Elementary Identities for Quocient of q-Series

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Abstract. We demonstrate some elementary identities for quocient of q-series.

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1. Introduction

In present paper, we derive some identities for quocient of two q-series, such as,

$$\frac{(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty}{(-a^2b;ab)_\infty(-ab^2;ab)_\infty(a^2b^2;ab)_\infty}=(1+a)(1+b)(1-ab)$$

and

$$z=\frac{(zq;q)_\infty(az;q)_\infty-(azq;q)_\infty(z;q)_\infty}{(zq;q)_\infty(az;q)_\infty-a\,(azq;q)_\infty(z;q)_\infty}.$$

2. The Quotient of two q-Series

Lemma 2.1. *For any complex numbers a,b, with $0<|a|,|b|<1$, then*

$$(a;b)_\infty=(1-a)(ab;b)_\infty, \tag{2.1}$$

where $(a;b)_\infty$ denotes the q-Pochhammer symbol.

Proof. In previous paper [1, Theorem 5.3.ii] we prove that

$$\frac{(q^{m+k};q^m)_\infty}{(q^k;q^m)_\infty}=\frac{1}{1-q^k}. \tag{2.2}$$

Replace q^k by a and q^m by b in (2.2)

$$\frac{(ab;b)_\infty}{(a;b)_\infty}=\frac{1}{1-a}\Rightarrow(a;b)_\infty=(1-a)(ab;b)_\infty,$$

which is the desired result. □

Theorem 2.2. *For any complex numbers a,b,c,d with $0<|a|,|b|,|c|,|d|<1$,*

$$A:=A(q):=\frac{(a;b)_\infty}{(c;d)_\infty} \tag{2.3}$$

and

$$B:=B(q):=\frac{(ab;b)_\infty}{(cd;d)_\infty}. \tag{2.4}$$

then,

$$\frac{A}{B}=\frac{1-a}{1-c}, \tag{2.5}$$

$$\frac{A}{B}-\frac{B}{A}=\frac{(a+c-2)(a-c)}{(1-a)(1-c)}, \tag{2.6}$$

$$\frac{A}{B}+\frac{B}{A}=\frac{2-(2-a)a-(2-c)c}{(1-a)(1-c)} \tag{2.7}$$

and

$$\frac{A^2-B^2}{A^2+B^2}=\frac{(a+c-2)(a-c)}{2-(2-a)a-(2-c)c}. \tag{2.8}$$

Proof. Replace a by c and b by d in Lemma 2.1

$$(c;d)_\infty=(1-c)(cd;d)_\infty. \tag{2.9}$$

Divide the result of Lemma 2.1 by (2.9) and encounter

$$\frac{(a;b)_\infty}{(c;d)_\infty}=\left(\frac{1-a}{1-c}\right)\frac{(ab;b)_\infty}{(cd;d)_\infty}, \tag{2.10}$$

From (2.3), (2.4) and (2.10), it follows that

$$\frac{A}{B}=\frac{1-a}{1-c},$$

whence, we form the equations

$$\frac{A}{B}-\frac{B}{A}=\frac{1-a}{1-c}-\frac{1-c}{1-a}=\frac{(1-a)^2-(1-c)^2}{(1-a)(1-c)}=\frac{(a+c-2)(a-c)}{(1-a)(1-c)}$$

and

$$\frac{A}{B}+\frac{B}{A}=\frac{1-a}{1-c}+\frac{1-c}{1-a}=\frac{(1-a)^2+(1-c)^2}{(1-a)(1-c)}=\frac{2-(2-a)a-(2-c)c}{(1-a)(1-c)}.$$

Divide (2.6) by (2.7), and obtain (2.8). This completes the proof. □

Corollary 2.3. *For any complex numbers a,b,c with $0<|a|,|b|,|c|<1$, then*

$$\frac{(a;b)_\infty}{(a;c)_\infty}=\frac{(ab;b)_\infty}{(ac;c)_\infty}, \tag{2.11}$$

Proof. In (2.10), replace c by a and d by c, and obtain (2.11). □

Corollary 2.4. *For any complex numbers a,b,c with $0<|a|,|b|,|c|<1$, then*

$$\frac{(a;c)_\infty}{(b;c)_\infty}=\left(\frac{1-a}{1-b}\right)\frac{(ac;c)_\infty}{(bc;c)_\infty}. \tag{2.12}$$

Proof. In (2.11) replace c by d

$$\frac{(a;b)_\infty}{(a;d)_\infty}=\frac{(ab;b)_\infty}{(ad;d)_\infty}\Rightarrow (ab;b)_\infty=\frac{(a;b)_\infty(ad;d)_\infty}{(a;d)_\infty}. \tag{2.13}$$

Substitute the right hand side of (2.13) in the right hand side of (2.10), and get

$$\frac{(a;b)_\infty}{(c;d)_\infty} = \left(\frac{1-a}{1-c}\right)\frac{(a;b)_\infty (ad;d)_\infty}{(cd;d)_\infty (a;d)_\infty} \Leftrightarrow \frac{(a;d)_\infty}{(c;d)_\infty} = \left(\frac{1-a}{1-c}\right)\frac{(ad;d)_\infty}{(cd;d)_\infty}. \tag{2.14}$$

Replace c by b and d by c in both members of (2.14) and encounter (2.12). □

Corollary 2.5. *For any complex numbers a, b, c with $0 < |a|, |b|, |c| < 1$, then*

$$\frac{(a;b)_\infty}{(b;c)_\infty} = \left(\frac{1-a}{1-b}\right)\frac{(ab;b)_\infty}{(bc;c)_\infty}. \tag{2.15}$$

Proof. From (2.11), we find

$$\frac{(a;c)_\infty}{(ac;c)_\infty} = \frac{(a;b)_\infty}{(ab;b)_\infty}, \tag{2.16}$$

On the other hand, from (2.12), we get

$$\frac{(a;c)_\infty}{(ac;c)_\infty} = \left(\frac{1-a}{1-b}\right)\frac{(b;c)_\infty}{(bc;c)_\infty}. \tag{2.17}$$

Eliminate $(a;c)_\infty / (ac;c)_\infty$ in (2.16) and (2.17); rearranging their terms, we encounter (2.15). □

Remark 2.6. With the equations (2.10), (2.11), (2.12) and (2.15), we transpose the classical q-series to $(ab;b)_\infty$-series and *vice-versa*; so we just repeat them below:

$$\frac{(a;b)_\infty}{(c;d)_\infty} = \left(\frac{1-a}{1-c}\right)\frac{(ab;b)_\infty}{(cd;d)_\infty},$$

$$\frac{(a;b)_\infty}{(a;c)_\infty} = \frac{(ab;b)_\infty}{(ac;c)_\infty},$$

$$\frac{(a;c)_\infty}{(b;c)_\infty} = \left(\frac{1-a}{1-b}\right)\frac{(ac;c)_\infty}{(bc;c)_\infty}$$

and

$$\frac{(a;b)_\infty}{(b;c)_\infty} = \left(\frac{1-a}{1-b}\right)\frac{(ab;b)_\infty}{(bc;c)_\infty}.$$

