{2n};q^n)_\infty(q^{\ell};q^{\ell})_\infty}{(q^n;q^n)_\infty(q^{2\ell};q^{\ell})_\infty}.$$

Proof. By Lemma 3.1, for $z\to1$ and $q\to q^{\ell}$, we obtain

$$\frac{1}{1-q^{\ell}}=\frac{(q^{2\ell};q^{\ell})_\infty}{(q^{\ell};q^{\ell})_\infty}, \tag{4.6}$$

and, again, by Lemma 3.1, for $z\to 1$ and $q\to q^n$, we find

$$\frac{1}{1-q^n}=\frac{(q^{2n};q^n)_\infty}{(q^n;q^n)_\infty}. \tag{4.7}$$

From Corollary 4.2, (4.6) and (4.7), we conclude that

$$\frac{(q;q)_{n-1}(q;q)_\ell}{(q;q)_n(q;q)_{\ell-1}}=\frac{(q^{2n};q^n)_\infty(q^\ell;q^\ell)_\infty}{(q^n;q^n)_\infty(q^{2\ell};q^\ell)_\infty},$$

which is the desired result. □

5. Application for the Generating Functions of (m,k)-Capsids and (m,r_1,r_2)-Capsids

Theorem 5.1. *We have*

$$C_{m,k}(q)=\frac{1-q^m}{(1-q^k)(1-q^{m-k})}\cdot\frac{(q^{2m};q^m)_\infty}{(q^{m+k};q^m)_\infty(q^{2m-k};q^m)_\infty},$$

where $C_{m,k}(n)$ denotes the generating function of (m,k)-capsids and $(a;q)_\infty$ denotes the q-Pochhammer symbol.

Proof. In [9, Proposition 1, p. 3], Frank Garvan and Michael Schlosser define the generating function of (m,k)-capsids by

$$C_{m,k}(q):=\frac{(q^m;q^m)_\infty}{(q^k;q^m)_\infty(q^{m-k};q^m)_\infty}. \tag{5.1}$$

Replace q by q^m and z by 1 in Lemma 3.1

$$\frac{1}{1-q^m}=\frac{(q^{2m};q^m)_\infty}{(q^m;q^m)_\infty}. \tag{5.2}$$

Replace q by q^m and z by q^{k-m} in Lemma 3.1

$$\frac{1}{1-q^k}=\frac{(q^{m+k};q^m)_\infty}{(q^k;q^m)_\infty}. \tag{5.3}$$

Replace q by q^m and z by q^{-k} in Lemma 3.1

$$\frac{1}{1-q^{m-k}}=\frac{(q^{2m-k};q^m)_\infty}{(q^{m-k};q^m)_\infty}. \tag{5.4}$$

From (5.1) at (5.4), it follows that

$$C_{m,k}(q)=\frac{1-q^m}{(1-q^k)(1-q^{m-k})}\cdot\frac{(q^{2m};q^m)_\infty}{(q^{m+k};q^m)_\infty(q^{2m-k};q^m)_\infty},$$

which is the desired result. □

Theorem 5.2. *We have*

$$C_{m,k}(q)=\frac{1-q^m}{1-q^k}\cdot C_{m,m,m}(q^k,q^{-k},q)$$

or

$$C_{m,k}(q)=\frac{1-q^m}{1-q^k}\cdot C_{m,m,m}(q^{-k},q^k,q),$$

where $C_{m,k}(n)$ denotes the generatinf function of (m,k)-capsids and $C_{m,r_1,r_2}(x,y,q)$ denotes the generatinf function of (m,r_1,r_2)-capsids.

Proof. In [9, p. 10, Proposition 2], Frank Garvan and Michael Schlosser define the generating function of (m,r_1,r_2)-capsids by

$$C_{m,r_1,r_2}(x,y,q)=\frac{(xyq^{r_1+r_2};q^m)_\infty}{(xq^{r_1};q^m)_\infty(yq^{r_2};q^m)_\infty}. \tag{5.5}$$

Let $r_1 = m, r_2 = m, x = q^k$ and $y = q^{-k}$ in (5.5) and encounter

$$C_{m,m,m}(q^k, \quad q^{-k}, \quad q) \quad = \quad \frac{(q^{2m}; q^m)_\infty}{(q^{m+k}; q^m)_\infty (q^{m-k}; q^m)_\infty} \quad \Rightarrow \quad (q^{m-k}; \quad q^m)_\infty C_{m,m,m}(q^k, \quad q^{-k}, \quad q) \quad = \frac{(q^{2m}; q^m)_\infty}{(q^{m+k}; q^m)_\infty} \tag{5.6}$$

or, setting $r_1 = m, r_2 = m, x = q^{-k}$ and $y = q^k$ in (5.5), we find

$$C_{m,m,m}(q^{-k}, \quad q^k, \quad q) \quad = \quad \frac{(q^{2m}; q^m)_\infty}{(q^{m-k}; q^m)_\infty (q^{m+k}; q^m)_\infty} \quad \Rightarrow \quad (q^{m-k}; \quad q^m)_\infty C_{m,m,m}(q^{-k}, \quad q^k, \quad q) \quad = \frac{(q^{2m}; q^m)_\infty}{(q^{m+k}; q^m)_\infty}. \tag{5.7}$$

Comparing (5.6) and (5.7), we conclude easily that

$$C_{m,m,m}(q^k, q^{-k}, q) = C_{m,m,m}(q^{-k}, q^k, q). \tag{5.8}$$

From Theorem 5.1, (5.7) and (5.8), we obtain

$$C_{m,k}(q) = \frac{1 - q^m}{(1 - q^k)(1 - q^{m-k})} \cdot \frac{(q^{m-k}; q^m)_\infty}{(q^{2m-k}; q^m)_\infty} \cdot C_{m,m,m}(q^k, q^{-k}, q) \tag{5.9}$$

or

$$C_{m,k}(q) = \frac{1 - q^m}{(1 - q^k)(1 - q^{m-k})} \cdot \frac{(q^{m-k}; q^m)_\infty}{(q^{2m-k}; q^m)_\infty} \cdot C_{m,m,m}(q^{-k}, q^k, q). \tag{5.10}$$

