A new concept for simultaneity in TSR (v6, 2018-03-21)

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Abstract. A main part is completely rewritten in this new version of the paper. We start by specifying a state vector ('time vector') in the theory of special relativity (TSR). This vector is well suited to specify various types of simultaneity. Moving clocks, being synchronized at a common 'point of initiation' play a crucial role. We may present the time vector as a complex variable, and there is a relation to the Minkowski distance. We exemplify the approach by including a short discussion of the 'travelling twin'.

Key words: Time dilation, simultaneity, Lorentz transformation, time vector, Minkowski distance, travelling twin.

1 Introduction

The concept of *simultaneity* becomes crucial when inertial reference frames (RFs) are moving relative to each other. Of course, we have the 'basic simultaneity'; *i.e.* simultaneity of events occurring at the same instant *and* same location, but these are rather the same event, just seen in the perspective of two different RFs. For events at a distance, we can essentially observe simultaneity from the 'perspective' of a certain RF: When the synchronized clocks of a specific RF show the same readings, we have simultaneous events *in the perspective of* this RF. The literature further refers to the 'relativity of simultaneity'. Here we actually present an approach, actually suggesting a definition of simultaneity even 'at a distance'. This is partly based on Hokstad (2016, 2018).

We start by repeating some material related to the Lorentz transformation (LT), and further give a short introduction to simultaneity. It is important here to observe our notion of the 'basic clocks' (BC), which at time 0 are (imagined to be) located at the common origin, and at this instant are all synchronized. This is the basis for specifying a two-dimensional state vector, which we denote a 'time vector'. Essential for the definition of this vector is the BC reading at the current location, and the distance from the RF's own BC. This time vector proves useful when investigating simultaneity also 'at a distance'. We finally include an exemplification utilizing the travelling twin paradox.

This paper gives a purely mathematical description of the phenomenon, and there is no attempt of a physical interpretation.

2 Foundation

We here present some assumptions, and give some background related to the LT, which proves useful.

2.1 Basic notation

We start out with a RF, K_0, where the position along the x-axis is denoted x_0. At virtually any position there are synchronized clocks with clock reading denoted, t_0. We will simply refer to (x_0, t_0) as an event. Further, there is a RF, K_v, moving along the x-axis of K_0 at velocity v. On K_v we have

x_v = The position on K_v, being identical to the location x_0 at a time t_0 on K_0

t_v = Clock reading at position x_v on K_v, when x_v corresponds to x_0, and the clock on K_0 reads t_0.

Observers (observational equipment) on both of these two RFs agree on these four observations. Further,

- There is a complete *symmetry* between the two RFs K_0 and K_v; these being identical in all respects.
- The clock at $x_v = 0$ and the clock at $x_0 = 0$ will when $t_v = t_0 = 0$ be at the same location, and they are then synchronized. We refer to this as the 'point of initiation', and these clocks as 'basic clocks' (BCs).

2.2 The Lorentz transformation (LT) and time dilation

The LT represents the fundament for our discussions. In the above notation the LT takes the form

$$t_v = \frac{t_0 - (v/c^2)x_0}{\sqrt{1-(v/c)^2}} \quad (1)$$

$$x_v = \frac{x_0 - vt_0}{\sqrt{1-(v/c)^2}} \quad (2)$$

We prefer a modified version of the LT. At any time, t_v and position, x_v we introduce w_v equal to $w_v = x_v/t_v$, (and therefore also $w_0 = x_0/t_0$). Then we insert $x_0 = w_0 t_0$, and (1) directly gives that the clock reading on the RF, K_v at this position equals:

$$t_v = t_v(w_0) = \frac{1 - vw_0/c^2}{\sqrt{1-(v/c)^2}} t_0 \quad (3)$$

Note that we – when appropriate – will write $t_v(w_0)$ rather than t_v to pinpoint its dependence on w_0, The new time dilation formula (3) will – for a given clock reading, t_0 on the primary system, K_0 – give the clock reading, $t_v(w_0)$ on the secondary system, K_v, as a linear, decreasing function of w_0. Observe that (3) shows that we can write t_v in the form

$$t_v = t_v(w_0) = \gamma_v(w_0)t_0.$$

Fig.1 provides an illustration of this time dilation formula. Here we give clock reading ('time') both on K_0 and K_v *in the perspective of* K_0; (*i.e.* all clocks on K_0 reading the same time). Therefore, the figure illustrates an instant when time equals t_0 all over this reference frame. The horizontal axis gives the 'position' $w_0 = x_0/t_0$ on K_0 at which the clock measurements are carried out. The vertical axis gives the actual clock readings. So as time on K_0 equals t_0 at any 'position', w_0, the clock readings on K_v at this instant, $t_v = t_v(w_0)$, is a linear function of w_0.

Further, by also inserting $x_0 = w_0 t_0$ and $x_v = w_v \cdot t_v$, in (2), we will from (1) and (2) obtain

$$w_v = \frac{x_v}{t_v(w_0)} = \frac{w_0 - v}{1 - \frac{w_0}{c}\frac{v}{c}} \quad (4)$$

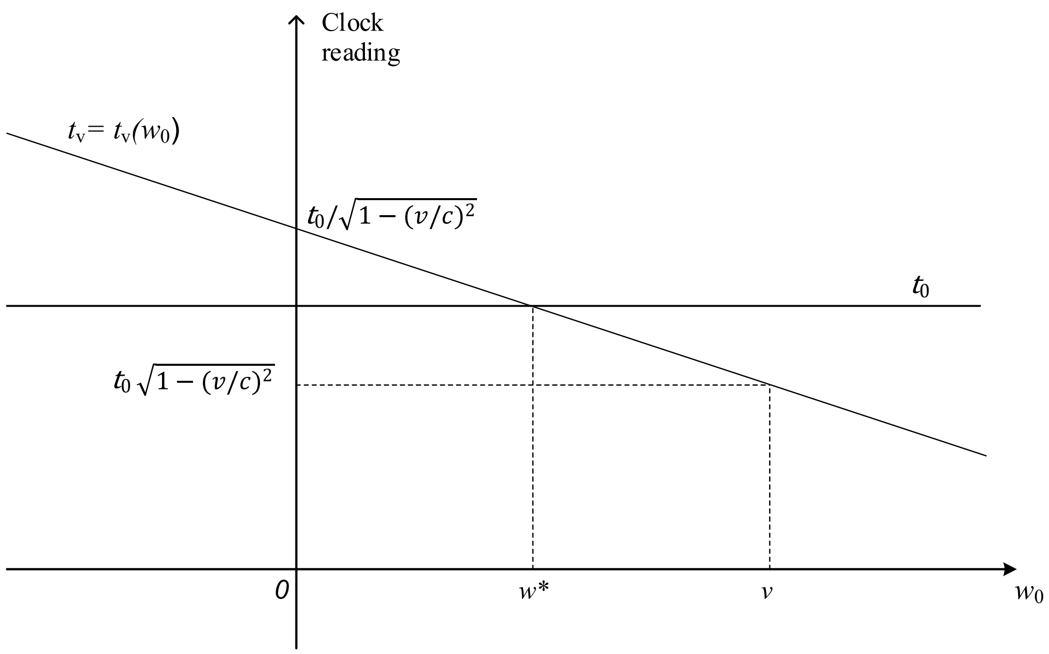

Figure 1. Clock readings in the perspective of K_0. Thus, 'time' all over K_0 equals t_0, while clock readings, $t_v(w_0)$ on the other RF is given as a function of w_0, where $w_0 = x_0/t_0$ provides the 'position' on K_0; *cf.* (3).

Now equations (3), (4) represent an alternative version of the LT, here expressed by parameters (*t, w*) rather than (*t, x*). The most striking feature of this new version is that it is a single equation, (3), which involves the time parameters, t_0 and t_v; giving t_v as a factor independent of t_0 multiplied with t_0.