Example 2.7. Set $a = q, b = q^2, c = q^3$ and $d = q^4$ in Remark 2.6 and encounter

$$\frac{(q;q^2)_\infty}{(q^3;q^4)_\infty} = \left(\frac{1}{1+q+q^2}\right)\frac{(q^3;q^2)_\infty}{(q^7;q^4)_\infty},$$

$$\frac{(q;q^2)_\infty}{(q;q^3)_\infty} = \frac{(q^3;q^2)_\infty}{(q^4;q^3)_\infty},$$

$$\frac{(q;q^3)_\infty}{(q^2;q^3)_\infty} = \left(\frac{1}{1+q}\right)\frac{(q^4;q^3)_\infty}{(q^5;q^3)_\infty}$$

and

$$\frac{(q;q^2)_\infty}{(q^2;q^3)_\infty} = \left(\frac{1}{1+q}\right)\frac{(q^3;q^2)_\infty}{(q^5;q^3)_\infty}.$$

3. Other Identities

Theorem 3.1. *For any complex numbers a, b, c, d with $0 < |a|, |b|, |c|, |d| < 1$, then*

$$\frac{(a;b)_\infty}{(a;d)_\infty (c;d)_\infty} = \left(\frac{1}{1-c}\right)\frac{(ab;b)_\infty}{(ad;d)_\infty (cd;d)_\infty}. \tag{3.1}$$

Proof. In Lemma 2.1, we have the identity

$$(a;b)_\infty = (1-a)(ab;b)_\infty \Rightarrow 1-a = \frac{(a;b)_\infty}{(ab;b)_\infty}. \tag{3.2}$$

The formula in (3.2) is wonderful. On the left side, there is only the variable a; on the right side, there are two variables: a, which we have already known, and b, which is a new variable. What can we say about b? The variable b is independent of a. This point is crucial! So, if b is independent of a, we can change it to any other variable and this would not affect the left side of this identity. That's what we'll do. Replace b by d in (3.1), and encounter

$$1-a = \frac{(a;d)_\infty}{(ad;d)_\infty}. \tag{3.3}$$

Substitute the right hand side of (3.3) in the right hand side of (2.10)

$$\frac{(a;b)_\infty}{(c;d)_\infty} = \left(\frac{1}{1-c}\right)\frac{(a;d)_\infty (ab;b)_\infty}{(ad;d)_\infty (cd;d)_\infty} \Rightarrow \frac{(a;b)_\infty}{(a;d)_\infty (c;d)_\infty} = \left(\frac{1}{1-c}\right)\frac{(ab;b)_\infty}{(ad;d)_\infty (cd;d)_\infty},$$

which is the desired result. □

Corollary 3.2. *For any complex numbers a,b,c,d with $0<|a|,|b|,|c|,|d|<1$, then*

$$\frac{(a;c)_\infty}{(a;d)_\infty (b;d)_\infty} = \left(\frac{1}{1-b}\right)\frac{(ac;c)_\infty}{(ad;d)_\infty (bd;d)_\infty}. \tag{3.4}$$

Proof. In Theorem 3.1, replace b by c and c by b. This completes the proof. □

Corollary 3.3. *For any complex numbers a,b,c with $0<|a|,|b|,|c|<1$, then*

$$\frac{(a;b)_\infty}{(a;c)_\infty (b;c)_\infty} = \left(\frac{1}{1-b}\right)\frac{(ab;b)_\infty}{(ac;c)_\infty (bc;c)_\infty}. \tag{3.5}$$

Proof. In Corollary 3.2, replace b by c. Now, replace c by b and d by c. This completes the proof. □

Theorem 3.4. *For any complex numbers a,b,c,d with $0<|a|,|b|,|c|,|d|<1$, then*

$$\frac{(a;c)_\infty}{(a;d)_\infty (b;c)_\infty} = \left(\frac{1}{1-b}\right)\frac{(ac;c)_\infty}{(ad;d)_\infty (bc;c)_\infty}. \tag{3.6}$$

Proof. Substitute the right hand side of (3.3) into the right hand side of (2.12), and encounter the desired result. □

Corollary 3.5. *For any complex numbers a,b,c with $0<|a|,|b|,|c|<1$, then*

$$\frac{(a;c)_\infty}{(a;b)_\infty (b;c)_\infty} = \left(\frac{1}{1-b}\right)\frac{(ac;c)_\infty}{(ab;b)_\infty (bc;c)_\infty}. \tag{3.7}$$

Proof. Replace d by b in Theorem 3.4. This completes the proof. □

4. THE DISCRETE CASES

Theorem 4.1. *For any complex numbers a,b,c,d with $0<|a|,|b|,|c|,|d|<1$, and $m,n\in\mathbb{N}_{\geq 0}$, then*

$$\frac{(a;b)_n}{(c;d)_m} = \left(\frac{1-a}{1-c}\right)\frac{(ab;b)_\infty}{(cd;d)_\infty}\cdot\frac{(cd^m;d)_\infty}{(ab^n;b)_\infty}. \tag{4.1}$$

Proof. In [2, p. 6, (1.1.7)], we have

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \tag{4.2}$$

for $-\infty<n<\infty$.

Replace q by b in (4.2), and obtain

$$(a;b)_n = \frac{(a;b)_\infty}{(ab^n;b)_\infty} \Rightarrow (a;b)_\infty = (a;b)_n(ab^n;b)_\infty. \tag{4.3}$$

Replace a by c, b by d and n by m, in (4.3), and get

$$(c;d)_\infty = (c;d)_m(cd^m;d)_\infty. \tag{4.4}$$

Substitute the right hand side of (4.3) and (4.4) in the left hand side of (2.10)

$$\frac{(a;b)_n(ab^n;b)_\infty}{(c;d)_m(cd^m;d)_\infty} = \left(\frac{1-a}{1-c}\right)\frac{(ab;b)_\infty}{(cd;d)_\infty} \Rightarrow \frac{(a;b)_n}{(c;d)_m} = \left(\frac{1-a}{1-c}\right)\frac{(ab;b)_\infty}{(cd;d)_\infty}\cdot\frac{(cd^m;d)_\infty}{(ab^n;b)_\infty},$$

which is the desired result. □

Corollary 4.2. *For any complex numbers a,b,c with $0<|a|,|b|,|c|<1$, and $m,n\in\mathbb{N}_{\geqq 0}$, then*

$$\frac{(a;b)_n}{(a;c)_m} = \frac{(ab;b)_\infty}{(ac;c)_\infty}\cdot\frac{(ac^m;c)_\infty}{(ab^n;b)_\infty}. \tag{4.5}$$

