

The Irrationality of Trigonometric and Hyperbolic Functions

Timothy W. Jones

Introduction

Niven shows that for rational, non-zero r , $\cos r$ and $\cosh r$ are irrational [6]. His method is similar to that of his famous irrationality of π proof: functions are defined, integrals involving integration by parts are used, and a contradiction is arrived at [5]. Parks makes a similar argument, arguably simpler, for the cosine case [7]. Zhou recently proved the cosine and hyperbolic cosine results using recursive integrals [8]. In this article, our pattern does not involve integrals; just multiplication and derivatives of polynomials are needed.

The pattern is to start with an exponential equation where a sum of exponential values equals a rational number. So, for \cosh this equation is $e^r + e^{-r} = a/b$. Then a polynomial, f , is defined. It has a zero root of multiplicity $p - 1$, p a prime, and the exponents in the sum, r , $-r$, for \cosh , as additional roots of multiplicity p . The sum of its derivatives is given by $F(x)$. Using $e^x F(0) = F(x) + \epsilon$, proven below, and simple multiplication, this gives $0 = F(0)(e^r + e^{-r} - a/b) = -a/bF(0) + F(r) + F(-r) + \epsilon$. As the ϵ value grows power wise in the degree of f and multiplicity in f translates into factorial values in F , division by $(p - 1)!$ gives a contradiction for large enough p . Details follow.

As the transform $e^x F(0) = F(x) + \epsilon$ is good for complex variables, as well as real, the identity $2 \cos r = e^{ri} + e^{-ri}$ is similar to the pattern for \cosh and its identity $2 \cosh r = e^r + e^{-r}$ just used. Using other identities the corresponding irrationality of other trigonometric and hyperbolic functions are obtained. Corresponding results for inverses of these functions are easily proven as well.

The methods used in this article are also used in [3, 4] to show the transcendence of e and π and the irrationality of their natural number powers.

Lemmas

All polynomials are integer polynomials, z is a complex number, n is a non-negative integer, and p is a prime number.

Definition 1. Given a polynomial $f(z)$, lowercase, the sum of all its derivatives is designated with $F(z)$, uppercase.

Definition 2. For non-negative integers n , let $\epsilon_n(z)$ denote the infinite series

$$\frac{z}{n+1} + \frac{z^2}{(n+1)(n+2)} + \cdots + \frac{z^j}{(n+1)(n+2)\cdots(n+j)} + \cdots$$

Lemma 1. If $f(z) = cz^n$, then

$$F(0)e^z = F(z) + \epsilon, \tag{1}$$

where ϵ has polynomial growth in n .

Proof. As $F(x) = c(x^n + nx^{n-1} + \cdots + n!)$, $F(0) = cn!$. Thus,

$$\begin{aligned} F(0)e^x &= cn!(1 + x/1 + x^2/2! + \cdots + x^n/n! + \dots) \\ &= cx^n + cnx^{(n-1)} + \cdots + cn! + cx^{n+1}/(n+1)! + \dots \\ &= F(x) + cx^n(x/(n+1) + x^2/(n+1)(n+2) + \dots) \\ &= F(x) + f(x)\epsilon_n(x). \end{aligned}$$

Now $f(x)$ has polynomial growth in n and $\epsilon_n(x) \leq e^x$, so the product has polynomial growth in n . □

Lemma 2. If F is the sum of the derivatives of the polynomial $f(z) = c_0 + c_1z + \cdots + c_nz^n$ of degree n , then

$$e^z F(0) = F(z) + \epsilon, \tag{2}$$

where ϵ has polynomial growth in the degree of f .