The other equation (4) has a direct interpretation related to velocities. According to standard results of TSR, *e.g.* Refs. [7]-[9], the velocities v_1 and v_2 sums up to v, given by the formula

$$v = v_1 \oplus v_2 \overset{\text{def}}{=} \frac{v_1+v_2}{1+\frac{v_1}{c}\frac{v_2}{c}} \tag{5}$$

So now defining the operator $\oplus$ this way, eq. (4) actually says that $w_v = w_0 \oplus (-v)$, (also implying $w_0 = w_v \oplus v$); thus, clearly interpreting w_0 and w_v as velocities along the x - axis. That is we have a moving position along the x-axis for clock comparisons. Therefore, this w_0 specifies what we refer to as the *observational principle,* pinpointing that this is an essential factor for the resulting observed time dilation. Note that we do not *need* to think of w_0 as a velocity; rather as a way to specify a certain position $x_0 = w_0 t_0$ on the primary RF, K_0.

2.3 Two standard special cases (observational principles)

Two special cases are of particular interest. Recall that the first clock comparison is carried out at the origins $x_v = x_0 = 0$ when $t_v = t_0 = 0$. Now repeating some essential (and well-known) arguments given in Hokstad (2016, 2018), we specify two choices for the second comparison of clock readings.

First we compare the clock located at $x_v = 0$ on K_v (with the passing clocks on K_0, showing time t_0). Thus, also $w_v = 0$, and (4) implies $w_0 = v$, and (3) gives the relation between the two clock readings at this position, (*cf.* Fig. 1):

$$t_v = t_v(v) = t_0 \sqrt{1-(v/c)^2} \tag{6}$$

This equals the standard 'time dilation formula'. Secondly, we can compare the clock located at $x_0 = 0$ on K_0 with a passing clock on K_v. For $x_0 = w_0 = 0$, (*i.e.* following the basic clock at the origin of K_0), *eq.* (3) gives the following relation, (again see Fig. 1):

$$t_v = t_v(0) = t_0/\sqrt{1-(v/c)^2} \tag{7}$$

The relations, (6), (7) are apparently contradictory; *eq.* (6) tells that the clock on K_v goes slower, and (7) tells that the clock on K_0 goes slower; *cf.* the Dingle's question, (McCausland 2008, 2012). Thus, the time dilation is not a feature of the RF, but follows from which single clock we choose to follow when we perform the second clock comparisons. Therefore, we prefer to formulate the time dilation formulas (6), (7) in compact form as

$$t^{BC} = t^{MC} \sqrt{1-(v/c)^2} \tag{8}$$

Here we have introduced the notation

t^{BC} = The clock reading of a basic clock (BC), *i.e.* clock located at the origin of a RF[1].

t^{MC} = The clock reading at the same location but on the other RF; *i.e.* the clock reading on a RF using multiple clocks (MC) for clock comparisons with the basic clock.

Therefore, both of the RFs can apply a BC for a certain clock comparison, and then conclude that 'time goes slower' on the RF which use BC. However, the same RF would apply MC for a clock comparison with a BC on the other RF; and we would then conclude that 'time goes slower' on this other RF. It is the observational principle, *i.e.* choice of clocks for the clock comparisons that matters; *cf.* discussion in Hokstad (2018). This is a well-known result. According to Petkov (2012) already Minkowski referred to proper time and coordinate time, corresponding the above two concepts of time. However, the underlying duality has perhaps not received the attention it deserves in standard literature.

[1] We have previously also used t^{SC} (where SC = Single Clock) to denote this clock reading

2.4 The symmetric case

There is another interesting special case of the LT, (3), (4). We can ask which value of w_0 (and thus w_v) will result in $t_v = t_0$. We easily find that this equality is obtained by choosing $w_0 = w^*$, where

$$w^* = \frac{c^2}{v}\left(1 - \sqrt{1-(v/c)^2}\right) = \frac{v}{1+\sqrt{1-(v/c)^2}} \tag{9}$$

Further, by this choice of w_0 we also get $w_v = -w^*$. This means that if we consistently consider the positions where simultaneously $x_0 = w^* t_0$ and $x_v = -w^* t_v = -w^* t_0$, then no time dilation will be observed at these positions. In other words (*cf.* Fig. 1):

$$t_v(w^*) = t_0$$

At this position we also find $x_v = -x_0$, and so we see this as the midpoint between the origins of the two RFs; thus, providing a nice symmetry. Note that when we choose the observational principle, (9), then absolutely everything is symmetric, and it should be no surprise that we get $t_v = t_0$.

Note that w^* has a simple interpretation. Recalling the definition of the operator $\oplus$ in eq. (5) for adding velocities in TSR, ($v = v_1 \oplus v_2$), it is easily verified that when w^* is given by (9), then we get $w^* \oplus w^* = v$. So this confirms that when our point of observation 'moves' with velocity w^* relative to K_0 and $-w^*$, relative to K_v, it corresponds exactly to the case that the relative speed between K_0 and K_v equals v.

3 Concepts of simultaneity

Within a single RF simultaneity is easily established by the synchronization of clocks, *e.g.* using light rays, for instance see Einstein (1924), Giulini (2005), Mermin (2005). Further, a specific event, (t, x) will be specified differently by the two RFs. However, this is rather the same event; described by different (time, space) parameters, and we refer to this as *basic simultaneity.*

However, for moving reference frames there is within the TSR no unique definition of simultaneity at a distance. Rather, one refers to *relativity of simultaneity*, *e.g.* see the discussion in Debs and Redhead (1996). In particular they argue for the *conventionality of simultaneity.* That is, when establishing simultaneity at a distance by the use of light signals, the definition of simultaneity is essentially a matter on convention; any time in a certain interval can be seen as simultaneous with a specified distant event.

We would in this respect comment that even if there are several possible definitions for simultaneity at a distance, this does not mean that all are equally valid. If, for instance, we want to model a symmetric situation, there should also be a certain symmetry with respect to simultaneity.

When the events occur at different locations one could refer to the rather weak concept of *simultaneity by perspective.* One can say that events with the same clock reading (t) measured on a specific RF are simultaneous in the perspective of this frame. As we know, this simultaneity depends on the chosen RF.

However, in Hokstad (2018) we found it useful to apply an auxiliary reference frame as a tool to obtain simultaneity at a distance. We simply postulate an auxiliary RF with origin always located at the midpoint between our two main RFs. Further, we utilized the symmetry of this model, so that simultaneous clock readings at the auxiliary RF implies a certain simultaneity at a distance for the two main RFs. In particular, we strongly argued that this approach provides logical and consistent solutions to the travelling twin example.

In the present paper we will pursue a slightly different approach. First we point out that an essential requirement for the use of the LT is that we start out with three sets of synchronizations.

1. All clocks on the first RF, K_0;
2. All clocks on the second RF, K_v;
3. The two clocks at the origins of K_0 and K_v at time 0, this represent *basic simultaneity*, and we refer to these as the *basic clocks* (BCs).

Usually one will here implicitly assume that all clocks on K_0 *remain* synchronized; as also do the clocks on K_v. We will now further argue that also the two BCs at the origins of K_0 and K_v - being synchronized at time 0 - will remain synchronized. They are moving away from each other at speed, v, but in a symmetric situation, there is no way to claim that one of the two clocks goes faster than the other. We have the standard phrase 'moving clock goes slower', but that is when the 'moving clock' is compared with passing clocks, and not with the other clock at the origin; *cf.* discussion of Section 2.2.

So our claim is that when the two 'basic clocks' at the origins of the two RFs show the same time, this corresponds to simultaneous events 'at a distance'. (We consider this rather to be a consequence of our assumption of symmetry between the RFs.) This actually leads to a rather strong form of simultaneity, as all observers can agree on this. This is a main basis for our discussions in the next chapter.