Replace q by q^m and z by q^{-k} in Lemma 3.1, we get

$$\frac{1}{1 - q^{m-k}} = \frac{(q^{2m-k}; q^m)_\infty}{(q^{m-k}; q^m)_\infty} \Rightarrow \frac{(q^{m-k}; q^m)_\infty}{(q^{2m-k}; q^m)_\infty} = 1 - q^{m-k}. \tag{5.11}$$

Now, from (5.9) at (5.11), we finally deduce that

$$C_{m,k}(q) = \frac{1 - q^m}{1 - q^k} \cdot C_{m,m,m}(q^k, q^{-k}, q)$$

or

$$C_{m,k}(q) = \frac{1 - q^m}{1 - q^k} \cdot C_{m,m,m}(q^{-k}, q^k, q),$$

which are the desired results. □

Theorem 5.3. *If*

$$P_{m,k}(q) := (q^k; q^m)_\infty (q^{m-k}; q^m)_\infty$$

and

$$P_{m,m,m}(q^k, q^{-k}, q) = (q^{m+k}; q^m)_\infty (q^{m-k}; q^m)_\infty,$$

then

(i)

$$\frac{1 - q^k}{P_{m,k}(q)} = \frac{1}{P_{m,m,m}(q^k, q^{-k}, q)}$$

and

(ii)

$$\frac{(q^{m+k}; q^m)_\infty}{(q^k; q^m)_\infty} = \frac{1}{1 - q^k}.$$

Proof. By Theorem 5.2, we obtain

$$C_{m,k}(q) = q^k C_{m,k}(q) + C_{m,m,m}(q^k, q^{-k}, q) - q^m C_{m,m,m}(q^k, q^{-k}, q). \tag{5.12}$$

Hence, following Garvan and Schlosser, we define

$$P_{m,k}(q) := (q^k; q^m)_\infty (q^{m-k}; q^m)_\infty \tag{5.13}$$

and define

$$P_{m,r_1,r_2}(x, y, q) := (x\, q^{r_1}; q^m)_\infty (y\, q^{r_2}; q^m)_\infty. \tag{5.14}$$

Setting $r_1 = m, r_2 = m, x = q^k$ and $y = q^{-k}$ in (5.14), we have

$$P_{m,m,m}(q^k, q^{-k}, q) = (q^{m+k}; q^m)_\infty (q^{m-k}; q^m)_\infty. \tag{5.15}$$

From (5.1), (5.6), (5.12), (5.13) and (5.15), we find

$$\frac{(q^m;q^m)_\infty}{P_{m,k}(q)} = q^k\frac{(q^m;q^m)_\infty}{P_{m,k}(q)} + \frac{(q^{2m};q^m)_\infty}{P_{m,m,m}(q^k,q^{-k},q)} - q^m\frac{(q^{2m};q^m)_\infty}{P_{m,m,m}(q^k,q^{-k},q)}. \tag{5.16}$$

Divide both members of (5.6) by $(q^m;q^m)_\infty$

$$\frac{1}{P_{m,k}(q)} = \frac{q^k}{P_{m,k}(q)} + \frac{(q^{2m};q^m)_\infty}{(q^m;q^m)_\infty P_{m,m,m}(q^k,q^{-k},q)} - q^m\frac{(q^{2m};q^m)_\infty}{(q^m;q^m)_\infty P_{m,m,m}(q^k,q^{-k},q)}. \tag{5.17}$$

Replace q by q^m and z by 1 in Lemma 3.1

$$\frac{1}{1-q^m} = \frac{(q^{2m};q^m)_\infty}{(q^m;q^m)_\infty}. \tag{5.18}$$

From (5.17) and (5.18), we get

$$\frac{1}{P_{m,k}(q)} = \frac{q^k}{P_{m,k}(q)} + \frac{1}{(1-q^m)P_{m,m,m}(q^k,q^{-k},q)} - \frac{q^m}{(1-q^m)P_{m,m,m}(q^k,q^{-k},q)}. \tag{5.19}$$

Multiply both member of (5.19) by $1-q^m$ and rearranging

$$\frac{(1-q^m)(1-q^k)}{P_{m,k}(q)} = \frac{1-q^m}{P_{m,m,m}(q^k,q^{-k},q)} \Rightarrow \frac{1-q^k}{P_{m,k}(q)} = \frac{1}{P_{m,m,m}(q^k,q^{-k},q)}. \tag{5.20}$$

From (5.13), (5.14) and (5.20), it follows that

$$\frac{1-q^k}{P_{m,k}(q)} = \frac{1}{P_{m,m,m}(q^k,q^{-k},q)} \Rightarrow \frac{1-q^k}{(q^k;q^m)_\infty(q^{m-k};q^m)_\infty} = \frac{1}{(q^{m+k};q^m)_\infty(q^{m-k};q^m)_\infty}. \tag{5.21}$$

Eliminate $(q^{m-k};q^m)_\infty$ in both members of (5.21) and encounter

$$\frac{(q^{m+k};q^m)_\infty}{(q^k;q^m)_\infty} = \frac{1}{1-q^k}.$$

This completes the proof. □

Corollary 5.4. *If*

$$P_{m,k}(q) := (q^k;q^m)_\infty(q^{m-k};q^m)_\infty,$$

then

(i)

$$\frac{(1-q^k)(1-q^{m-k})}{P_{m,k}(q)} = \frac{1}{(q^{m+k};q^m)_\infty(q^{2m-k};q^m)_\infty},$$