Proof. Let $f_j(x) = c_j x^j$, for $0 \leq j \leq n$. Using the derivative of the sum is the sum of the derivatives,

$$F = \sum_{k=0}^n (f_0 + f_1 + \cdots + f_n)^{(k)} = F_0 + F_1 + \cdots + F_n,$$

where F_j is the sum of the derivatives of f_j . Using Lemma 1,

$$e^x F_j(0) = F_j(x) + \epsilon \quad (3)$$

and summing (3) from $k = 0$ to n , gives

$$e^x F(0) = F(x) + n\epsilon.$$

As the finite sum of functions with polynomial growth in n also has polynomial growth in n , we arrive at (2). \square

Lemma 3. *If polynomial $f(x)$ of degree n has root r of multiplicity p , then $f^{(k)}(r) = 0$ for $0 \leq k \leq p - 1$ and each term of $f^{(k)}$, $p \leq k \leq n$ is a multiple of $p!$.*

Proof. Suppose $r = 0$ then $f(x) = x^p(b_n x^n + \cdots + b_0)$ has $b_0 x^p$ as its term with minimal exponent. Using the derivative operator, $D(x^n) = nx^{n-1}$, repeatedly, we see the 0 through $p - 1$ derivatives of $f(x)$ will have a positive exponent of x in each term. This implies that $r = 0$ is a root, as needed for this case. Using the product of p consecutive natural numbers is divisible by $p!$, terms of subsequent derivatives will be multiples of $p!$, .

If $r \neq 0$, then $f(x) = (x - r)^p Q(x)$, for some polynomial $Q(x)$. Let $g(x) = f(x + r) = x^p Q(x + r)$. As $g^{(k)} = f^{(k)}$ for all k , $g^{(k)}(0) = f^{(k)}(r)$, and the $r = 0$ case applies. \square

A Leibniz table can be used to give an example of the result of Lemma 3. Suppose $f(z) = (z - r)^3 Q(z)$, where $Q(z)$ is a polynomial of degree 2. Table 1 indicates $f^{(k)}(r) = 0$ for $0 \leq k \leq 2$ and $3!$ divides the terms of $f^{(k)}(r)$ for $3 \leq k \leq 5$. For more on Leibniz tables see [2].

Lemma 4. *If a and b are Gaussian integers and p is a prime, $p > |a|$, then $|a(p - 1)! + bp!|$ is a non-zero integer divisible by $(p - 1)!$, but not by p .*

Proof. As $a(p - 1)! + bp!$ is of the form $A - B + (C - D)i$ with $A - B \neq 0$ or $C - D \neq 0$, A , B , C , and D integers, both results follow. \square

| | | | | |
|----------|-----------|------------|----------|---------------|
| | $(x-r)^3$ | $3(x-r)^2$ | $6(x-r)$ | $3!$ |
| $Q(z)$ | 0 0 | 0 1 | 0 2 | $3!bQ(r)$ 3 |
| $Q'(z)$ | 0 1 | 0 2 | 0 3 | $3!bQ'(r)$ 4 |
| $Q''(z)$ | 0 2 | 0 3 | 0 4 | $3!bQ''(r)$ 5 |

Table 1: First interior cell values give evaluations at $x = r$ and the second the *order* of the derivative. The 'b' factor is a binomial coefficient.

Lemma 5. *Let polynomial $f(z)$ have root $r = 0$ of multiplicity $p - 1$ then, for large enough p , $p \nmid |F(0)|$.*

Proof. We can write

$$f(x) = x^{p-1}(b_j x^j + \dots + b_0) \quad (4)$$

$$= x^{p-1}(b_j x x^{j-1} + \dots + b_1 x) + x^{p-1} b_0 \quad (5)$$

$$= x^p(b_j x^{j-1} + \dots b_1) + x^{p-1} b_0. \quad (6)$$

Using Lemma 3, as $r = 0$ is a root of multiplicity p of $x^p(b_j x^{j-1} + \dots b_1)$, its first $p - 1$ derivatives, evaluated at $r = 0$ are 0 and then its terms are multiples of $p!$. Similarly the first $p - 2$ derivatives of $x^{p-1} b_0$ are 0 at 0 and the $p - 1$ derivative is $(p - 1)! b_0$ and subsequent derivatives are multiples of $p!$. If $p > |b_0|$, as all terms but one in $F(0)$ is divisible by p , the result follows. \square