4 The time vector and simultaneity

We now introduce at wo-dimensional state vector. We refer to this as a time vector and will utilize it to define simultaneity. We first introduce the time vector of a single RF; next look at two related RFs.

4.1 Time vector of a single RF

Since we now consider just one RF, we make slight change in the notation by dropping all subscripts v (and 0). Thus we consider a RF, K, with clock readings, t and positions, x. Further, we look at a specific a clock reading, $t = t^0$, and the position, $x = w \cdot t^0$. Now, by letting $w = x/t^0$ we define the time vector for the event (t^0, x) as[2] $\vec{t}(t^0, w) = \begin{pmatrix} \sqrt{1-(w/c)^2} \\ w/c \end{pmatrix} t^0$ with the following alternative expressions

$$\vec{t}(t^0, w) = \begin{pmatrix} \sqrt{1-(w/c)^2} \\ w/c \end{pmatrix} t^0 = \begin{pmatrix} t^0\sqrt{1-(w/c)^2} \\ x/c \end{pmatrix} = \begin{pmatrix} \sqrt{(t^0)^2-(x/c)^2} \\ x/c \end{pmatrix} \quad (10)$$

As stated, its magnitude, t^0, equals the clock reading at the chosen position. We can further interpret the other parameter, $w = x/t^0$, as the velocity relative to K of a (possibly imagined) RF, K_w, (which origin has the same location as the origin of K at the 'point of initiation'). So the point is that at time t^0 on K, the origin of K_w is located in $x = w \cdot t^0$. In other words, the basic clock (BC) located in x at time t^0 is the BC of K_w. This means that we have the relation, (*cf.* (8))

$$t^{BC} = t^0\sqrt{1-(w/c)^2}$$

Where we take t^{BC} to be the clock reading of the BC of K_w. We further introduce

$$\sin\varphi = w/c.$$

Then we can also write $\vec{t}(t^0, w)$ as

$$\vec{t}(t^0, w) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} t^0$$

Further, by utilizing the above relation between t^0 and t^{BC} we also get

$$\vec{t}(t^0, w) = \begin{pmatrix} 1 \\ \tan\varphi \end{pmatrix} t^{BC}$$

Now the vector, $\vec{t}(t^0, w)$ corresponds to a point on the semicircle with radius, t^0 in the $(t, x/c)$ space; see Fig. 2. In summary, for a specific clock time, t^0 and position, $x = wt^0$ on K we interpret the two components of this time vector as follows:

- The first component, $t^{BC} = t^0\sqrt{1-(w/c)^2}$ equals the clock reading of the BC of the (possibly imagined) RF, K_w; which by now is located at the position $x = w\,t^0$ on K. We call this the 'basic time' ('BC reading') at this position.
- The second component, x/c, equals the distance, x from the origin (and thus from the BC) of K to the given position, measured as the time $t^0 \cdot w/c = x/c$ required for a light flash to go to this distance.

[2] An argument for this choice can be found in Appendix B; also see Hokstad (2017).

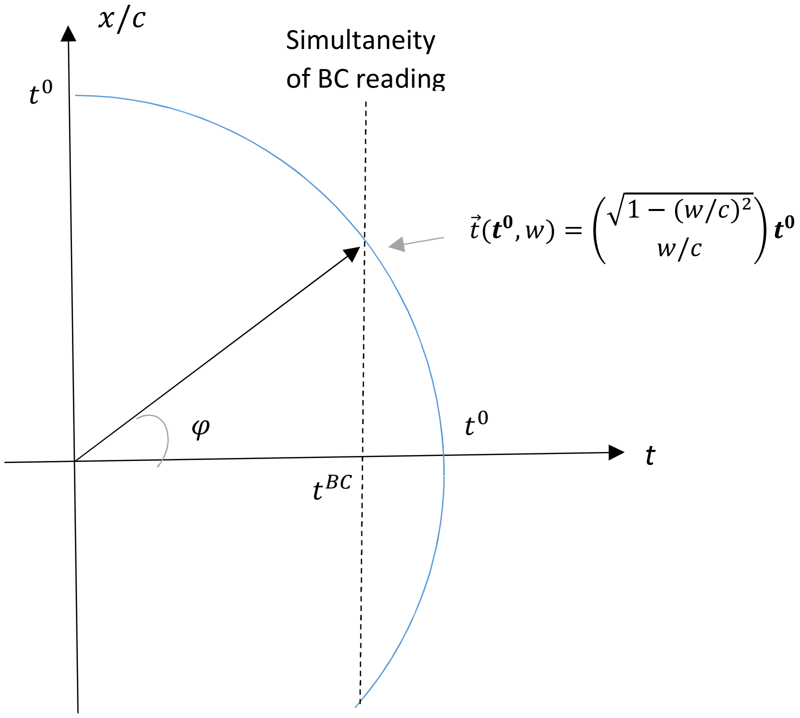

Figure 2 Time vector, $\vec{t}(\boldsymbol{t^0}, \boldsymbol{w}) = \begin{pmatrix} \sqrt{1-(w/c)^2} \\ w/c \end{pmatrix} \boldsymbol{t^0}$ for a specific clock reading t^0 at the position, $x = wt^0$; (here sin φ = w/c).

Thus, both components refer to a distance from the 'point of initiation', ($x = t = 0$). Only for $x = w = 0$, the BC of the first component refers to the BC at *K* itself; and in that case the second component equals zero. At all other positions, the first component refers to an 'imagined BC'. Thus, we implicitly assume that we initially synchronized the basic clocks of *K* and K_w. We further repeat that the absolute value of our time vector is independent of *w* (and *x*):

$$\left| \vec{t}(t^0, w) \right| = t^0 \tag{11}$$

In the perspective of *K* this is not only the clock reading at the chosen position, *x,* but –due to the clock synchronization - equals the 'simultaneous' clock reading at *any* position, on *K, cf.* the blue semicircle of Fig. 2 representing all time vectors on *K* with this absolute value. Thus, events on *K* which have time vector with the same absolute value, have the same clock time on *K,* and are therefore simultaneous 'in the perspective of *K'*.

However, as we know, this is indeed a weak form of simultaneity. Events that are simultaneous in the perspective of one RF, are usually *not* simultaneous in the perspective of another RF. So we should look for a strong form of simultaneity. Actually we will claim that - locally on a specific RF - we should rather consider events with the same BC reading t^{BC} to be simultaneous. This follows from our argument regarding simultaneity of BCs presented in Chapter 3. Therefore, we will consider events having time vectors with identical first components to be simultaneous; *cf.* stipled vertical line in Fig. 2.

We here observe that there is a strong link between this approach and Minkowski's approach to space-time; *cf.* space-time distance as $\sqrt{c^2t^2 - x^2 - y^2 - z^2}$ in his four-dimensional space, Minkowski (1909). As stated in Petkov (2012), Minkowski refers to the time of the basic clock, (*cf.* $\sqrt{(t^0)^2 - (x/c)^2}$ of (10)), as 'proper time, and our t^0 as 'coordinate time'. However, to my knowledge this has not been applied in the discussion of simultaneity.

Finally we note that the BC reading, t^{BC} is not just the BC reading of a specific, RF, *K.* It must be the BC reading at this position for *any* RF. Thus, the first component, t^{BC}, of the time vector is also most relevant when we discuss simultaneity 'across RFs'. We return to this in Section 4.3 below.

4.2 Time formulated as a complex variable

We can of course formulate the time vector for the event (t^0, x) as a complex variable. In polar form, we can write the vector $\vec{t}(t^0, w)$ in (10) as

$$\mathbf{t}(t^0, w) = t^0 e^{i\varphi}, \quad (w = x/t^0) \tag{12}$$

Here the magnitude, t^0 still equals the clock reading, and we interpret $w = x/t^0$ as the velocity relative to K of an (imagined) basic clock (BC), now having arrived at the position, x. Further, the argument, $\varphi \in (-\pi/2,\ \pi/2)$, is given by

$$\sin\varphi = w/c.$$

When $\varphi = 0$, we have $w = x = 0$. Then we are at the origin of K, and the relevant BC is the one located on K itself. In this case the time variable becomes a real number.