Proof. Replace c by a and d by c in Theorem 4.1. This completes the proof. □

Corollary 4.3. *For any complex numbers a,b,c with $0<|a|,|b|,|c|<1$, and $m,n\in\mathbb{N}_{\geqq 0}$,then*

$$\frac{(a;c)_n}{(b;c)_m} = \left(\frac{1-a}{1-b}\right)\frac{(ac;c)_\infty}{(bc;c)_\infty}\cdot\frac{(bc^m;c)_\infty}{(ac^n;c)_\infty}. \tag{4.6}$$

Proof. Replace b by c, c by b and d by c in Theorem 4.1. This completes the proof. □

Corollary 4.4. *For any complex numbers a,b,c with $0<|a|,|b|,|c|<1$, and $m,n\in\mathbb{N}_{\geqq 0}$, then*

$$\frac{(a;b)_n}{(b;c)_m} = \left(\frac{1-a}{1-b}\right)\frac{(ab;b)_\infty}{(bc;c)_\infty}\cdot\frac{(bc^m;c)_\infty}{(ab^n;b)_\infty}. \tag{4.7}$$

Proof. Replace c by b and d by c in Theorem 4.1. This completes the proof. □

5. Other Discrete Cases

Theorem 5.1. *For any complex numbers a,b,c,d with $0<|a|,|b|,|c|,|d|<1$, and $k,m,n\in\mathbb{N}_{\geqq 0}$, then*

$$\frac{(a;b)_n}{(a;d)_m(c;d)_k} = \left(\frac{1}{1-c}\right)\frac{(ab;b)_\infty}{(ad;d)_\infty(cd;d)_\infty}\cdot\frac{(ad^m;d)_\infty(cd^k;d)_\infty}{(ab^n;b)_\infty}. \tag{5.1}$$

Proof. Replace b by d and n by m, in (4.2), and get

$$(a;d)_\infty = (a;d)_m(ad^m;d)_\infty \tag{5.2}$$

Replace a by c, b by d and n by k, in (4.2), and obtain

$$(c;d)_\infty = (c;d)_k(cd^k;d)_\infty. \tag{5.3}$$

From (3.1), (4.2), (5.2) and (5.3), it follows that

$$\frac{(a;b)_n(ab^n;b)_\infty}{(a;d)_m(ad^m;d)_\infty(c;d)_k(cd^k;d)_\infty} = \left(\frac{1}{1-c}\right)\frac{(ab;b)_\infty}{(ad;d)_\infty(cd;d)_\infty}$$
$$\Rightarrow \frac{(a;b)_n}{(a;d)_m(c;d)_k} = \left(\frac{1}{1-c}\right)\frac{(ab;b)_\infty}{(ad;d)_\infty(cd;d)_\infty}\cdot\frac{(ad^m;d)_\infty(cd^k;d)_\infty}{(ab^n;b)_\infty},$$

which is the desired result. □

Corollary 5.2. *For any complex numbers a,b,c,d with $0<|a|,|b|,|c|,|d|<1$, and $k,m,n\in\mathbb{N}_{\geqq 0}$, then*

$$\frac{(a;c)_n}{(a;d)_m(b;d)_k}=\left(\frac{1}{1-b}\right)\frac{(ac;c)_\infty}{(ad;d)_\infty(bd;d)_\infty}\cdot\frac{(ad^m;d)_\infty(bd^k;d)_\infty}{(ac^n;c)_\infty}. \tag{5.4}$$

Proof. In Theorem 5.1, replace b by c and c by b. This completes the proof. □

Corollary 5.3. *For any complex numbers a,b,c with $0<|a|,|b|,|c|<1$, and $k,m,n\in\mathbb{N}_{\geqq 0}$, then*

$$\frac{(a;b)_n}{(a;c)_m(b;c)_k}=\left(\frac{1}{1-b}\right)\frac{(ab;b)_\infty}{(ac;c)_\infty(bc;c)_\infty}\cdot\frac{(ac^m;c)_\infty(bc^k;c)_\infty}{(ab^n;b)_\infty}. \tag{5.5}$$

Proof. In Corollary 5.2, replace b by c. Now, replace c by b and d by c. This completes the proof. □

Theorem 5.4. *For any complex numbers a,b,c,d with $0<|a|,|b|,|c|,|d|<1$, and $k,m,n\in\mathbb{N}_{\geqq 0}$, then*

$$\frac{(a;c)_n}{(a;d)_m(b;c)_k}=\left(\frac{1}{1-b}\right)\frac{(ac;c)_\infty}{(ad;d)_\infty(bc;c)_\infty}\cdot\frac{(ad^m;d)_\infty(bc^k;c)_\infty}{(ac^n;c)_\infty}. \tag{5.6}$$

Proof. Replace b by c in (4.2)

$$(a;c)_\infty=(a;c)_n(ac^n;c)_\infty. \tag{5.7}$$

Replace c by d and n by m in (5.8)

$$(a;d)_\infty=(a;d)_m(ad^m;d)_\infty. \tag{5.8}$$

Replace a by b and n by k in (5.8)

$$(b;c)_\infty=(b;c)_k(bc^k;c)_\infty. \tag{5.9}$$

From (3.6), (5.8), (5.9) and (5.10), we conclude that

$$\begin{aligned}&\frac{(a;c)_n(ac^n;c)_\infty}{(a;d)_m(ad^m;d)_\infty(b;c)_k(bc^k;c)_\infty}=\left(\frac{1}{1-b}\right)\frac{(ac;c)_\infty}{(ad;d)_\infty(bc;c)_\infty}\\ \Rightarrow&\frac{(a;c)_n}{(a;d)_m(b;c)_k}=\left(\frac{1}{1-b}\right)\frac{(ac;c)_\infty}{(ad;d)_\infty(bc;c)_\infty}\cdot\frac{(ad^m;d)_\infty(bc^k;c)_\infty}{(ac^n;c)_\infty},\end{aligned}$$

which is the desired result. □

Corollary 5.5. *For any complex numbers a,b,c with $0<|a|,|b|,|c|<1$, and $k,m,n\in\mathbb{N}_{\geqq 0}$, then*

$$\frac{(a;c)_n}{(a;b)_m(b;c)_k}=\left(\frac{1}{1-b}\right)\frac{(ac;c)_\infty}{(ab;b)_\infty(bc;c)_\infty}\cdot\frac{(ab^m;b)_\infty(bc^k;c)_\infty}{(ac^n;c)_\infty}. \tag{5.10}$$

Proof. Replace d by b in Theorem 5.4. This completes the proof. □

6. APPLICATION FOR THE CLASSICAL q-BINOMIAL THEOREM

Theorem 6.1. *For any complex a,q,z, with $0<|a|,|q|,|z|<1$, then*

$$\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_\infty}{(aq^n;q)_\infty}z^n=\frac{(q;q)_\infty(az;q)_\infty}{(a;q)_\infty(z;q)_\infty}. \tag{6.1}$$

Proof. In [3], the q-binomial theorem assures us that

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n=\frac{(az;q)_\infty}{(z;q)_\infty}={}_1\phi_0(a;;q,z), \tag{6.2}$$

for $|q|<1$ and $|z|<1$.