(ii)

$$\frac{1}{P_{m,k}(q)} = \frac{q^k}{P_{m,k}(q)} + \frac{1}{(q^{m+k};q^m)_\infty(q^{m-k};q^m)_\infty}$$

and

(iii)

$$\frac{1}{P_{m,k}(q)} = \frac{q^{m-k}}{P_{m,k}(q)} + \frac{1}{(q^k;q^m)_\infty(q^{2m-k};q^m)_\infty}.$$

Proof. From Theorem 5.3.ii, we obtain

$$\frac{1-q^k}{(q^k;q^m)_\infty} = \frac{1}{(q^{m+k};q^m)_\infty}. \tag{5.22}$$

and, replacing k by ℓ, in (5.22), we have

$$\frac{1-q^\ell}{(q^\ell;q^m)_\infty} = \frac{1}{(q^{m+\ell};q^m)_\infty}. \tag{5.23}$$

Multiply (5.22) by (5.23), member by member, we find

$$\frac{(1-q^k)(1-q^\ell)}{(q^k;q^m)_\infty(q^\ell;q^m)_\infty} = \frac{1}{(q^{m+k};q^m)_\infty(q^{m+\ell};q^m)_\infty}. \tag{5.24}$$

Set $\ell = m - k$ into (5.24)

$$\frac{(1-q^k)(1-q^{m-k})}{(q^k;q^m)_\infty(q^{m-k};q^m)_\infty} = \frac{1}{(q^{m+k};q^m)_\infty(q^{2m-k};q^m)_\infty}. \tag{5.25}$$

From (5.13) and (5.25), we obtain

$$\frac{(1-q^k)(1-q^{m-k})}{P_{m,k}(q)} = \frac{1}{(q^{m+k};q^m)_\infty(q^{2m-k};q^m)_\infty}. \tag{5.26}$$

From Theorem 5.3.i and (5.15), we have

$$\frac{1-q^k}{P_{m,k}(q)} = \frac{1}{(q^{m+k};q^m)_\infty(q^{m-k};q^m)_\infty}. \tag{5.27}$$

From (5.26) and (5.27), we find

$$\frac{q^m - q^{m-k}}{P_{m,k}(q)} = \frac{1}{(q^{m+k};q^m)_\infty(q^{2m-k};q^m)_\infty} - \frac{1-q^k}{P_{m,k}(q)} \Rightarrow \frac{q^{m-k}(1-q^k)}{P_{m,k}(q)} = \frac{1}{(q^{m+k};q^m)_\infty(q^{m-k};q^m)_\infty} - \frac{1}{(q^{m+k};q^m)_\infty(q^{2m-k};q^m)_\infty}. \tag{5.28}$$

Using the following identity, see Theorem 5.3.ii,

$$\frac{(q^{m+k};q^m)_\infty}{(q^k;q^m)_\infty} = \frac{1}{1-q^k},$$

in (5.28), we obtain

$$\begin{aligned} \frac{q^{m-k}(1-q^k)}{P_{m,k}(q)} &= \frac{1-q^k}{(q^k;q^m)_\infty(q^{m-k};q^m)_\infty} - \frac{1-q^k}{(q^k;q^m)_\infty(q^{2m-k};q^m)_\infty} \\ \Rightarrow \frac{q^{m-k}}{P_{m,k}(q)} &= \frac{1}{(q^k;q^m)_\infty(q^{m-k};q^m)_\infty} - \frac{1}{(q^k;q^m)_\infty(q^{2m-k};q^m)_\infty} \\ \Leftrightarrow \frac{1}{P_{m,k}(q)} &= \frac{q^{m-k}}{P_{m,k}(q)} + \frac{1}{(q^k;q^m)_\infty(q^{2m-k};q^m)_\infty}, \end{aligned}$$

which are the desired results. □

Corollary 5.5. *For any complex numbers* a,b,c,d,e,f *with* $0<|a|,|b|,|c|,|d|,|e|,|f|<1$, *then*

$$\frac{(a;b)_\infty(c;d)_\infty}{(a;e)_\infty(c;f)_\infty} = \frac{(ab;b)_\infty(cd;d)_\infty}{(ae;e)_\infty(cf;f)_\infty}.$$

Proof. Replace m by n and k by ℓ into (5.27), and obtain

$$\frac{1-q^\ell}{P_{n,\ell}(q)} = \frac{1}{(q^{n+\ell};q^n)_\infty(q^{n-\ell};q^n)_\infty}. \tag{5.29}$$

Multiply (5.27) by (5.29)

$$\begin{aligned} &\frac{(1-q^k)(1-q^\ell)}{P_{m,k}(q)P_{n,\ell}(q)} = \frac{1}{(q^{m+k};q^m)_\infty(q^{m-k};q^m)_\infty(q^{n+\ell};q^n)_\infty(q^{n-\ell};q^n)_\infty} \\ \Rightarrow &\frac{1}{P_{m,k}(q)P_{n,\ell}(q)} - \frac{1}{(q^{m+k};q^m)_\infty(q^{m-k};q^m)_\infty(q^{n+\ell};q^n)_\infty(q^{n-\ell};q^n)_\infty} = \frac{q^k+q^\ell-q^{k+\ell}}{P_{m,k}(q)P_{n,\ell}(q)}. \end{aligned} \tag{5.30}$$

From (5.13) and (5.30), after simplification, we have

$$\frac{1}{(q^k;q^m)_\infty(q^\ell;q^n)_\infty} - \frac{1}{(q^{m+k};q^m)_\infty(q^{n+\ell};q^n)_\infty} = \frac{q^k+q^\ell-q^{k+\ell}}{(q^k;q^m)_\infty(q^\ell;q^n)_\infty}. \tag{5.31}$$

Replace q^k by a, q^m by b, q^ℓ by c and q^n by d in (5.31)

$$\frac{1}{(a;b)_\infty(c;d)_\infty} - \frac{1}{(ab;b)_\infty(cd;d)_\infty} = \frac{a+c-ac}{(a;b)_\infty(c;d)_\infty}. \tag{5.32}$$