As specified in Section 4.1, the real part, $\mathrm{Re}(\mathbf{t}(t^0, w)) = t^0\sqrt{1-(w/c)^2} = t^0\cos\varphi = t^{BC}$ gives the BC reading at the location, and this identifies simultaneity. The imaginary part, $\mathrm{Im}(\mathbf{t}(t^0, w)) = t^0(w/c) = t^0 \sin\varphi = x/c$ represents the distance (measured in terms of time required for a light flash) from the RFs own BC to the position in question.

Finally, we can generalize (12) to hold for a three-dimensional space, with coordinates (x, y, z). We then define w by $w = \sqrt{x^2+y^2+z^2}\ /t^0$. Thus, w still specifies a position (distance from origin) at time t^0 on the RF K, and we still have $\sin\varphi = w/c$, using the new definition of w.

4.3 Time vectors of two related RFs

We now consider how to relate the time vectors, $\vec{t}(t^0, w)$ of two different RFs, moving relative to each other. We return to the notation of Chapter 2, with two RFs K_0 and K_v. Relative to these RFs we have the 'corresponding' positions $x_0 = w_0 t_0$ and $x_v = w_v t_v$, and the LT is given as (3), (4). We further introduce

$$\sin\varphi_v = w_v/c \text{ (all } v\text{; also } v=0) \tag{13}$$

This defines an angle φ_v relative to an event $(x_v,\ t_v)$ on any K_v, (including K_0). As suggested in Section 4.2 we can write the time vector as

$$\mathbf{t}(t_v, w_v) = t_v e^{i\varphi_v}$$

We will now utilized the LT to find the relation between the time variables $\mathbf{t}(t_0, w_0)$ and $\mathbf{t}(t_v, w_v)$ of two 'corresponding' events, (having basic simultaneity). Now first replace v in the LT, (3), (4) with an angle, θ_v, given by

$$\sin\theta_v = v/c \tag{14}$$

Which implies that $\cos\theta_v = \sqrt{1-(v/c)^2}$. Then we can formulate the LT, (3), (4) as:

$$t_v = \frac{1-\sin\varphi_0 \sin\theta_v}{\cos\theta_v}\cdot t_0 \tag{15}$$

$$\sin\varphi_v = \frac{\sin\varphi_0 - \sin\theta_v}{1-\sin\varphi_0\sin\theta_v} \tag{16}$$

This new version of the LT tells how we can express $\mathbf{t}(t_v, w_v) = t_v e^{i\varphi_v}$ in terms of $\mathbf{t}(t_0, w_0)$ and θ_v. A calculations first gives the result that the real part is constant under the LT:

$$t_v \cos\varphi_v = t_0 \cos\varphi_0 = Const.$$

Actually, this just equals our earlier result that $t_v\sqrt{1-(w_v/c)^2}$ (all v) equals the unique clock reading of the BC at this position and this time; *i.e.,* t^{BC}. Thus

$$Re(\mathbf{t}(t_v, w_v)) = t_v\cos\varphi_v = t_v\sqrt{1-(w_v/c)^2} = t^{BC}, \text{ (all } v\text{)} \tag{17}$$

Further, we have various expressions for the imaginary part of the complex time vector:

$$Im\big(\mathbf{t}(t_v, w_v)\big) = \frac{x_v}{c} = \left(\frac{w_v}{c}\right) t_v = t_v \sin\varphi_v = \tan\varphi_v\, t^{BC} \tag{18}$$

Utilizing the LT, (15), (16), we obtain an expression for $t_v \sin\varphi_v$, and get

$$Im(\mathbf{t}(t_v, w_v)) = \frac{\sin\varphi_0 - \sin\theta_v}{\cos\theta_v} t_0 = \frac{w_0/c - v/c}{\sqrt{1-(v/c)^2}} t_0$$

(where we could also insert $t_0 = t^{BC}/\sqrt{1-(w_0/c)^2}$). In particular, for v =0 we have $Im(\mathbf{t}(t_0, w_0)) = \sin\varphi_0\, t_0$. To summarize, we may express the LT for K_v, and K_0 as

$$Re\big(\mathbf{t}(t_v, w_v)\big) = Re\big(\mathbf{t}(t_0, w_v)\big) = t^{BC} \text{ (= BC reading of given event)}$$

$$Im\big(\mathbf{t}(t_v, w_v)\big) = \frac{\sin\varphi_0 - \sin\theta_v}{\sin\varphi_0 \cos\theta_v}\, Im(\mathbf{t}(t_0, w_0))$$

Here also, $\frac{\sin\varphi_0 - \sin\theta_v}{\sin\varphi_0\cos\theta_v} = \frac{\tan\varphi_v}{\tan\varphi_0}$. Finally, we can now also write the time vector as

$$\vec{t}(t_v, w_v) = \begin{pmatrix}\cos\varphi_v \\ \sin\varphi_v\end{pmatrix} t_v = \begin{pmatrix}1 \\ \tan\varphi_v\end{pmatrix} t^{BC} \tag{19}$$

Fig. 3 illustrates the two time vectors at K_0 and K_v. Recall that we consider one and the same event in the perspective of K_0 and K_v, respectively, at the instant when the BC on the given location reads time t^{BC}. The blue semicircle gives the time vectors on K_0 corresponding to events being simultaneous with the chosen on 'in the perspective of K_0' (that is, showing clock time, t_0). The red semicircle gives the time vectors on K_v corresponding to events being simultaneous with the chosen on 'in the perspective of K_v' (that is, showing clock time, t_v). The t_0 and t_v are related through (15). The vectors are here given in a common coordinate system, (t, x/c).

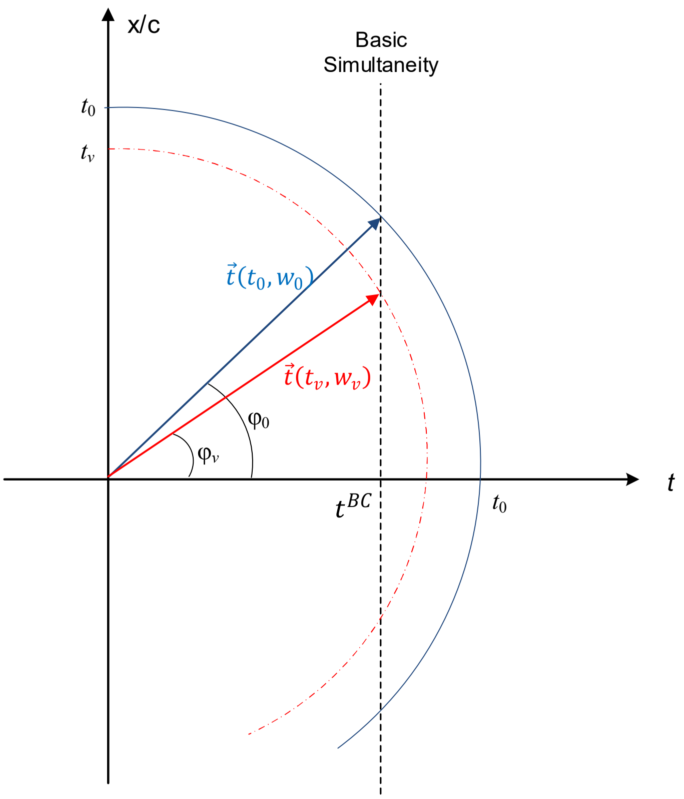

Figure 3 Time vector, $\vec{t}(t_0, w_0)$ on K_0 when its clocks read time t_0 (blue), and time vector, $\vec{t}(t_v, w_v)$ on K_v at the same position (red). The given position has a BC reading equal to t^{BC}

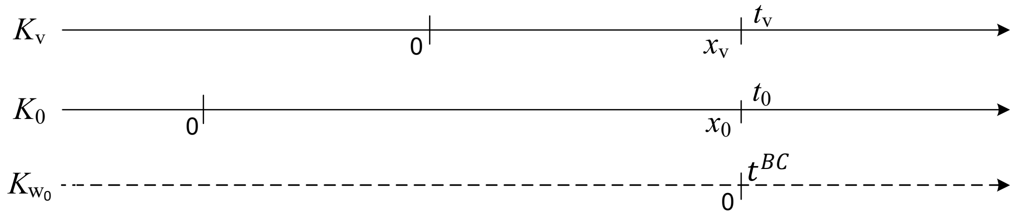

Figure 4 Two events (x_0, t_0) and (x_v, t_v) representing basic simultaneity, and clock reading, t^{BC} of a BC at the same position. Origins of all RFs are marked with a zero, 0.

