The Squared Case of π^n is Irrational Gives π is Transcendental

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Two recent articles use transcendence techniques to prove e^n and π are irrational [1, 2]. In these articles the mean value theorem is used for e and complex integration for π to give equivalents to Lemma 2 below. As Lemma 2 drops the necessity of separate real and complex cases, some slight gain of simplicity and efficiency is achieved. In addition, the natural number powers of π can now be proven irrational with the transcendence of this constant an easy generalization. The missing proofs for lemmas in this article can be found in the e companion to this article [3].

In what follows, x is complex number, all polynomials are integer polynomials, and p is a prime.

Definition 1. Given a polynomial $f(x)$, lowercase, the sum of all its derivatives is designated with $F(x)$, uppercase.

Definition 2. For non-negative integers n, let $\epsilon_n(x)$ denote the infinite series

$$\frac{x}{n+1} + \frac{x^2}{(n+1)(n+2)} + \cdots + \frac{x^j}{(n+1)(n+2)\ldots(n+j)} + \cdots.$$

Lemma 1. *If $f(x) = cx^n$, then*

$$F(0)e^x = F(x) + \epsilon, \tag{1}$$

where ϵ has polynomial growth in n.

Lemma 2. *If $f(x) = c_0 + c_1x + \cdots + c_nx^n$, then*

$$e^xF(0) = F(x) + \epsilon, \tag{2}$$

where ϵ has polynomial growth in the degree of f.

Lemma 3. *If the polynomial $f(x)$ has a root of multiplicity p and dr^k is a term of $F(r)$ then $p!|d$.*

Lemma 4. *Let polynomial $f(x)$ have root $r = 0$ of multiplicity $p-1$ then, for large enough p, $p \nmid F(0)$.*

Lemma 5. *If a and b are Gaussian integers with $p > |a| + |b|$ then $|a(p-1)!| + |bp!|$ is a non-zero integer divisible by $(p-1)!$.*

Proof. Suppose, to obtain a contradiction, that $|a(p-1)!| + |bp!| = 0$. Then $p||a|$ or $p|(p-1)!$, a contradiction. Clearly, $(p-1)!|p!$. □

Theorem 1. *π^2 is irrational.*

Proof. Suppose $\pi^2 = a/b$, with a and b natural numbers, $a > b$. Let $a_2(z) = z^2 - (\pi i)^2$, then a_2 has two roots: $r_1 = \pi i$ and $r_2 = -\pi i$, one of which is πi.① As one root is πi, we have

$$0 = (1 + e^{r_1})(1 + e^{r_2}) = 1 + e^{r_1} + e^{r_2} + e^{r_1 + r_2} = 2 + e^{r_1} + e^{r_2}.② \tag{3}$$

Form a polynomial for roots 0 with multiplicity $p-1$ and the non-zero exponents in (3) with multiplicity p. Multiply it by a power of b that makes it an integer polynomial:

$$f_2(z) = b^{2p-1} z^{p-1}[(z - \pi i)(z + \pi i)]^p = (bz)^{p-1}(bz + a)^p.③ \tag{4}$$

We then have, using (3) with (4),

$$0 = F(0)(1 + e^{r_1})(1 + e^{r_2}) = 2F(0) + F(r_1) + F(r_2) + \epsilon.$$

Using 4, for $p > \max\{2, b\}$, $p \nmid 2F(0)$ and $(p-1)!|2F(0)$. Now, per Lemma 3, the coefficients of $F(r_1)$ and $F(r_2)$ will be of the form $(p+j)!c_j$. We can observe the sum of the powers of the non-zero roots involved in $F(r_1) + F(r_2)$ will be integers as well: odd powers cancel to zero and even powers are under the rationality assumption of π^2. For example,

$$(b\pi i)^{2n} + (-b\pi i)^{2n} = (bi)^{2n}(a/b)^n + (bi)^{2n}(a/b)^n = 2(i)^{2n} a^n b^n,$$

a power of i makes it real, in this case.④

Finally,

$$0 = \frac{2F(0) + F(r_1) + F(r_2) + \epsilon}{(p-1)!}$$

gives a contradiction for large enough p. □

① In general, $a_n(z) = z^n - (\pi i)^n$ will have n roots, r_j, one of which is πi.
② In general, the exponents will consist of sums of r_j roots taken one through n at a time, with some adding to 0 and being absorbed in the A value.
③ In general, the fundamental theorem of symmetric functions insures that the sum of roots polynomial will have coefficients that are integer polynomials of the *by assumption* polynomial, $a_n(z)$; that is the sum of the roots, as they are symmetric, generate a polynomial with coefficients that are integer polynomials of the coefficients of $a_n(z)$. Consequently, as the only coefficient of $a_n(z)$ is a/b, a power of b will work. Making the power of b the maximum exponent of z works for this purpose.
④ In general, Newton's identities show that the sum of the powers of the roots are symmetric functions and as such can be expressed as integer polynomials of the coefficients of the polynomial they are roots of. So, the pattern is coefficients of $a_n(z)$ form the coefficients of $f_n(z)$ and thus the sums of the powers of the roots of $f_n(z)$ are, in turn, integer polynomials of a/b, the only coefficient in $a_n(z)$.

The annotations of the square case of the irrationality of π shows the general π^n case. With a slight adjustment of this π^n case, π is proven transcendental.

Theorem 2. *π is transcendental.*

Proof. A number is transcendental if it doesn't solve an integer polynomial. Suppose πi solves an nth degree integer polynomial $a_t(z)$ with roots r_j, then the roots in the proof of the irrationality of π^n are replaced with these roots: all the steps are the same and lead, as in the irrationality case, to a contradiction. □

References

[1] T. Beatty and T.W. Jones, A Simple Proof that $e^{p/q}$ is Irrational, *Math. Magazine*, **87**, (2014) 50–51.

[2] T. W. Jones, Euler's Identity, Leibniz Tables, and the Irrationality of Pi, *Coll. Math. J.*, **43** (2012) 361–364.

[3] ______, The Irrationality and Transcendence of e Connected (2017), available at http://vixra.org/abs/1711.0130