Replace b by e and d by f in (5.32)

$$\frac{1}{(a;e)_\infty(c;f)_\infty} - \frac{1}{(ae;e)_\infty(cf;f)_\infty} = \frac{a+c-ac}{(a;e)_\infty(c;f)_\infty}. \tag{5.33}$$

Eliminate $a+c-ac$ in (5.32) and (5.33)

$$\frac{(a;b)_\infty(c;d)_\infty}{(a;e)_\infty(c;f)_\infty}=\frac{(ab;b)_\infty(cd;d)_\infty}{(ae;e)_\infty(cf;f)_\infty},$$

which is the desired result. □

Example 5.6. Set $a=q, b=q^2, c=q^3, d=q^4, e=q^5$ and $f=q^6$ in previous Corollary, and get

$$\frac{(q;q^2)_\infty(q^3;q^4)_\infty}{(q;q^5)_\infty(q^3;q^6)_\infty}=\frac{(q^3;q^2)_\infty(q^7;q^4)_\infty}{(q^6;q^5)_\infty(q^9;q^6)_\infty}.$$

Example 5.7. The Corollary above also serves to find this type of equality

$$\begin{aligned}
\frac{(q^4;q^3)_\infty(q^7;q^4)_\infty}{(q^7;q^5)_\infty(q^9;q^7)_\infty}&=\frac{(qq^3;q^3)_\infty(q^3q^4;q^4)_\infty}{(q^2q^5;q^5)_\infty(q^2q^7;q^7)_\infty}\\
&=\frac{(q^3q^4;q^4)_\infty(q^3q^7;q^7)_\infty}{(q^3q^7;q^7)_\infty(q^2q^7;q^7)_\infty}\cdot\frac{(q^2q^3;q^3)_\infty(qq^3;q^3)_\infty}{(q^2q^3;q^3)_\infty(qq^5;q^5)_\infty}\cdot\frac{(qq^5;q^5)_\infty(q^2q^4;q^4)_\infty}{(q^2q^5;q^5)_\infty(q^2q^4;q^4)_\infty}\\
&=\frac{(q^3q^4;q^4)_\infty(q^2q^3;q^3)_\infty}{(q^3q^7;q^7)_\infty(q^2q^7;q^7)_\infty}\cdot\frac{(qq^3;q^3)_\infty(q^2q^4;q^4)_\infty}{(qq^5;q^5)_\infty(q^2q^5;q^5)_\infty}\cdot\frac{(q^3q^7;q^7)_\infty(qq^5;q^5)_\infty}{(q^2q^3;q^3)_\infty(q^2q^4;q^4)_\infty}\\
&=\frac{(q;q^3)_\infty(q^2;q^3)_\infty(q^2;q^4)_\infty(q^3;q^4)_\infty}{(q;q^5)_\infty(q^2;q^5)_\infty(q^2;q^7)_\infty(q^3;q^7)_\infty}\cdot\frac{(q^{10};q^7)_\infty(q^6;q^5)_\infty}{(q^5;q^3)_\infty(q^6;q^4)_\infty}\\
\Rightarrow\frac{(q;q^3)_\infty(q^2;q^3)_\infty(q^2;q^4)_\infty(q^3;q^4)_\infty}{(q;q^5)_\infty(q^2;q^5)_\infty(q^2;q^7)_\infty(q^3;q^7)_\infty}&=\frac{(q^4;q^3)_\infty(q^5;q^3)_\infty(q^6;q^4)_\infty(q^7;q^4)_\infty}{(q^6;q^5)_\infty(q^7;q^5)_\infty(q^9;q^7)_\infty(q^{10};q^7)_\infty}.
\end{aligned}$$

Theorem 5.8. *For any complex number q, with $0<|q|<1$, $n,r,x\in\mathbb{N}^+$, $r<n$ and $x>0$, then*

$$\begin{bmatrix} n \\ r \end{bmatrix}_q=\prod_{k=0}^{x-1}\frac{(q^{k+r+1};q^x)_\infty(q^{k+n-r+1};q^x)_\infty}{(q^{k+1};q^x)_\infty(q^{k+n+1};q^x)_\infty},$$

where $\begin{bmatrix} n \\ r \end{bmatrix}_q$ denotes the q-binomial coefficient and $(a;q)_\infty$ denotes the q-Pochhammer symbol.

Proof. In previous paper [6, p. 8], we prove that

$$\begin{bmatrix} n \\ r \end{bmatrix}_q=\frac{(q^{r+1};q)_\infty(q^{n-r+1};q)_\infty}{(q;q)_\infty(q^{n+1};q)_\infty}. \tag{5.34}$$

On the other hand, we know [10, p. 13, Entry 1(iii)] that

$$(a;q)_\infty=\prod_{k=0}^{x-1}(aq^k;q^x)_\infty. \tag{5.35}$$

Replace a by q in (5.35)

$$(q;q)_\infty=\prod_{k=0}^{x-1}(q^{k+1};q^x)_\infty. \tag{5.36}$$

Replace a by q^{n+1} in (5.35)

$$(q^{n+1};q)_\infty=\prod_{k=0}^{x-1}(q^{k+n+1};q^x)_\infty. \tag{5.37}$$

Replace a by q^{r+1} in (5.35)

$$(q^{r+1};q)_\infty=\prod_{k=0}^{x-1}(q^{k+r+1};q^x)_\infty. \tag{5.38}$$

Replace a by q^{n-r+1} in (5.35)

$$(q^{n-r+1};q)_\infty=\prod_{k=0}^{x-1}(q^{k+n-r+1};q^x)_\infty. \tag{5.39}$$

From (5.34), (5.36) at (5.39), we conclude that

$$\begin{bmatrix} n \\ r \end{bmatrix}_q=\prod_{k=0}^{x-1}\frac{(q^{k+r+1};q^x)_\infty(q^{k+n-r+1};q^x)_\infty}{(q^{k+1};q^x)_\infty(q^{k+n+1};q^x)_\infty},$$