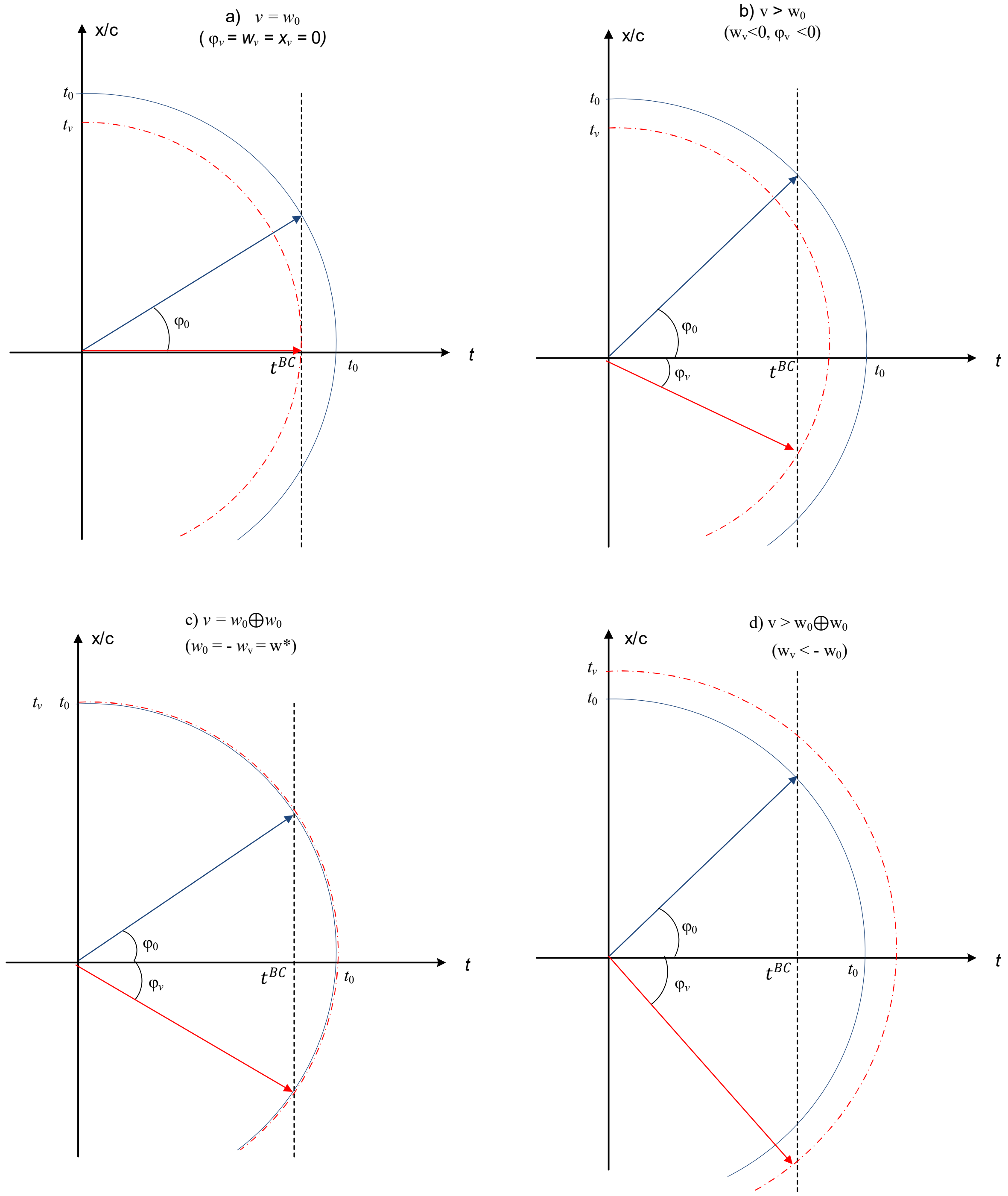

Figure 5 Further examples of a time vector, $\vec{t}(t_0, w_0)$ on K_0 (blue), and 'corresponding' time vector $\vec{t}(t_v, w_v)$ on K_v (red).

Fig. 3 presents an example where $0 < v < w_0$, and using (16), also $0 < w_v < w_0$; thus, $0 < \varphi_v < \varphi_0$. This is the situation illustrated in Fig. 4. The next Fig. 5 illustrates some further cases (all with $v > 0$):

a) $v = w_0$. In this case $x_v = w_v = \varphi_v = 0$, and the origin of K_v is located at the relevant position; *i.e.* the BC of this event is actually located on K_v itself.
b) $v > w_0 > 0$. Using (16) we get $\varphi_v < 0$. So also $w_v < 0, x_v < 0$.
c) $v = w_0 \oplus w_0$. In this case we know $w_0 = -w_v = w^*$, (and also $x_v = -x_0$). Here the blue and red semicircle coincide.
d) $v > w_0 \oplus w_0$. In this case, $w_v < -w_0$.

4.4 Summary of time vector and simultaneity

To summarize: On any RF, *K,* we introduce a time vector related to an event (t, x), where $x = wt$, which splits the clock reading ('clock time'), t in two components:

1. $t \cdot \sqrt{1-(w/c)^2} = t^{BC}$; the clock reading, of the (imagined) BC currently at this position
2. $t \cdot w/c = x/c$; clock time required for a light flash to go from the RF's own BC (at its origin) to the current position.

So, both components represent 'distance in time' from the 'point of initiation'. The absolute value equals the clock reading of the event. It is rather sensible to take the time vector as a complex variable, with the first component as the real part, and the second as the imaginary.

We point out that this time vector is suitable for specifying various forms of simultaneity:

1. Simultaneity 'in the perspective of a RF' occurs when time vectors (on this RF) have the same absolute value; *cf.* semicircles of previous figures. This represents simultaneity in a 'weak sense'. So synchronization of clocks does not imply that identical clock readings (even at the same location) infers simultaneity.
2. Simultaneity in a 'strong sense' occurs for two events having time vectors with the same real part. This means that the (possibly imagined) BCs present at the location of the events have the same clock reading.
   i. When we refer to two events of the same RF, there are necessarily different BCs involved (at different locations); *cf.* Section 4.1. (These events are of course *not* simultaneous *in the perspective* of *K*.)
   ii. When we refer to two events of different RFs, (moving relative to each other at speed, *v*), it could be one and the same BC involved; *cf.* Section 4.3. In this case, we have what we call 'basic simultaneity'; it is the same event just specified in two different RFs. In this case the LT describes the relation between the two time vecors.
3. We can further combine the two version of the above (strong) simultaneity concept 2, to define simultaneity 'at a distance' also for events related to different RFs. Assume that we at a location A have basic simultaneity (type 2.ii) defined by the BC at this location. Further, on both these RFs we have strong simultaneity with an event at another location B on the same RF (type 2.i). When both these conditions are satisfied, it follows that we also have simultaneity between the two events at A and B, as specified by the two different RFs. Again it is the real part of the time vector (the local t^{BC}) that defines this simultaneity. A discussion in Appendix B elaborates on this case, and we further refer to the example on the travelling twin in the next chapter.