Applications

Theorem 1. *For non-zero rational r , $\cosh r$ is irrational.*

Proof. Suppose not. Suppose $2 \cosh r = a/b$ where a/b is a rational number. As 0 is not in the range of \cosh , we can assume $a/b \neq 0$. Using the exponents of $2 \cosh = e^r + e^{-r}$, define

$$f(z) = d^{2p-1}[(z+r)(z-r)]^p = (dz)^{p-1}(dz - c^2)^p$$

where $r = c/d$. Then $f(z)$ is an integer polynomial. Next

$$0 = F(0) \left(e^r + e^{-r} - \frac{a}{b} \right).$$

Using Lemma 2

$$0 = bF(r) + bF(-r) - aF(0) + b\epsilon$$

and, using Lemmas 3, 4, and 5, this gives a contradiction. \square

As Lemma 3 applies to complex polynomials as well as real, the proof for $\cos r$ is similar.

Theorem 2. *For non-zero rational r , $\cos r$ is irrational.*

Proof. Suppose not. Suppose $2\cos r = a/b$ where a/b is a rational number. We exclude $a/b = 0$ as $\cos k\pi/2 = 0$ has \cos with an irrational argument. Using the exponents of $2\cos = e^{ri} + e^{-ri}$, define

$$f(z) = d^{2p+p-1} z^{p-1} [(z + ri)(z - ri)]^p = (dz)^{p-1} ((dz)^2 + c^2)^p$$

where $r = c/d$. Then $f(z)$ is an integer polynomial. Next

$$0 = F(0) \left(e^{ri} + e^{-ri} - \frac{a}{b} \right).$$

Using Lemma 2 and multiplying by b gives

$$0 = bF(ri) + bF(-ri) - aF(0) + b\epsilon$$

and, using Lemmas 3, 4, and 5, this gives a contradiction. \square

Note: As $\cos \pi = -1$, Theorem 2 does imply that π is irrational. This does imply that $k\pi/2$ is also irrational – the case missing from Theorem 2.

Other functions

Once $\cos r$ and $\cosh r$ are proven irrational, $\sec r$ and $\operatorname{sech} r$ are easy consequences, being reciprocals of these functions [6]. As $\cos 2r = \cos^2 r - \sin^2 r = 1 - 2\sin^2 r$, the rationality of $\sin r$ would imply that of \cos , a contradiction. Assume $\tan r$ is rational. Then using

$$\cos 2r = \frac{1 - \tan^2 r}{1 + \tan^2 r},$$

$\cos r$ would be rational too, a contradiction. As $\csc r$ and $\cot r$ are the reciprocals of $\sin r$ and $\tan r$, the former two are proven irrational. Similarly, the hyperbolic functions all follow the same program. Inverse functions have an easy proof: if $f^{-1}(r) = a/b$ then $f(f^{-1}(r)) = r = f(a/b)$, a contradiction of $f(r)$ is irrational.

References

- [1] T. Beatty and T.W. Jones, A Simple Proof that $e^{p/q}$ is Irrational, *Math. Magazine*, **87**, (2014) 50–51.
- [2] T. W. Jones, Euler’s Identity, Leibniz Tables, and the Irrationality of Pi, *Coll. Math. J.*, **43** (2012) 361–364.
- [3] ———, The Irrationality and Transcendence of e Connected (2017), available at <http://vixra.org/abs/1711.0130>
- [4] ———, The Squared Case of π^n is Irrational Gives Pi is Transcendental (2017), available at <http://vixra.org/abs/1711.0258>
- [5] I. Niven, A simple proof that π is irrational, *Bull. Amer. Math. Soc.* **53** (1947) 509.
- [6] ———, *Irrational Numbers*, Carus Mathematical Monographs, no. 11, Mathematical Association of America, Washington, DC, 1985.
- [7] A. E. Parks, e , π , and other irrational numbers, *Amer. Math. Monthly*, **93** (1986) 722- 723.
- [8] L. Zhou and L. Markov, Recurrent proofs of the irrationality of certain trigonometric values, *Amer. Math. Monthly*, **117** (2010) 360–362.