which is the desired result. □

Example 5.9. Set $n=5$, $r=2$ and $x=3$ in Theorem 5.8

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix}_q = \prod_{k=0}^{2} \frac{(q^{k+3};q^3)_\infty (q^{k+4};q^3)_\infty}{(q^{k+1};q^3)_\infty (q^{k+6};q^3)_\infty} = \frac{(q^4;q^3)_\infty}{(q;q^3)_\infty} \cdot \frac{(q^4;q^3)_\infty (q^5;q^3)_\infty}{(q^2;q^3)_\infty (q^7;q^3)_\infty} \cdot \frac{(q^5;q^3)_\infty}{(q^8;q^3)_\infty}$$
$$\Rightarrow \frac{(q^4;q^3)_\infty^2 (q^5;q^3)_\infty^2}{(q;q^3)_\infty (q^2;q^3)_\infty (q^7;q^3)_\infty (q^8;q^3)_\infty} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q$$
$$\Leftrightarrow \frac{(q^4;q^3)_\infty^2 (q^5;q^3)_\infty^2}{(q;q^3)_\infty (q^2;q^3)_\infty (q^7;q^3)_\infty (q^8;q^3)_\infty} = (1+q^2)(1+q+q^2+q^3+q^4).$$

Example 5.10. Set $n=2$, $r=1$ and $x=5$ in Theorem 5.8

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = \prod_{k=0}^{4} \frac{(q^{k+2};q^5)_\infty^2}{(q^{k+1};q^5)_\infty (q^{k+3};q^5)_\infty}$$
$$= \frac{(q^2;q^5)_\infty^2}{(q;q^5)_\infty (q^3;q^5)_\infty} \cdot \frac{(q^3;q^5)_\infty^2}{(q^2;q^5)_\infty (q^4;q^5)_\infty} \cdot \frac{(q^4;q^5)_\infty^2}{(q^3;q^5)_\infty (q^5;q^5)_\infty} \cdot \frac{(q^5;q^5)_\infty^2}{(q^4;q^5)_\infty (q^6;q^5)_\infty} \cdot \frac{(q^6;q^5)_\infty^2}{(q^5;q^5)_\infty (q^7;q^5)_\infty}$$
$$\Rightarrow \frac{(q^2;q^5)_\infty}{(q;q^5)_\infty} = (1+q) \frac{(q^7;q^5)_\infty}{(q^6;q^5)_\infty}.\text{ok!}$$

Example 5.11. Set $n=5$, $r=2$ and $x=5$ in Theorem 5.8

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix}_q = \prod_{k=0}^{4} \frac{(q^{k+3};q^5)_\infty (q^{k+4};q^5)_\infty}{(q^{k+1};q^5)_\infty (q^{k+6};q^5)_\infty}$$
$$= \frac{(q^3;q^5)_\infty (q^4;q^5)_\infty}{(q;q^5)_\infty (q^6;q^5)_\infty} \cdot \frac{(q^4;q^5)_\infty (q^5;q^5)_\infty}{(q^2;q^5)_\infty (q^7;q^5)_\infty} \cdot \frac{(q^5;q^5)_\infty (q^6;q^5)_\infty}{(q^3;q^5)_\infty (q^8;q^5)_\infty} \cdot \frac{(q^6;q^5)_\infty (q^7;q^5)_\infty}{(q^4;q^5)_\infty (q^9;q^5)_\infty} \cdot \frac{(q^7;q^5)_\infty (q^8;q^5)_\infty}{(q^5;q^5)_\infty (q^{10};q^5)_\infty}$$
$$\Rightarrow \frac{(q^4;q^5)_\infty (q^5;q^5)_\infty}{(q;q^5)_\infty (q^2;q^5)_\infty} = (1+q^2)(1+q+q^2+q^3+q^4) \frac{(q^9;q^5)_\infty (q^{10};q^5)_\infty}{(q^6;q^5)_\infty (q^7;q^5)_\infty}.$$

6. Expansion for Some Bilateral Lambert Series

In this section, we demonstrate how the theory previously developed can be used for the expansion of some bilateral Lambert series.

Theorem 6.1. *For* $|q|<1$, *we have*

$$\sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{7n+1}} = \frac{(q^2;q^7)_\infty (q^5;q^7)_\infty (q^7;q^7)_\infty^2}{(q;q^7)_\infty^2 (q^6;q^7)_\infty^2}.$$

Proof. Define the bilateral Lambert series

$$L(q) := \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{7n+1}}. \tag{6.1}$$

On the other hand, we know the Ramanujan's notable ${}_1\psi_1$ summation [11, p. 118, (4.4.6)]

$${}_1\psi_1(a;b;q,z) := \sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n = \frac{(az;q)_\infty (q/(az);q)_\infty (q;q)_\infty (b/a;q)_\infty}{(z;q)_\infty (b/(az);q)_\infty (b;q)_\infty (q/a;q)_\infty}, \tag{6.2}$$

for any complex numbers a,b,z with $|z|<1$ and $|b/a|<1$.

Replace k by $7n+1$ and m by 7 in Theorem 5.3.ii

$$\frac{1}{1-q^{7n+1}} = \frac{(q^{7n+8};q^7)_\infty}{(q^{7n+1};q^7)_\infty}, \tag{6.3}$$

We know the identity [11, p. 118, (4.4.5)], for all integers n,

$$(a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty} \Rightarrow (aq^n;q)_\infty = \frac{(a;q)_\infty}{(a;q)_n}. \tag{6.4}$$

Replace q by q^7 and a by q^8 in (6.4)

$$(q^{7n+8};q^7)_\infty = \frac{(q^8;q^7)_\infty}{(q^8;q^7)_n}. \tag{6.5}$$

Replace q by q^7 and a by q^2 in (6.4)

$$(q^{7n+1};q^7)_\infty = \frac{(q;q^7)_\infty}{(q;q^7)_n}. \tag{6.6}$$

From (6.1), (6.3), (6.5) and (6.6), it follows that

$$L(q) = \sum_{n=-\infty}^{\infty} \frac{(q^{7n+8};q^7)_\infty}{(q^{7n+1};q^7)_\infty} q^n = \frac{(q^8;q^7)_\infty}{(q;q^7)_\infty} \sum_{n=-\infty}^{\infty} \frac{(q;q^7)_n}{(q^8;q^7)_n} q^n. \tag{6.7}$$