We note the limitation of the given approach: So far we have just defined simultaneity relative to a specific 'point of initiation'. However, if we first identify simultaneous events 'at a distance' relative to a given 'point of initiation' (as described above), it should in addition be possible to define further simultaneities based on these events, now treated as new 'common' points of initiation. This could represent a considerable extension.

5 The travelling twin

The travelling twin paradox is frequently discussed, *e.g.* see Schuler and Robert (2014). As stated for instance in Mermin (2005) this illustrates that two identical clocks, initially in the same place and reading the same time, can end up with different readings if they move apart from each other and then back together. We gave a lengthy discussion in Hokstad (2018), and now restrict to a short comment.

We start out with two synchronized clocks at the origins of two reference frames; the RF of the earth, and the RF of the rocket of the travelling twin. This is exactly the situation described in Ch. 3. We note that both clocks are located at the origin of their RFs, and so both are basic clocks (BCs).

Fig. 6 gives an illustration of relevant time vectors by the arrival at the star, using the numerical example of Mermin (2005). The distance from the earth to the star equals $x_0 = 3$ light years, *i.e.* $x_0/c = 3$ years, and the velocity of the rocket is $v = 0.6c$, giving $\sqrt{1-(v/c)^2} = 0.8$. Then the clock readings of the two RFs at the star by the twin's arrival are 4 and 5 years, respectively. First, Fig. 6.a) gives the time vectors 'in the perspective' of the earthbound twin. We know that on the star the clock of his RF reads 5 years, and so 'in his perspective' this is the clock readings all over his RF, (blue semicircle). However, the BC on the star at this instant, (that is the clock of his travelling twin) reads 4 years. Thus, this event on the star is simultaneous with his own clock (also being a BC) showing 4 years. Thus, on the RF of the earthbound twin the event that the clock at the star reads 5 years will occur simultaneously (in the strong sense) with the clock remaining on the earth showing 4 years. At both those two locations the BC reading equals 4 years.

Fig 6.b) illustrates the two time vectors at the star: Red time vector for the travelling twin, and again the (same) blue vector for the earthbound twin; (also see Fig. 5.a). These two time vectors of Fig. 6.b) represent 'basic simultaneity'. Now by combining the two results of Fig 6. a) and b), we get that the clock of the travelling twin showing 4 years (on the star) is an event being simultaneous with the clock of the earthbound twin showing 4 years (on the earth); *cf.* argument in Section 4.4. – Or we could simply claim these events to be simultaneous because the real part of their time vectors are equal.

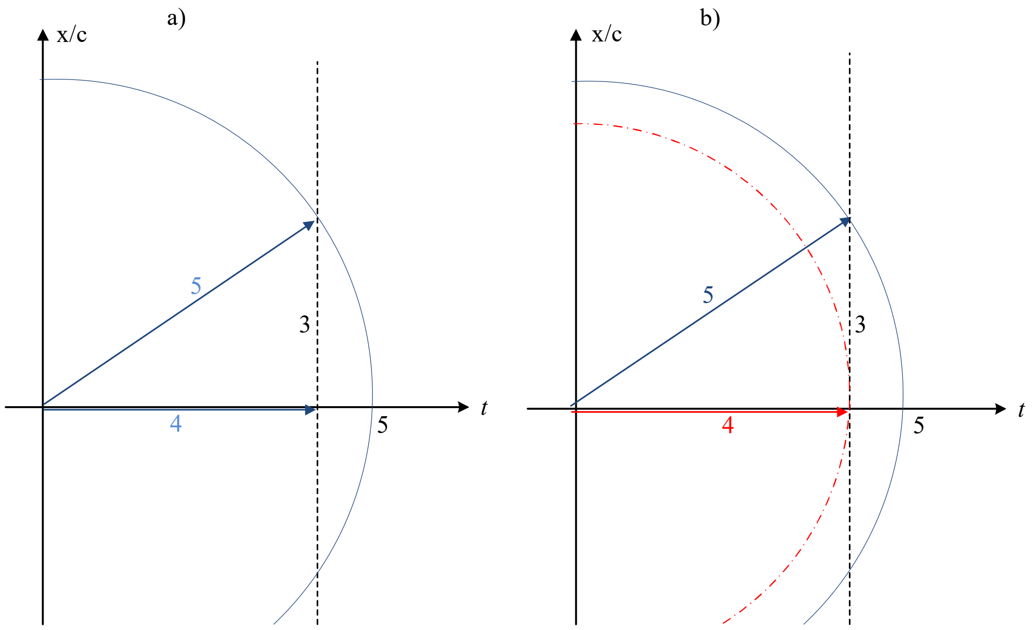

Figure 6 The time vectors $\vec{t} = \binom{3}{4} = 5\binom{0{,}6}{0{,}8}$ and $\vec{t} = \binom{4}{0} = 4\binom{1}{0}$ **by the arrival at the star of the travelling twin. a) Time vectors at the star and on the earth on the RF of the earthbound twin. b) Time vectors at the star: Red = travelling twin. Blue = RF of earthbound twin.**

I think this statement regarding the simultaneity at a distance is crucial for a proper handling of the travelling twin paradox. Note that our result does not imply that the twins will have the same age also by their reunion. Starting out from the above two simultaneous events, we will find that the actual clock readings by the reunion of the twins will depend of the further experimental set-up. In particular, if the travelling twin immediately start his return, and the earthbound twin remains at rest on the earth, we arrive at the standard result, referred in Mermin (2005): The travelling twin will have aged just 8 years, as compared to 10 years for the earthbound; *cf.* last version of Hokstad (2018).

6 Conclusions

In the present work - considering the model of special relativity with just one space coordinate - we define a state vector, where we can interpret both components as aspects of time. Thus, we refer to it as a time vector, even if it also involves location. It appears sensible to formulate the vector as a complex variable, and there is a rather close link to the time-space of Minkowski.

The absolute value of this time vector equals the clock reading ('clock time') at the relevant position, and we can see this as a measure for the overall distance in time from the 'point of initiation', *i.e.* the time, 0 and the origins of the reference frames (RFs) having a common location.

This vector provides a means to define various forms of simultaneity. Time vectors on the same RF, having same absolute value will specify events that are simultaneous 'in the perspective of this RF. This is known to be a rather weak form of simultaneity.

However, we strongly argue that we - without the use of light rays – can also use this time vector to define a strong form of simultaneity 'at a distance'. Events having a time vector with the same real part will exhibit this strong form of simultaneity.

The main simultaneity result is based on one fundamental claim: Due to the inherent symmetry of the given RFs we have that the clocks at the origin of the RFs at time 0 will remain 'synchronized'. We refer to these as BCs, and identical clock readings of these will represent simultaneity at a distance.

Considering the time vector for any event (t , x) of an actual RF, we may introduce an 'imagined' RF with its BC currently located at the chosen position, x. The first component of the time vector equals the clock reading of this (imagined) BC. The second component equals the distance to the BC of the actual RF itself, measured as the time required for a light signal to go this distance.

The results apply for simultaneity relative to a common 'point of initiation'; it seems the derivations must always be based on such an event. However, extensions of the given presentation seem possible.

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Appendix A. Some notation

K_v = RF moving relative to a RF, K_0, at velocity v

x_v = The position on K_v

t_v = Clock reading at position x_v on K_v

$w_v = x_v/t_v$ (Used to specify a location x_v on K_v at a specific clock time t_v. Can also be interpreted as the velocity of an imagined RF K_{w_v}.)