Replace q by b in (6.2)

$$\sum_{n=0}^{\infty} \frac{(a;b)_n}{(b;b)_n} z^n = \frac{(az;b)_\infty}{(z;b)_\infty} = {}_1\phi_0(a;;b;z). \tag{6.3}$$

In Corollary 4.3, replace c by b, m by n, multiply by z^n and summing from 0 at infinity with respect to n, we get

$$\sum_{n=0}^{\infty} \frac{(a;b)_n}{(b;b)_n} z^n = \left(\frac{1-a}{1-b}\right) \frac{(ab;b)_\infty}{(b^2;b)_\infty} \cdot \sum_{n=0}^{\infty} \frac{(b^{n+1};b)_\infty}{(ab^n;b)_\infty} z^n. \tag{6.4}$$

From (6.3) and (6.4), it follows that

$$\begin{gathered} \frac{(az;b)_\infty}{(z;b)_\infty} = \left(\frac{1-a}{1-b}\right) \frac{(ab;b)_\infty}{(b^2;b)_\infty} \cdot \sum_{n=0}^{\infty} \frac{(b^{n+1};b)_\infty}{(ab^n;b)_\infty} z^n \\ \Rightarrow \sum_{n=0}^{\infty} \frac{(b^{n+1};b)_\infty}{(ab^n;b)_\infty} z^n = \left(\frac{1-b}{1-a}\right) \frac{(b^2;b)_\infty (az;b)_\infty}{(ab;b)_\infty (z;b)_\infty}. \end{gathered} \tag{6.5}$$

On the other hand, from Lemma 2.1, we conclude that

$$1-a = \frac{(a;b)_\infty}{(ab;b)_\infty}. \tag{6.6}$$

Replace a by b in (6.6)

$$1-b = \frac{(b;b)_\infty}{(b^2;b)_\infty}. \tag{6.7}$$

From (6.5), (6.6) and (6.7), it follows that

$$\sum_{n=0}^{\infty} \frac{(b^{n+1};b)_\infty}{(ab^n;b)_\infty} z^n = \frac{(ab;b)_\infty (b;b)_\infty (b^2;b)_\infty (az;b)_\infty}{(a;b)_\infty (b^2;b)_\infty (ab;b)_\infty (z;b)_\infty} \Rightarrow \sum_{n=0}^{\infty} \frac{(b^{n+1};b)_\infty}{(ab^n;b)_\infty} z^n = \frac{(b;b)_\infty (az;b)_\infty}{(a;b)_\infty (z;b)_\infty}.$$

Replace b by q in previous equation and obtain the desired result. □

Corollary 6.2. *For any complex a, q, z, with $0<|a|, |b|, |q|<1$, then*

$$\sum_{n=0}^{\infty} \frac{(q^{n+1};q)_\infty}{((a/b)\,q^n;q)_\infty} b^n = \frac{(q;q)_\infty (a;q)_\infty}{(a/b;q)_\infty (b;q)_\infty}. \tag{6.8}$$

Proof. A slighty modified version of the q-binomial theorem is given by [2, p.6, (1.2.2)]

$$\sum_{n=0}^{\infty} \frac{(a/b;q)_n}{(q;q)_n} b^n = \frac{(a;q)_\infty}{(b;q)_\infty}, \tag{6.9}$$

where $|b|<1$.

In Corollary 4.3, replace b by q, c by q, a by a/b, m by n, multiply by b^n and summing from 0 at infinity with respect to n, we get

$$\sum_{n=0}^{\infty} \frac{(a/b;q)_n}{(q;q)_n} b^n = \left(\frac{1-(a/b)}{1-q}\right) \frac{((a/b)\,q;q)_\infty}{(q^2;q)_\infty} \cdot \sum_{n=0}^{\infty} \frac{(q^{n+1};q)_\infty}{((a/b)\,q^n;q)_\infty} b^n. \tag{6.10}$$

Substitute the right hand side of (6.9) into the left hand side of (6.10) and find

$$\begin{gathered} \frac{(a;q)_\infty}{(b;q)_\infty} = \left(\frac{1-(a/b)}{1-q}\right) \frac{((a/b)\,q;q)_\infty}{(q^2;q)_\infty} \cdot \sum_{n=0}^{\infty} \frac{(q^{n+1};q)_\infty}{((a/b)\,q^n;q)_\infty} b^n \\ \Rightarrow \sum_{n=0}^{\infty} \frac{(q^{n+1};q)_\infty}{((a/b)\,q^n;q)_\infty} b^n = \left(\frac{1-q}{1-(a/b)}\right) \frac{(a;q)_\infty (q^2;q)_\infty}{(b;q)_\infty ((a/b)\,q;q)_\infty}. \end{gathered} \tag{6.11}$$

Replace b by q and a by q in (6.6)

$$1-q=\frac{(q;q)_\infty}{(q^2;q)_\infty}. \tag{6.12}$$

Replace b by q and a by a/b in (6.6)

$$1-(a/b)=\frac{(a/b;q)_\infty}{((a/b)q;q)_\infty}. \tag{6.13}$$

From (6.11), (6.12) and (6.13), it follows that

$$\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_\infty}{((a/b)\,q^n;q)_\infty}\,b^n=\frac{((a/b)q;q)_\infty(q;q)_\infty(a;q)_\infty(q^2;q)_\infty}{(a/b;q)_\infty(q^2;q)_\infty(b;q)_\infty((a/b)\,q;q)_\infty}$$
$$\Rightarrow\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_\infty}{((a/b)\,q^n;q)_\infty}\,b^n=\frac{(q;q)_\infty(a;q)_\infty}{(a/b;q)_\infty(b;q)_\infty},$$

which is the desired result. □

Note 6.3. We get another simpler proof of Corollary 6.2: replace z by b and a by a/b in Theorem 6.1.