Replace q by q^7, a by q, b by q^8 and z by q in (6.2)

$$\psi_1(q;q^8;q^7,q) = \sum_{n=-\infty}^{\infty} \frac{(q;q^7)_n}{(q^8;q^7)_n} q^n = \frac{(q^2;q^7)_\infty (q^5;q^7)_\infty (q^7;q^7)_\infty (q^7;q^7)_\infty}{(q;q^7)_\infty (q^6;q^7)_\infty (q^8;q^7)_\infty (q^6;q^7)_\infty}. \tag{6.8}$$

From (6.7) and (6.8), we find

$$L(q) = \frac{(q^2;q^7)_\infty (q^5;q^7)_\infty (q^7;q^7)_\infty^2}{(q;q^7)_\infty^2 (q^6;q^7)_\infty^2},$$

which is the desired result. □

Theorem 6.2. *For $|q|<1$, we have*

$$\sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{7n+2}} = \frac{(q^3;q^7)_\infty (q^4;q^7)_\infty (q^7;q^7)_\infty^2}{(q;q^7)_\infty (q^2;q^7)_\infty (q^5;q^7)_\infty (q^6;q^7)_\infty}.$$

Proof. Define the bilateral Lambert series

$$H(q) := \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{7n+2}}. \tag{6.9}$$

On the other hand, we know the Ramanujan's notable ${}_1\psi_1$ summation [11, p. 118, (4.4.6)]

$${}_1\psi_1(a;b;q,z) := \sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n = \frac{(az;q)_\infty (q/az;q)_\infty (q;q)_\infty (b/a;q)_\infty}{(z;q)_\infty (b/az;q)_\infty (b;q)_\infty (q/a;q)_\infty}, \tag{6.10}$$

for any complex numbers a,b,z with $|z|<1$ and $|b/a|<1$.

Replace k by $7n+2$ and m by 7 in Theorem 5.3.ii

$$\frac{1}{1-q^{7n+2}} = \frac{(q^{7n+9};q^7)_\infty}{(q^{7n+2};q^7)_\infty}, \tag{6.11}$$

We know the identity [11, p. 118, (4.4.5)], for all integers n,

$$(a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty} \Rightarrow (aq^n;q)_\infty = \frac{(a;q)_\infty}{(a;q)_n}. \tag{6.12}$$

Replace q by q^7 and a by q^9 in (6.12)

$$(q^{7n+9};q^7)_\infty = \frac{(q^9;q^7)_\infty}{(q^9;q^7)_n}. \tag{6.13}$$

Replace q by q^7 and a by q^2 in (6.12)

$$(q^{7n+2};q^7)_\infty = \frac{(q^2;q^7)_\infty}{(q^2;q^7)_n}. \tag{6.14}$$

From (6.9), (6.11), (6.13) and (6.14), it follows that

$$H(q) = \sum_{n=-\infty}^{\infty} \frac{(q^{7n+9};q^7)_\infty}{(q^{7n+2};q^7)_\infty} q^n = \frac{(q^9;q^7)_\infty}{(q^2;q^7)_\infty} \sum_{n=-\infty}^{\infty} \frac{(q^2;q^7)_n}{(q^9;q^7)_n} q^n. \tag{6.15}$$

Replace q by q^7, a by q^2, b by q^9 and z by q in (6.10)

$${}_1\psi_1(q^2;q^9;q^7,q) = \sum_{n=-\infty}^{\infty} \frac{(q^2;q^7)_n}{(q^9;q^7)_n} q^n = \frac{(q^3;q^7)_\infty (q^4;q^7)_\infty (q^7;q^7)_\infty^2}{(q;q^7)_\infty (q^5;q^7)_\infty (q^6;q^7)_\infty (q^9;q^7)_\infty}. \tag{6.16}$$

From (6.15) and (6.16), we find

$$H(q) = \frac{(q^3;q^7)_\infty (q^4;q^7)_\infty (q^7;q^7)_\infty^2}{(q;q^7)_\infty (q^2;q^7)_\infty (q^5;q^7)_\infty (q^6;q^7)_\infty},$$

which is the desired result. □

Remark 6.3. In [12, p. 59, (3.3.12)], we encounter the following formula, for $|q|<|x|<1$ and any number y,

$$\frac{(xy;q)_\infty(q/(xy);q)_\infty(q;q)_\infty^2}{(x;q)_\infty(q/x;q)_\infty(y;q)_\infty(q/y;q)_\infty}=\sum_{n=-\infty}^{\infty}\frac{x^n}{1-yq^n}. \tag{6.17}$$

If we replace q by q^7, y by q and x by q, then, (6.17) reduces to the formula in Theorem 6.1; therewithal, if we replace q by q^7, y by q^2 and x by q, then, (6.17) reduces to the formula in Theorem 6.2.