$\varphi_v = \sin^{-1} w_v/c$

t^{BC} = The clock reading of the basic clock (BC) of a given event

v = Velocity of RF K_v, relative to RF K_0

$\theta_v = \sin^{-1} v/c$

When we have two RFs we apply the notation for K_0, x_0, … *etc.* for all parameters related to the 'primary' RF. However in cases of having just one RF, we use the notation, *K, x,* ... rather than K_0, x_0, ... (θ_v is not relevant for v=0.)

Appendix B

Here we provide some material from the previous version of the paper. Here the formulation of time vector/simultaneity focuses on the relation between two symmetric events of the RFs, K_0 and K_v, and so the presentation deviates somewhat from that of the present work.

B.1 An alternative formulation of the Lorentz transformation (LT)

We start out from this version of the LT:

$$t_v \cdot \cos\theta_v = t_0 - (x_0/c) \cdot \sin\theta_v \tag{B1}$$

$$x_v/c \cdot \cos\theta_v = x_0/c - t_0 \cdot \sin\theta_v \tag{B2}$$

As we restrict to consider one space coordinate the LT involves four state variables, t_0, x_0, t_v, x_v. If we specify any two of these four variables, the other two will be given by the LT. The standard version of the LT, (1), (2) (or as here (B1), (B2)) gives (t_v, x_v) expressed by (t_0, x_0), or *vice versa*. But similarly, we could reformulate the LT to give a relation between (t_0, t_v) and (x_0, x_v). And – as a third possibility – we can formulate the LT as a relation between (t_0, x_v) and (t_v, x_0). In the present work we follow up on this third possibility. First, by combining (B1) and (B2), we can replace (B2) by

$$x_v/c = (x_0/c) \cdot \cos\theta_v - t_v \sin\theta_v \tag{B3}$$

To give the resulting modified version of the LT we introduce the matrix

$$C_v = \begin{bmatrix} \cos\theta_v & \sin\theta_v \\ -\sin\theta_v & \cos\theta_v \end{bmatrix} \tag{B4}$$

This is an orthogonal matrix as

$$C_v^{-1} = C_v^T = C_{-v} = \begin{bmatrix} \cos\theta_v & -\sin\theta_v \\ \sin\theta_v & \cos\theta_v \end{bmatrix}$$

Now (B1) and (B3) give a new version of the LT, which we in matrix form can write[3]

[3] Again observe some change in the notation, as compared to previous versions

$$\begin{pmatrix} t_0 \\ x_v/c \end{pmatrix} = C_v \begin{pmatrix} t_v \\ x_0/c \end{pmatrix} \quad \text{(B5)}$$

Now also introduce the two 'time vectors' related to our two RFs, K_0 and K_v

$$\vec{t} = \begin{pmatrix} t_v \\ x_0/c \end{pmatrix} \quad \text{(B6)}$$

$$\vec{t'} = \begin{pmatrix} t_0 \\ x_v/c \end{pmatrix} \quad \text{(B7)}$$

and then we write the relation (11) as

$$\vec{t'} = C_v \vec{t} \quad \text{(B8)}$$

We will denote this the orthogonal version of the Lorentz transformation. A nice feature of this formulation is that it represents a rotation, θ_v, of the $(t_v, x_0/c)$ plane, (with the components t_v and x_0/c being orthogonal). So also the vector $\vec{t} = \begin{pmatrix} t_v \\ x_0/c \end{pmatrix}$ will be given by $\vec{t'} = \begin{pmatrix} t_0 \\ x_v/c \end{pmatrix}$, using the same rotation in opposite direction, *i.e.* we replace -*v* by *v*, (and applying $C_{-v} = C_v^{-1}$ here rather than C_v. We note that he vectors, $\vec{t}$ and $\vec{t'}$ provide identical information, as they actually define the same event. We will see that it becomes rather natural to link $\vec{t}$ to the RF, K_0, and $\vec{t'}$ to the RF, K_v.

Both components of the vectors, $\vec{t}$ and $\vec{t'}$ represent time. The first component equals the clock reading of one of the RFs. The second component equals the position of the other RF for the event in question, divided by *c*; so it equals the time for a light flash to go from the origin to this position.

B.2 Moving clocks and simultaneity

We will now apply the general result of Section 2.3 for two specific positions of the RFs, K_0 and K_v. Position A equals the origin of K_0, *i.e.* location $x_0 = 0$. Position B is at the other origin, *i.e.* $x_v = 0$ of K_v. In other words, A and B are the positions of the 'basic clocks' at the origins of K_0 and K_v, respectively. Further we consider the events that time equal to $t^0 \cdot \sqrt{1-(v/c)^2}$ on both these 'basic clocks'. Thus, we consider two events, that are not simultaneous, neither in the perspective of K_0, nor of K_v. However, they would be simultaneous in a certain auxiliary RF, *cf.* Hokstad (2018); demonstrating a strong sense of symmetry. The remaining parameters are easily specified as (3) and (4) apply here. We summarize the parameter values at these positions as follows (*cf.* Fig. 1):

Position A:

$x_0 = 0, \quad t_0 = t^0\sqrt{1-(v/c)^2}$ (on K_0)

$x_v = -v\, t^0, \quad t_v = t^0$ (on K_v)

Position B:

$x_0 = vt_0 \quad t_0 = t^0$ (on K_0)

$x_v = 0, \quad t_v = t^0\sqrt{1-(v/c)^2}$ (on K_v)

The corresponding time vectors, as introduced in Section 2.3 become:

Position A:

$$\vec{t} = \begin{pmatrix} t_v \\ x_0/c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^0 \quad \text{(B9)}$$

$$\vec{t'} = C_v\vec{t} = \begin{pmatrix} t_0 \\ x_v/c \end{pmatrix} = \begin{pmatrix} \sqrt{1-(v/c)^2} \\ -v/c \end{pmatrix} t^0 = \begin{pmatrix} \cos\theta_v \\ -\sin\theta_v \end{pmatrix} t^0 \quad \text{(B10)}$$

Position B:

$$\vec{t} = \begin{pmatrix} t_v \\ x_0/c \end{pmatrix} = \begin{pmatrix} \sqrt{1-(v/c)^2} \\ v/c \end{pmatrix} t^0 = \begin{pmatrix} \cos\theta_v \\ \sin\theta_v \end{pmatrix} t^0 \quad \text{(B11)}$$

$$\vec{t'} = C_v\vec{t} = \begin{pmatrix} t_0 \\ x_v/c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t^0 \quad \text{(B12)}$$

We observe that the two time vectors (B11), (B12) are essentially identical to the two time vectors (B9), (B10); (just interchanging their order). There is one minor asymmetry, as the position, $x_v = -v\, t^0$ in *eq.* (B10), whilst the position $x_v = v\, t^0$ (without the minus sign) in *eq.* (B11). However, this difference appears since K_0 and K_v here have the same orientation. Therefore, the clock at the origin of K_0 moves along the negative axis of K_v, (thus, $x_v < 0$ at A); while the origin of K_v moves along the positive axis of K_0, (thus $x_v > 0$ at B), but actually there is a complete symmetry in the results of positions A and B.

This symmetry is our main argument for simultaneity at a distance (at A and B). Following the argument of Chapter 3, the event $(x_0,\ t_0) = (0,\ t^0\sqrt{1-(v/c)^2})$ at A and the event $(x_v,\ t_v) = (0,\ t^0\sqrt{1-(v/c)^2})$ at B are simultaneous at a distance, even if they are related to two different RFs. They both provide the clock readings of the basic clocks at these positions, (the origins of K_0 and K_v, respectively).

Further, the two (identical) events, $(x_0,\ t_0)$ and $(x_v,\ t_v)$ at Position A represent 'basic simultaneity', ('same location, same time'). The same is valid for these two events at Position B. In total, we will conclude that all four events illustrated in Fig. B1 are simultaneous (at least in some sense).