7. Application for the q-Gauss Summation

Theorem 7.1. *For any complex a,b,c,q and t, with $0<|a|,|b|,|c|,|q|,|t|<1$, then*

$${}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix};q,t\right)=\frac{(a;q)_\infty(b;q)_\infty}{(q;q)_\infty(c;q)_\infty}\cdot\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_\infty(cq^n;q)_\infty}{(aq^n;q)_\infty(bq^n;q)_\infty}\,t^n. \tag{7.1}$$

Proof. In [2, p. 5, (1.1.1)], we define the Gauss's q-hypergeometric series by

$${}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix};q,t\right):=\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n}\,t^n, \tag{7.2}$$

where $|q|<1$ and $|t|<1$.

In Corollary 4.3, replace b by q, c by q and m by n

$$\frac{(a;q)_n}{(q;q)_n}=\left(\frac{1-a}{1-q}\right)\frac{(aq;q)_\infty}{(q^2;q)_\infty}\cdot\frac{(q^{n+1};q)_\infty}{(aq^n;q)_\infty}. \tag{7.3}$$

In Corollary 4.3, replace c by q, b by c, a by b and m by n

$$\frac{(b;q)_n}{(c;q)_n}=\left(\frac{1-b}{1-c}\right)\frac{(bq;q)_\infty}{(cq;q)_\infty}\cdot\frac{(cq^n;q)_\infty}{(bq^n;q)_\infty}. \tag{7.4}$$

Multiply (7.3) by (7.4) and t^n, sum from 0 at infinity with respect to n, and get

$$\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n}\,t^n=\frac{(1-a)(1-b)}{(1-q)(1-c)}\cdot\frac{(aq;q)_\infty(bq;q)_\infty}{(q^2;q)_\infty(cq;q)_\infty}\cdot\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_\infty(cq^n;q)_\infty}{(aq^n;q)_\infty(bq^n;q)_\infty}\,t^n. \tag{7.5}$$

Replace b by q in (6.6)

$$1-a=\frac{(a;q)_\infty}{(aq;q)_\infty}. \tag{7.6}$$

Replace a by b in (7.6)

$$1-b=\frac{(b;q)_\infty}{(bq;q)_\infty}. \tag{7.7}$$

Replace a by c in (7.6)

$$1-c=\frac{(c;q)_\infty}{(cq;q)_\infty}. \tag{7.8}$$

From (6.12), (7.2), (7.6), (7.7) and (7.8), it follows that

$$ {}_2\phi_1\left(\begin{matrix} a,b \\ c \end{matrix};q,t\right)=\frac{(a;q)_\infty(q^2;q)_\infty(b;q)_\infty(cq;q)_\infty(aq;q)_\infty(bq;q)_\infty}{(aq;q)_\infty(q;q)_\infty(bq;q)_\infty(c;q)_\infty(q^2;q)_\infty(cq;q)_\infty}\cdot\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_\infty(cq^n;q)_\infty}{(aq^n;q)_\infty(bq^n;q)_\infty}t^n $$

$$ \Rightarrow {}_2\phi_1\left(\begin{matrix} a,b \\ c \end{matrix};q,t\right)=\frac{(a;q)_\infty(b;q)_\infty}{(q;q)_\infty(c;q)_\infty}\cdot\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_\infty(cq^n;q)_\infty}{(aq^n;q)_\infty(bq^n;q)_\infty}t^n, $$

which is the desired result. □

Corollary 7.2. *For any complex a,b,c,q and t, with $0<|a|,|b|,|c|,|q|,|t|<1$, then*

$$ \sum_{n=0}^{\infty}\frac{(q^{n+1};q)_\infty(cq^n;q)_\infty}{(aq^n;q)_\infty(bq^n;q)_\infty}\left(\frac{c}{ab}\right)^n=\frac{(q;q)_\infty(c/a;q)_\infty(c/b;q)_\infty}{(a;q)_\infty(b;q)_\infty(c/(ab);q)_\infty}. \tag{7.9} $$

Proof. In [2, p. 10, (1.3.1)], we encounter the q-Gauss summation theorem, given in the form

$$ {}_2\phi_1\left(\begin{matrix} a,b \\ c \end{matrix};q,\frac{c}{ab}\right)=\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n}\left(\frac{c}{ab}\right)^n=\frac{(c/a;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/(ab);q)_\infty}, \tag{7.10} $$

where $|c/ab|<1$.

Replace t by c/ab in Theorem 7.1 and obtain

$$ {}_2\phi_1\left(\begin{matrix} a,b \\ c \end{matrix};q,\frac{c}{ab}\right)=\frac{(a;q)_\infty(b;q)_\infty}{(q;q)_\infty(c;q)_\infty}\cdot\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_\infty(cq^n;q)_\infty}{(aq^n;q)_\infty(bq^n;q)_\infty}\left(\frac{c}{ab}\right)^n. \tag{7.11} $$

Substitute the right hand side of (7.10) into the left hand side of (7.11) and get the desired result by rearranging the terms and canceling $(c;q)_\infty$ in numerator and denominator. □

Corollary 7.3. (The Ramanujan's version for q-Gauss Summation Theorem) *For any complex a,b,c and q, with $0<|abc|<1$ and $bc\neq 0$, then*

$$ \sum_{n=0}^{\infty}\frac{(q^{n+1};q)_\infty(aq^n;q)_\infty}{((1/b)\,q^n;q)_\infty((1/c)\,q^n;q)_\infty}(abc)^n=\frac{(q;q)_\infty(ac;q)_\infty(ab;q)_\infty}{(1/b;q)_\infty(1/c;q)_\infty(abc;q)_\infty}. \tag{7.12} $$

Proof. In [2, p. 12, (1.3.8)], we encounter the Ramanujan's version for q-Gauss Summation Theorem, given by

$$ \frac{(ac;q)_\infty}{(abc;q)_\infty}=\frac{(a;q)_\infty}{(ab;q)_\infty}\sum_{n=0}^{\infty}\frac{(1/b;q)_n(1/c;q)_n}{(a;q)_n(q;q)_n}(abc)^n\Rightarrow {}_2\phi_1\left(\begin{matrix} 1/b,1/c \\ a \end{matrix};q,abc\right)=\frac{(ac;q)_\infty(ab;q)_\infty}{(abc;q)_\infty(a;q)_\infty}, \tag{7.13} $$

for $|abc|<1$ and $bc\neq 0$.