7. More Elementary Identities

Theorem 7.1. *For any complex numbers a,b,c,d with $0<|a|,|b|,|c|,|d|<1$ and $|d|>|c|$ and $|b|>|a|$, then*

$$\frac{(a;b)_\infty(cd;d)_\infty}{(c;d)_\infty(ab;b)_\infty}=\frac{(a;b/a)_\infty(d;d/c)_\infty}{(b;b/a)_\infty(c;d/c)_\infty}.$$

Proof. We consider the identity, see Theorem 5.3.ii,

$$\frac{(q^{m+k};q^m)_\infty}{(q^k;q^m)_\infty}=\frac{1}{1-q^k}\Rightarrow(q^k;q^m)_\infty=(1-q^k)(q^{m+k};q^m)_\infty. \tag{7.1}$$

Let $m\to m-k$ in (7.1) and encounter

$$\frac{(q^m;q^{m-k})_\infty}{(q^k;q^{m-k})_\infty}=\frac{1}{1-q^k}\Rightarrow(q^k;q^{m-k})_\infty=(1-q^k)(q^m;q^{m-k})_\infty. \tag{7.2}$$

Eliminate $1-q^k$ in (7.1) and (7.2)

$$(q^k;q^m)_\infty=\frac{(q^k;q^{m-k})_\infty(q^{m+k};q^m)_\infty}{(q^m;q^{m-k})_\infty}. \tag{7.3}$$

Replace k by ℓ and m by n in (7.3)

$$(q^\ell;q^n)_\infty=\frac{(q^\ell;q^{n-\ell})_\infty(q^{n+\ell};q^n)_\infty}{(q^n;q^{n-\ell})_\infty}. \tag{7.4}$$

Divide (7.3) by (7.4) and find

$$\frac{(q^k;q^m)_\infty}{(q^\ell;q^n)_\infty}=\frac{(q^k;q^{m-k})_\infty(q^{m+k};q^m)_\infty(q^n;q^{n-\ell})_\infty}{(q^m;q^{m-k})_\infty(q^{n+\ell};q^n)_\infty(q^\ell;q^{n-\ell})_\infty}. \tag{7.5}$$

Replace q^k by a, q^m by b, q^ℓ by c and q^n by d in (7.5)

$$\frac{(a;b)_\infty}{(c;d)_\infty}=\frac{(a;b/a)_\infty(ab;b)_\infty(d;d/c)_\infty}{(b;b/a)_\infty(cd;d)_\infty(c;d/c)_\infty}\Leftrightarrow\frac{(a;b)_\infty(cd;d)_\infty}{(c;d)_\infty(ab;b)_\infty}=\frac{(a;b/a)_\infty(d;d/c)_\infty}{(b;b/a)_\infty(c;d/c)_\infty},$$

which is the desired result. □

Example 7.2. Set $k=1$ and $m=3$ in (7.3)

$$\frac{(q;q^3)_\infty}{(q;q^2)_\infty}=\frac{(q^4;q^3)_\infty}{(q^3;q^2)_\infty}.$$

Theorem 7.3. *For any complex numbers a,b,c,d,e,f with $0<|a|,|b|,|c|,|d|,|e|,|f|<1$ and $|d|>|c|$ and $|b|>|a|$, then*

$$\frac{(a;e)_\infty(cf;f)_\infty}{(c;f)_\infty(ae;e)_\infty}=\frac{(a;b/a)_\infty(d;d/c)_\infty}{(b;b/a)_\infty(c;d/c)_\infty}.$$

Proof. In Corollary 5.5, replace f by d and d by f and encounter

$$\frac{(a;b)_\infty(c;f)_\infty}{(a;e)_\infty(c;d)_\infty}=\frac{(ab;b)_\infty(cf;f)_\infty}{(ae;e)_\infty(cd;d)_\infty}\Rightarrow\frac{(a;b)_\infty}{(c;d)_\infty}=\frac{(a;e)_\infty(ab;b)_\infty(cf;f)_\infty}{(c;f)_\infty(cd;d)_\infty(ae;e)_\infty}. \tag{7.6}$$

Substitute the right hand side of (7.6) in the left hand side of the Theorem 7.1 and find

$$\frac{(a;e)_\infty(cf;f)_\infty}{(c;f)_\infty(ae;e)_\infty}=\frac{(a;b/a)_\infty(d;d/c)_\infty}{(b;b/a)_\infty(c;d/c)_\infty},$$

which is the desired result. □

Example 7.4. Set $a=q, b=q^2, c=q^3, d=q^4, e=q^5$ and $f=q^6$ in previous Theorem, and get

$$\frac{(q;q^5)_\infty(q^9;q^6)_\infty}{(q^6;q^5)_\infty(q^3;q^6)_\infty}=\frac{(q;q)_\infty(q^4;q)_\infty}{(q^2;q)_\infty(q^3;q)_\infty}.$$

Theorem 7.5. *For any complex numbers a,b,c with $0<|a|,|b|,|c|<1$, then*

$$1-ab=\frac{(a;c)_\infty}{(ac;c)_\infty}+\frac{(b;c)_\infty}{(bc;c)_\infty}-\frac{(a;c)_\infty(b;c)_\infty}{(ac;c)_\infty(bc;c)_\infty}.$$

Proof. In Theorem 5.3.ii, replace q^k by a and q^m by c, finding

$$\frac{(ac;c)_\infty}{(a;c)_\infty}=\frac{1}{1-a}\Rightarrow 1-a=\frac{(a;c)_\infty}{(ac;c)_\infty}. \tag{7.7}$$

Replace a by b in (7.7)

$$\frac{(bc;c)_\infty}{(b;c)_\infty}=\frac{1}{1-b}\Rightarrow 1-b=\frac{(b;c)_\infty}{(bc;c)_\infty}. \tag{7.8}$$

Multiply (7.7) by (7.8) and encounter

$$1-a-b+ab=\frac{(a;c)_\infty(b;c)_\infty}{(ac;c)_\infty(bc;c)_\infty}\Rightarrow 1-a+1-b-(1-ab)=\frac{(a;c)_\infty(b;c)_\infty}{(ac;c)_\infty(bc;c)_\infty}. \tag{7.9}$$

From (7.7), (7.8) and (7.9), it follows that

$$1-ab=\frac{(a;c)_\infty}{(ac;c)_\infty}+\frac{(b;c)_\infty}{(bc;c)_\infty}-\frac{(a;c)_\infty(b;c)_\infty}{(ac;c)_\infty(bc;c)_\infty},$$

which is the desired result. □

Example 7.6. Set $a=q, b=q^2$ and $c=q^3$ in previous Theorem, and get

$$1-q^3=\frac{(q;q^3)_\infty}{(q^4;q^3)_\infty}+\frac{(q^2;q^3)_\infty}{(q^5;q^3)_\infty}-\frac{(q;q^3)_\infty(q^2;q^3)_\infty}{(q^4;q^3)_\infty(q^5;q^3)_\infty}.$$

