On Expansion of Convergence Domain of Dirichlet Series Determining the Riemann Zeta Function

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Abstract

This paper proves that the Dirichlet series determining the Riemann zeta function converges within a domain of a real component of complex variable equal to one, with an imaginary component non-equal to zero.

Keywords: Dirichlet series, Riemann zeta function.

It is known that the Dirichlet series determining the Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

converges within a $\Re[s] > 1$ domain [1]. This paper proves that the Dirichlet series determined as:

$$S(s) = \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n^s} \right), \tag{1}$$

converges within the $\Re[s] \geq 1$ domain, excluding the point $s = 1$.

We shall assume that the infinity in (1) is exactly divisible by any finite real integer number. This definition shall not influence the properties of Dirichlet series (either convergence or divergence) within the $\Re[s] > 0$ domain. In this case, the number of terms of sum, either dropped or added, with N indivisible by some finite real integer number, shall be finite, and the moduli of each term of this sum shall approach zero; hence, the sum of either dropped or added terms at $N \to \infty$ and $\Re[s] > 0$ shall approach zero.

We shall multiple both right and left parts of the (1) by a $(1 - k^{1-s})$ function where k is a finite real integer number which is greater than or equal to two:

$$\left(1 - k^{1-s}\right) S(s) = \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n^s} - \sum_{n=1}^{N} \frac{k}{(kn)^s} \right), \quad k = 2, 3, \ldots, K. \tag{2}$$

Then, we shall represent the last series of the right part of (2) as two series and rewrite the (2):

$$\left(1 - k^{1-s}\right) S(s) = \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n^s} - \sum_{n=1}^{\frac{N}{k}} \frac{k}{(kn)^s} - \sum_{n=\frac{N}{k}+1}^{N} \frac{k}{(kn)^s} \right). \tag{3}$$

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Let us introduce two functional series of two variables and determine them with series of the right part of (3):

$$A\left(k,s\right)=\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n^{s}}-\sum_{n=1}^{\frac{N}{k}}\frac{k}{\left(kn\right)^{s}}\right). \tag{4}$$

$$B\left(k,s\right)=\lim_{N\to\infty}\left(\sum_{n=\frac{N}{k}+1}^{N}\frac{k}{\left(kn\right)^{s}}\right). \tag{5}$$

After that, we can represent the (3) as follows:

$$\left(1-k^{1-s}\right)S\left(s\right)=A\left(k,s\right)-B\left(k,s\right). \tag{6}$$

We shall write down a trivial equality where N/k is integer:

$$\sum_{n=1}^{N}\frac{1}{n}-\sum_{n=1}^{\frac{N}{k}}\frac{1}{n}=\sum_{n=\frac{N}{k}+1}^{N}\frac{1}{n}. \tag{7}$$

Then we shall determine two limits:

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n}-\ln N\right)=\gamma_0,\quad \lim_{N\to\infty}\left(\sum_{n=1}^{\frac{N}{k}}\frac{1}{n}-\ln\frac{N}{k}\right)=\gamma_0, \tag{8}$$

where γ_0 is the Euler constant. While transiting to limits, proceeding from the (7) and using the (8), we shall obtain the following:

$$\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n}-\sum_{n=1}^{\frac{N}{k}}\frac{1}{n}\right)=\lim_{N\to\infty}\left(\sum_{n=\frac{N}{k}+1}^{N}\frac{1}{n}\right)=\lim_{N\to\infty}\left(\ln N-\ln\frac{N}{k}\right)=\ln k. \tag{9}$$

With $s=1$, the following arises from (4), (5) and (9):

$$A\left(k,1\right)=B\left(k,1\right)=\ln k. \tag{10}$$

With $k=2$, we shall have the following from the (4):

$$A\left(2,s\right)=\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n^{s}}-2\sum_{n=1}^{\frac{N}{2}}\frac{1}{\left(2n\right)^{s}}\right)=$$

$$\lim_{N\to\infty}\left(\sum_{n=1}^{\frac{N}{2}}\frac{1}{\left(2n-1\right)^{s}}+\sum_{n=1}^{\frac{N}{2}}\frac{1}{\left(2n\right)^{s}}-2\sum_{n=1}^{\frac{N}{2}}\frac{1}{\left(2n\right)^{s}}\right)=$$

$$\lim_{N\to\infty}\left(\sum_{n=1}^{\frac{N}{2}}\frac{1}{\left(2n-1\right)^{s}}-\sum_{n=1}^{\frac{N}{2}}\frac{1}{\left(2n\right)^{s}}\right)=\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{\left(-1\right)^{n-1}}{n^{s}}\right). \tag{11}$$

As follows from the (11):

$$A(2,s) = \eta(s),$$

where $\eta(s)$ is the Dirichlet eta function. With $k = 3$, we shall obtain from the (4):

$$A(3,s) = \lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n^s} - 3\sum_{n=1}^{\frac{N}{3}}\frac{1}{(3n)^s}\right) =$$

$$\lim_{N\to\infty}\left(\sum_{n=1}^{\frac{N}{3}}\frac{1}{(3n-2)^s} + \sum_{n=1}^{\frac{N}{3}}\frac{1}{(3n-1)^s} + \sum_{n=1}^{\frac{N}{3}}\frac{1}{(3n)^s} - 3\sum_{n=1}^{\frac{N}{3}}\frac{1}{(3n)^s}\right) =$$

$$\lim_{N\to\infty}\left(\sum_{n=1}^{\frac{N}{3}}\frac{1}{(3n-2)^s} + \sum_{n=1}^{\frac{N}{3}}\frac{1}{(3n-1)^s} - 2\sum_{n=1}^{\frac{N}{3}}\frac{1}{(3n)^s}\right). \tag{12}$$

In the general case, for each k we can obtain the following from the (4):

$$A(k,s) = \lim_{N\to\infty}\left(\sum_{n=1}^{\frac{N}{k}}\sum_{m=1}^{k-1}\left(\frac{1}{(kn-m)^s} - \frac{1}{(kn)^s}\right)\right). \tag{13}$$

According to theorem on uniform convergence of Dirichlet series [2], and as follows from the (13), the $A(k,s)$ functional series within the $\Re[s] > 0$ domain converges. The $B(k,s)$ functional series (5) with $s = 1$ equals to $\ln k$ (10), with that, all terms of the sum which determines the value of series in the point $s = 1$ are positive and uniformly decrease, thus, the $B(k,s)$ functional series converges within the $\Re[s] \geq 1$ domain.

We shall rewrite the equality (6):

$$\left(1 - k^{1-s}\right) S(s) = A(k,s) - B(k,s). \tag{14}$$

The value of Dirichlet series in the point $s = 1$ cannot be determined from the (14) since the equality (14) is trivial. But, as follows from the (14) and from the convergence of $(1 - k^{1-s})$ function and $A(k,s)$ and $B(k,s)$ functional series within the $\Re[s] \geq 1$ domain, the Dirichlet series determined as (1) converges in the rest points of the $\Re[s] \geq 1$ domain.

References

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[2] Cahen E., *Sur la fonction $\zeta(s)$ de Riemann et sur des fonctions analogues*, Annales scientifiques de l'Ecole Normale Superieure, **11**, (1894), 75–164.