In Theorem 7.1, replace a by $1/b$, b by $1/c$, c by a and t by abc, simultaneously,

$$ {}_2\phi_1\left(\begin{matrix} 1/b,1/c \\ a \end{matrix};q,abc\right)=\frac{(1/b;q)_\infty(1/c;q)_\infty}{(q;q)_\infty(a;q)_\infty}\cdot\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_\infty(aq^n;q)_\infty}{((1/b)\,q^n;q)_\infty((1/c)\,q^n;q)_\infty}(abc)^n. \tag{7.14} $$

Replace the right hand side of (7.13) into the left hand side of (7.14) and get the desired result by rearranging the terms and canceling $(a;q)_\infty$ in numerator and denominator. □

8. Application for the Rogers-Fine Identity

Theorem 8.1. *We have*

$$ \sum_{n=0}^{\infty}\frac{(\beta q^n;q)_\infty}{(\alpha q^n;q)_\infty}\tau^n=\frac{(\alpha\tau q/\beta;q)_\infty}{(\tau;q)_\infty}\cdot\sum_{n=0}^{\infty}\frac{(\beta q^n;q)_\infty(\tau q^{n+1};q)_\infty}{(\alpha q^n;q)_\infty((\alpha\tau q/\beta)\,q^n;q)_\infty}\beta^n\tau^n q^{n^2-n}(1-\alpha\tau q^{2n}). \tag{8.1} $$

Proof. In [2, p. 28, (1.7.1)], we encounter the Rogers-Fine identity given by

$$\sum_{n=0}^{\infty}\frac{(\alpha;q)_n}{(\beta;q)_n}\tau^n=\sum_{n=0}^{\infty}\frac{(\alpha;q)_n(\alpha\tau q/\beta;q)_n}{(\beta;q)_n(\tau;q)_{n+1}}\beta^n\tau^n q^{n^2-n}(1-\alpha\tau q^{2n}). \tag{8.2}$$

In Corollary 4.3, replace a by α, b by β, c by q and m by n

$$\frac{(\alpha;q)_n}{(\beta;q)_n}=\left(\frac{1-\alpha}{1-\beta}\right)\frac{(\alpha q;q)_\infty}{(\beta q;q)_\infty}\cdot\frac{(\beta q^n;q)_\infty}{(\alpha q^n;q)_\infty}. \tag{8.3}$$

Again, in Corollary 4.3, replace a by $\alpha\tau q/\beta$, b by τ, c by q and m by $n+1$

$$\frac{(\alpha\tau q/\beta;q)_n}{(\tau;q)_{n+1}}=\left(\frac{1-\alpha\tau q/\beta}{1-\tau}\right)\frac{((\alpha\tau q/\beta)q;q)_\infty}{(\tau q;q)_\infty}\cdot\frac{(\tau q^{n+1};q)_\infty}{((\alpha\tau q/\beta)\,q^n;q)_\infty}. \tag{8.4}$$

From (8.2), (8.3) and (8.4), it follows that

$$\begin{aligned}&\left(\frac{1-\alpha}{1-\beta}\right)\frac{(\alpha q;q)_\infty}{(\beta q;q)_\infty}\cdot\sum_{n=0}^{\infty}\frac{(\beta q^n;q)_\infty}{(\alpha q^n;q)_\infty}\tau^n\\ &=\left(\frac{1-\alpha}{1-\beta}\right)\left(\frac{1-\alpha\tau q/\beta}{1-\tau}\right)\frac{(\alpha q;q)_\infty((\alpha\tau q/\beta)q;q)_\infty}{(\beta q;q)_\infty(\tau q;q)_\infty}\cdot\sum_{n=0}^{\infty}\frac{(\beta q^n;q)_\infty(\tau q^{n+1};q)_\infty}{(\alpha q^n;q)_\infty((\alpha\tau q/\beta)\,q^n;q)_\infty}\beta^n\tau^n q^{n^2-n}(1-\alpha\tau q^{2n})\\ &\Rightarrow\sum_{n=0}^{\infty}\frac{(\beta q^n;q)_\infty}{(\alpha q^n;q)_\infty}\tau^n\\ &=\left(\frac{1-\alpha\tau q/\beta}{1-\tau}\right)\frac{((\alpha\tau q/\beta)q;q)_\infty}{(\tau q;q)_\infty}\cdot\sum_{n=0}^{\infty}\frac{(\beta q^n;q)_\infty(\tau q^{n+1};q)_\infty}{(\alpha q^n;q)_\infty((\alpha\tau q/\beta)\,q^n;q)_\infty}\beta^n\tau^n q^{n^2-n}(1-\alpha\tau q^{2n}).\end{aligned} \tag{8.5}$$

On the other hand, replace a by $\alpha\tau q/\beta$ and b by q in Lemma 2.1

$$(\alpha\tau q/\beta;q)_\infty=(1-\alpha\tau q/\beta)((\alpha\tau q/\beta)\,q;q)_\infty\Rightarrow 1-\alpha\tau q/\beta=\frac{(\alpha\tau q/\beta;q)_\infty}{((\alpha\tau q/\beta)q;q)_\infty}. \tag{8.6}$$

Replace a by τ and b by q in Lemma 2.1

$$(\tau;q)_\infty=(1-\tau)(\tau q;q)_\infty\Rightarrow 1-\tau=\frac{(\tau;q)_\infty}{(\tau q;q)_\infty}. \tag{8.7}$$

From (8.5), (8.6) and (8.7), we conclude that

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(\beta q^n;q)_\infty}{(\alpha q^n;q)_\infty}\tau^n\\ &=\left(\frac{(\alpha\tau q/\beta;q)_\infty(\tau q;q)_\infty((\alpha\tau q/\beta)q;q)_\infty}{((\alpha\tau q/\beta)q;q)_\infty(\tau;q)_\infty(\tau q;q)_\infty}\right)\cdot\sum_{n=0}^{\infty}\frac{(\beta q^n;q)_\infty(\tau q^{n+1};q)_\infty}{(\alpha q^n;q)_\infty((\alpha\tau q/\beta)\,q^n;q)_\infty}\beta^n\tau^n q^{n^2-n}(1-\alpha\tau q^{2n})\\ &\Rightarrow\sum_{n=0}^{\infty}\frac{(\beta q^n;q)_\infty}{(\alpha q^n;q)_\infty}\tau^n=\frac{(\alpha\tau q/\beta;q)_\infty}{(\tau;q)_\infty}\cdot\sum_{n=0}^{\infty}\frac{(\beta q^n;q)_\infty(\tau q^{n+1};q)_\infty}{(\alpha q^n;q)_\infty((\alpha\tau q/\beta)\,q^n;q)_\infty}\beta^n\tau^n q^{n^2-n}(1-\alpha\tau q^{2n}),\end{aligned}$$

which is the desired result. □

9. APPLICATION FOR RAMANUJAN'S THETA FUNCTIONS

Theorem 9.1. *For any complex a and b, with $|ab|<1$, then*

$$\frac{f(a,b)}{(1+a)(1+b)(1-ab)}=(-a^2b;ab)_\infty(-ab^2;ab)_\infty(a^2b^2;ab)_\infty \tag{9.1}$$

or

$$\frac{(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty}{(-a^2b;ab)_\infty(-ab^2;ab)_\infty(a^2b^2;ab)_\infty}=(1+a)(1+b)(1-ab). \tag{9.2}$$