Example 7.7. Set $a=q, b=q^2$ and $c=q^4$ in previous Theorem, and get

$$1-q^3=\frac{(q;q^4)_\infty}{(q^5;q^4)_\infty}+\frac{(q^2;q^4)_\infty}{(q^6;q^4)_\infty}-\frac{(q;q^4)_\infty(q^2;q^4)_\infty}{(q^5;q^4)_\infty(q^6;q^4)_\infty}.$$

Example 7.8. Eliminate $1-q^3$ in the Examples 7.6 and 7.7, and get

$$\frac{(q;q^3)_\infty(q^2;q^3)_\infty}{(q^4;q^3)_\infty(q^5;q^3)_\infty}-\frac{(q;q^4)_\infty(q^2;q^4)_\infty}{(q^5;q^4)_\infty(q^6;q^4)_\infty}=\frac{(q;q^3)_\infty}{(q^4;q^3)_\infty}-\frac{(q;q^4)_\infty}{(q^5;q^4)_\infty}+\frac{(q^2;q^3)_\infty}{(q^5;q^3)_\infty}-\frac{(q^2;q^4)_\infty}{(q^6;q^4)_\infty}.$$

Theorem 7.9. *For any complex numbers a,b with $0<|a|,|b|<1$, then*

$$\frac{(ab;b)_\infty(ab^4;b^4)_\infty}{(ab^2;b)_\infty(ab^8;b^4)_\infty}-\frac{(ab^2;b^2)_\infty(ab^3;b^3)_\infty}{(ab^4;b^2)_\infty(ab^6;b^3)_\infty}=\frac{(ab;b)_\infty}{(ab^2;b)_\infty}-\frac{(ab^2;b^2)_\infty}{(ab^4;b^2)_\infty}-\frac{(ab^3;b^3)_\infty}{(ab^6;b^3)_\infty}+\frac{(ab^4;b^4)_\infty}{(ab^8;b^4)_\infty}.$$

Proof. We know the elementary identities

$$1-q^v=(1-xq^{2v})-q^v(1-xq^v) \tag{7.10}$$

and

$$1-q^{2v}=(1-xq^{3v})-q^{2v}(1-xq^v). \tag{7.11}$$

Replace q^v by q^{2v} in (7.10)

$$1-q^{2v}=(1-xq^{4v})-q^{2v}(1-xq^{2v}). \tag{7.12}$$

Replace q^v by α, $1-xq^v$ by A, $1-xq^{2v}$ by B, $1-xq^{3v}$ by C and $1-xq^{4v}$ by D in (7.11) and (7.12), and encounter the system of equations

$$\begin{cases} 1-\alpha^2 = C - \alpha^2 A \\ 1-\alpha^2 = D-\alpha^2 B \end{cases} \tag{7.13}$$

Eliminate α in (7.13) and obtain

$$A - B = (1-B)C - (1-A)\,D. \tag{7.14}$$

Replace A by $1-xq^v$, B by $1-xq^{2v}$, C by $1-xq^{3v}$ and D by $1-xq^{4v}$ in (7.14)

$$(1-xq^v)-(1-xq^{2v}) = [1-(1-xq^{2v})](1-xq^{3v}) - [1-(1-xq^v)]\,(1-xq^{4v}). \tag{7.15}$$

Replace $x=a$ and $q^v=b$ in (7.15)

$$(1-ab)-(1-ab^2) = [1-(1-ab^2)](1-ab^3) - [1-(1-ab)]\,(1-ab^4). \tag{7.16}$$

In Lemma 3.1, replace z by a and q by b, and find

$$1-ab = \frac{(ab;b)_\infty}{(ab^2;b)_\infty}. \tag{7.17}$$

From (7.16) and (7.17), we conclude that

$$\frac{(ab;b)_\infty}{(ab^2;b)_\infty} - \frac{(ab^2;b^2)_\infty}{(ab^4;b^2)_\infty} = \left[1-\frac{(ab^2;b^2)_\infty}{(ab^4;b^2)_\infty}\right]\frac{(ab^3;b^3)_\infty}{(ab^6;b^3)_\infty} - \left[1-\frac{(ab;b)_\infty}{(ab^2;b)_\infty}\right]\frac{(ab^4;b^4)_\infty}{(ab^8;b^4)_\infty}. \tag{7.18}$$

Multiply both members of (7.18) by $(ab^2;b)_\infty(ab^4;b^2)_\infty(ab^6;b^3)_\infty(ab^8;b^4)_\infty$, and encounter

$$\frac{(ab;b)_\infty(ab^4;b^4)_\infty}{(ab^2;b)_\infty(ab^8;b^4)_\infty} - \frac{(ab^2;b^2)_\infty(ab^3;b^3)_\infty}{(ab^4;b^2)_\infty(ab^6;b^3)_\infty} = \frac{(ab;b)_\infty}{(ab^2;b)_\infty} - \frac{(ab^2;b^2)_\infty}{(ab^4;b^2)_\infty} - \frac{(ab^3;b^3)_\infty}{(ab^6;b^3)_\infty} + \frac{(ab^4;b^4)_\infty}{(ab^8;b^4)_\infty},$$

which is the desired result. □

8. Conclusion

In this article, we developement the theory of $(ab;b)_\infty$ and made some applications in the elementary identities in classical q-series theory; including, we evaluate some bilateral Lambert series and we show a new identity for the generating functions of (m,k)-capsids and (m,r_1,r_2)-capsids.

We hope that, in the future, with more research, we will be able to apply this theory in the q-hypergeometric series; or, at least, for the q-binomial theorem. For now, that's all.

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