We further argue that there are reasons to associate these four simultaneous events by the time vectors, (B9) - (B12), rather than the apparently more 'natural' choice of using say $(t_0,\ x_0/c)$ and$(t_v, x_v/c)$. The main reasons for our choice of time vector are that

- The vectors, $\vec{t}$ and $\vec{t'}$ of (B9) - (B12) all have the same absolute value, t^0, which seems a sensible feature for times of simultaneity.
- The *orthogonal* transformation C_v plays a crucial role in relating the four events of positions A and B. It relates the two events at the same position (either A or B), thus, representing 'basic' simultaneity. Further, it relates the two events being on the same RF (either K_0 and K_v), but at different positions.
- The two components of the time vector, relates to the two BC readings relevant for the event.
- The time vectors, $\vec{t}$ at A, and $\vec{t'}$ at B, specifying the 'simultaneity as a distance', are identical, as both are equal to $\vec{t} = \binom{1}{0} t^0$. (However, also a traditional choice, like $(t_0,\ x_0/c)$ and $(t_v, x_v/c)$ satisfies such an equality.)

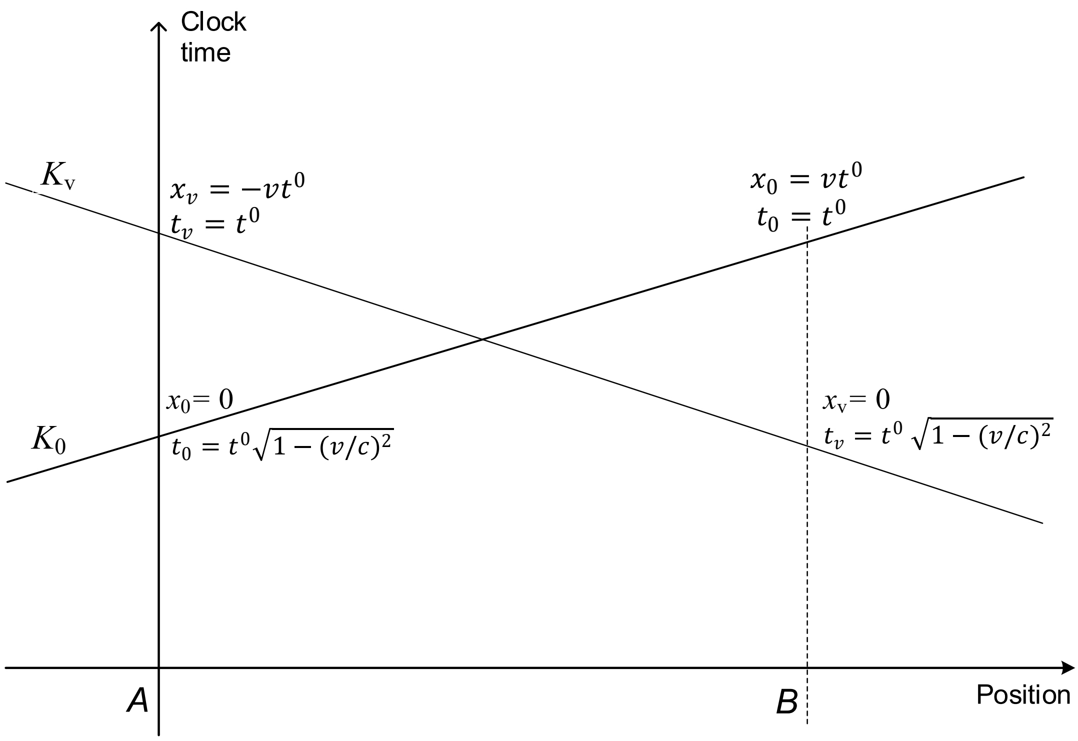

Figure B1 Positions A (at origin of K_0) and B (at origin of K_v) at time $t^0\sqrt{1-(v/c)^2}$ at both origins.

Now also observe that we can write all the four time vectors, (B9)-(B12), expressed by one single vector

$$\vec{t}(t^0, w) = \begin{pmatrix} \sqrt{1-(w/c)^2} \\ w/c \end{pmatrix} t^0 = \begin{pmatrix} \cos\theta_w \\ \sin\theta_w \end{pmatrix} t^0 \tag{B13}$$

This vector will equal either $\vec{t}$ or $\vec{t'}$ of (B9) - (B12) by inserting either $w = 0$, $w = v$, or $w = -v$. Here the values, $\pm v$, correspond to the distance between A and B, which equals $x = vt^0$.

A further illustration of our choice of time vectors is given in Fig. B2. this shows the time vector, $\vec{t} = \vec{t}(t^0, w)$ in the coordinate system, $(t_v, x_0/c)$, both at Positions A, (w=0), and Position B, ($w = v$). Also the $(t_0, x_v/c)$ coordinate system is given (equals $(t_v, x_0/c)$, rotated an angle θ_v). Of course, the rotation from Position A to B is equal to the rotation performed by the matrix, C_v.

We mention that both components of this time vector has a rather simple interpretation. Both at A and B, the first component is the clock reading of the 'basic clock' at the position in question. The second component equals the position (divided by c) of the RF, (here K_0). This is the time required to for the light to go from the origin of K_0 to the current position (A or B). So, both represent 'distance in time'.

The vectors, $\vec{t'} = \vec{t}\,(t^0, w)$, for $w = -v$ and w=0, equal the $\vec{t}$ - vectors in the rotated coordinate system, $(t_0, x_v/c)$. These $\vec{t'}$- vectors are not inserted in Fig. B2, but are easily identified and located, using (B10) and (B12).

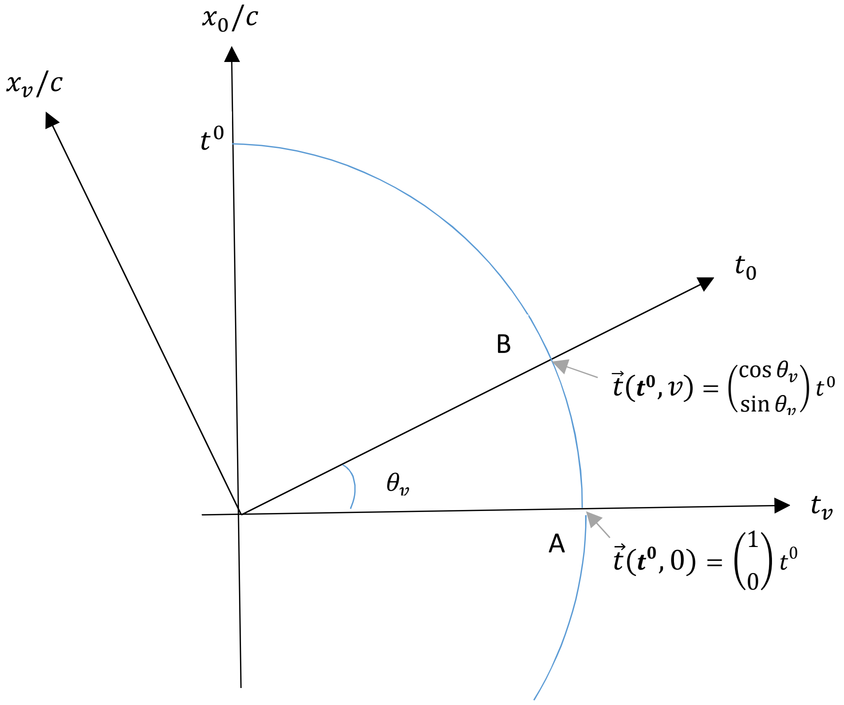

Figure B2 Time vectors, $\vec{t}$ in coordinate system $(t_v, x_0/c)$ at the positions A and B for RFs, K_0 and K_v in the symmetric case when $t_v = t_0$ $(= t^0)$. We can also read the same two vectors in the rotated coordinate system, $(t_0, x_v/c)$.