Proof. Replace a by c and b by cd in (7.2)

$$(c;cd)_\infty = (1-c)(c^2 d;cd)_\infty. \tag{9.3}$$

Replace a by d and b by cd in (7.2)

$$(d;cd)_\infty = (1-d)(cd^2;cd)_\infty. \tag{9.4}$$

Replace a by cd and b by cd in (7.2)

$$(cd;cd)_\infty = (1-cd)(c^2 d^2;cd)_\infty. \tag{9.5}$$

Replace c by $-a$ and d by $-b$ in (9.2), (9.3) and (9.4)

$$(-a;ab)_\infty = (1+a)(-a^2 b;ab)_\infty, \tag{9.6}$$

$$(-b;ab)_\infty = (1+b)(-ab^2;ab)_\infty \tag{9.7}$$

and

$$(ab;ab)_\infty = (1-ab)(a^2 b^2;ab)_\infty. \tag{9.8}$$

Multiply (9.5) by (9.6) and (9.7)

$$(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty = (1+a)(1+b)(1-ab)(-a^2 b;ab)_\infty(-ab^2;ab)_\infty(a^2 b^2;ab)_\infty. \tag{9.9}$$

On the other hand, we know that [2, p. 17, (1.4.8)]

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{9.10}$$

From (9.8) and (9.9), it follows that

$$f(a,b) = (1+a)(1+b)(1-ab)(-a^2 b;ab)_\infty(-ab^2;ab)_\infty(a^2 b^2;ab)_\infty$$
$$\Rightarrow \frac{f(a,b)}{(1+a)(1+b)(1-ab)} = (-a^2 b;ab)_\infty(-ab^2;ab)_\infty(a^2 b^2;ab)_\infty$$

or

$$\frac{(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty}{(-a^2 b;ab)_\infty(-ab^2;ab)_\infty(a^2 b^2;ab)_\infty} = (1+a)(1+b)(1-ab),$$

which are the desired results. □

10. The back to the Discrete Case

Lemma 10.1. *We have*

$$\frac{(a;q)_{n+1}(aq^n;q)_\infty}{(a;q)_\infty} = 1-aq^n. \tag{10.1}$$

Proof. We well-know identity [4, p. 300, (12.1.3)]

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \tag{10.2}$$

whence we obtain

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^{n-1}q;q)_\infty}, \tag{10.3}$$

In Lemma 2.1, replace a by aq^{n-1} and b by q

$$(aq^{n-1};q)_\infty = (1-aq^{n-1})(aq^{n-1}\,q;q)_\infty \Rightarrow (aq^{n-1}\,q;q)_\infty = \frac{(aq^{n-1};q)_\infty}{1-aq^{n-1}}. \tag{10.4}$$

From (10.3) and (10.4), it follows that

$$(a;q)_n=\frac{(1-aq^{n-1})(a;q)_\infty}{(aq^{n-1};q)_\infty}.$$

Replace n by $n+1$ in (10.5), and rearrange the terms of the above equation. This completes the proof. □

Theorem 10.2. *We have*

$$\frac{(azq;q)_\infty}{(zq;q)_\infty}+z\frac{(az;q)_\infty}{(z;q)_\infty}=\frac{(az;q)_\infty}{(z;q)_\infty}+az\frac{(azq;q)_\infty}{(zq;q)_\infty}. \tag{10.5}$$

Proof. Replace a by b in (10.1)

$$\frac{(b;q)_{n+1}(bq^n;q)_\infty}{(b;q)_\infty}=1-bq^n. \tag{10.6}$$

Divide (10.1) by (10.6) and rearrange the terms

$$\frac{(a;q)_{n+1}}{(b;q)_{n+1}}-b\frac{(a;q)_{n+1}}{(b;q)_{n+1}}\,q^n=\frac{(a;q)_\infty}{(b;q)_\infty}\left[\frac{(bq^n;q)_\infty}{(aq^n;q)_\infty}-a\frac{(bq^n;q)_\infty}{(aq^n;q)_\infty}\,q^n\right]. \tag{10.7}$$

Replace b by q in (10.7)

$$\frac{(a;q)_{n+1}}{(q;q)_{n+1}}-q\frac{(a;q)_{n+1}}{(q;q)_{n+1}}\,q^n=\frac{(a;q)_\infty}{(q;q)_\infty}\left[\frac{(q^{n+1};q)_\infty}{(aq^n;q)_\infty}-a\frac{(q^{n+1};q)_\infty}{(aq^n;q)_\infty}\,q^n\right]. \tag{10.8}$$

Multiply (10.8) by z^n and summing from 0 at infinity with respect to n, we encounter

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(a;q)_{n+1}}{(q;q)_{n+1}}z^n-q\sum_{n=0}^{\infty}\frac{(a;q)_{n+1}}{(q;q)_{n+1}}\,z^nq^n\\ =&\frac{(a;q)_\infty}{(q;q)_\infty}\left[\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_\infty}{(aq^n;q)_\infty}z^n-a\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_\infty}{(aq^n;q)_\infty}\,z^nq^n\right].\end{aligned} \tag{10.9}$$

Employing (6.1) in (10.9), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(a;q)_{n+1}}{(q;q)_{n+1}}z^n-q\sum_{n=0}^{\infty}\frac{(a;q)_{n+1}}{(q;q)_{n+1}}\,z^nq^n&=\frac{(a;q)_\infty}{(q;q)_\infty}\left[\frac{(q;q)_\infty(az;q)_\infty}{(a;q)_\infty(z;q)_\infty}-a\frac{(q;q)_\infty(azq;q)_\infty}{(a;q)_\infty(zq;q)_\infty}\right]\\ \Rightarrow\sum_{n=0}^{\infty}\frac{(a;q)_{n+1}}{(q;q)_{n+1}}z^n-\sum_{n=0}^{\infty}\frac{(a;q)_{n+1}}{(q;q)_{n+1}}\,z^nq^{n+1}&=\frac{(az;q)_\infty}{(z;q)_\infty}-a\frac{(azq;q)_\infty}{(zq;q)_\infty}\\ \Leftrightarrow\frac{1}{z}\sum_{n=0}^{\infty}\frac{(a;q)_{n+1}}{(q;q)_{n+1}}z^{n+1}-\frac{1}{z}\sum_{n=0}^{\infty}\frac{(a;q)_{n+1}}{(q;q)_{n+1}}\,z^{n+1}q^{n+1}&=\frac{(az;q)_\infty}{(z;q)_\infty}-a\frac{(azq;q)_\infty}{(zq;q)_\infty}\\ \Leftrightarrow\frac{1}{z}\sum_{n=1}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n-\frac{1}{z}\sum_{n=1}^{\infty}\frac{(a;q)_n}{(q;q)_n}\,z^nq^n&=\frac{(az;q)_\infty}{(z;q)_\infty}-a\frac{(azq;q)_\infty}{(zq;q)_\infty}\\ \Leftrightarrow\sum_{n=1}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n-\sum_{n=1}^{\infty}\frac{(a;q)_n}{(q;q)_n}\,z^nq^n&=z\frac{(az;q)_\infty}{(z;q)_\infty}-az\frac{(azq;q)_\infty}{(zq;q)_\infty}\\ \Leftrightarrow1+\sum_{n=1}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n-\left[1+\sum_{n=1}^{\infty}\frac{(a;q)_n}{(q;q)_n}\,z^nq^n\right]&=z\frac{(az;q)_\infty}{(z;q)_\infty}-az\frac{(azq;q)_\infty}{(zq;q)_\infty}\\ \Leftrightarrow\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n-\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}\,z^nq^n&=z\frac{(az;q)_\infty}{(z;q)_\infty}-az\frac{(azq;q)_\infty}{(zq;q)_\infty}.\end{aligned} \tag{10.10}$$

From (6.2) and (10.10), we easily conclude that

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^nq^n=\frac{(az;q)_\infty}{(z;q)_\infty}-z\frac{(az;q)_\infty}{(z;q)_\infty}+az\frac{(azq;q)_\infty}{(zq;q)_\infty}. \tag{10.11}$$

On the other hand, replace z by zq in (6.2)

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^nq^n=\frac{(azq;q)_\infty}{(zq;q)_\infty}. \tag{10.12}$$

From (10.11) and (10.12), it follows that

$$\begin{aligned}\frac{(azq;q)_\infty}{(zq;q)_\infty}&=\frac{(az;q)_\infty}{(z;q)_\infty}-z\frac{(az;q)_\infty}{(z;q)_\infty}+az\frac{(azq;q)_\infty}{(zq;q)_\infty}\\ \Leftrightarrow\frac{(azq;q)_\infty}{(zq;q)_\infty}+z\frac{(az;q)_\infty}{(z;q)_\infty}&=\frac{(az;q)_\infty}{(z;q)_\infty}+az\frac{(azq;q)_\infty}{(zq;q)_\infty},\end{aligned}$$

which is the desired result. □

Example 10.3. Replace q by q^5, a by q^{-1} and z by q^4 in (10.11)

$$\frac{(q^8;q^5)_\infty}{(q^9;q^5)_\infty}+q^4\frac{(q^3;q^5)_\infty}{(q^4;q^5)_\infty}=\frac{(q^3;q^5)_\infty}{(q^4;q^5)_\infty}+q^3\frac{(q^8;q^5)_\infty}{(q^9;q^5)_\infty}.$$

Corollary 10.4. *For any complex a, q and z, with $0<|az|<1$, then*

$$\frac{(azq;q)_\infty(z;q)_\infty}{(zq;q)_\infty(az;q)_\infty}=\frac{1-z}{1-az}. \tag{10.13}$$

Proof. Rearrange the terms of (10.5), as follows

$$\begin{aligned}\frac{(azq;q)_\infty}{(zq;q)_\infty}-az\frac{(azq;q)_\infty}{(zq;q)_\infty}&=\frac{(az;q)_\infty}{(z;q)_\infty}-z\frac{(az;q)_\infty}{(z;q)_\infty}\\ \Rightarrow(1-az)\frac{(azq;q)_\infty}{(zq;q)_\infty}&=(1-z)\frac{(az;q)_\infty}{(z;q)_\infty}\\ \Leftrightarrow\frac{(azq;q)_\infty(z;q)_\infty}{(zq;q)_\infty(az;q)_\infty}&=\frac{1-z}{1-az},\end{aligned}$$

which is the desired result. □

Example 10.5. Replace q by q^7, a by q^{-1} and z by q^6 in (10.13)

$$\frac{(q^{12};q^7)_\infty(q^6;q^7)_\infty}{(q^{13};q^7)_\infty(q^5;q^7)_\infty}=\frac{1-q^6}{1-q^5}.$$

Corollary 10.6. *We have*

$$z=\frac{(zq;q)_\infty(az;q)_\infty-(azq;q)_\infty(z;q)_\infty}{(zq;q)_\infty(az;q)_\infty-a\,(azq;q)_\infty(z;q)_\infty}. \tag{10.14}$$

Proof. Solve (10.13) for z and then delete the common denominator. This completes the proof. □

Example 10.7. Replace q by q^5, a by q^{-1} and z by q^4 in (10.14)

$$q^4=\frac{(q^9;q^5)_\infty(q^3;q^5)_\infty-(q^8;q^5)_\infty(q^4;q^5)_\infty}{(q^9;q^5)_\infty(q^3;q^5)_\infty-q^{-1}\,(q^8;q^5)_\infty(q^4;q^5)_\infty}.$$

Corollary 10.8. *For any complex a, q and z, with $0<|az|<1$, then*

$$\sum_{n=0}^{\infty}\frac{(1/a;q)_n}{(zq;q)_n}a^nz^n=\frac{1-z}{1-az}. \tag{10.15}$$

Proof. In (7.10), replace a by $1/a$, b by q and c by zq

$$\sum_{n=0}^{\infty} \frac{(1/a;q)_n(q;q)_n}{(q;q)_n(zq;q)_n} a^n z^n = \frac{(azq;q)_\infty(z;q)_\infty}{(zq;q)_\infty(az;q)_\infty} \Leftrightarrow \sum_{n=0}^{\infty} \frac{(1/a;q)_n}{(zq;q)_n} a^n z^n = \frac{(azq;q)_\infty(z;q)_\infty}{(zq;q)_\infty(az;q)_\infty}. \tag{10.16}$$

Substitute the left hand side of (10.16) into the left hand side of (10.13). This completes the proof. □

11. CONCLUSION

In this paper, we complete the theory of $(ab;b)_\infty$ and made some applications in the elementary identities in classical q-series theory; including, we evaluate the q-binomial theorem, q-Gauss summation, Rogers-Fine identity and Ramanujan's Theta function, besides other things.

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