A time vector and simultaneity in TSR (v8, 2018-12-17)

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Abstract. We specify a time vector for an event in the theory of special relativity (TSR). This vector is well suited to specify various types of simultaneity. Moving (possibly imagined) clocks, which are synchronized at a common 'point of initiation', play a crucial role. We can present the time vector as a complex variable, and there is a close relation to the Minkowski distance. We exemplify the approach by including a short discussion of the 'travelling twin'.

Key words: Time dilation, simultaneity, Lorentz transformation, time vector, Minkowski distance, travelling twin.

1 Introduction

The concept of *simultaneity* is crucial in the theory of special relativity (TSR). Within a single reference frame (RF) simultaneity is easily established by the synchronization of clocks, *e.g.* using light rays, for instance see textbooks like Giulini (2005) and Mermin (2005). Then we say that events with the same clock reading, 'time' (t) on a specific RF, are *simultaneous in the perspective* of this frame. So this type of simultaneity depends on the chosen RF.

The situation is more complex when we have several inertial RFs which are moving relative to each other. However, in the case there is actually just a single event, which of course will have different (time, space) parameters – say (t, x) – within the different RFs. We could refer to this trivial case as *basic simultaneity*.

However, when we have moving RFs, there is in TSR no unique definition of simultaneity of events occurring 'at a distance'. The various RFs will disagree with respect to simultaneity. We have *relativity of simultaneity*, *e.g.* see the discussion in Debs and Redhead (1996). They argue for the *conventionality of simultaneity*; the definition of simultaneity is essentially a matter on convention.

We will here argue that one can provide a single, sensible and consistent definition of simultaneity also 'at a distance'. Hokstad (2018) introduced an approach to obtain such a simultaneity for two RFs, postulating an auxiliary RF with origin always located at the midpoint between the two main RFs, and use a symmetry argument. In the present paper we pursue a slightly different approach.

First, we point out that an essential requirement for the use of the fundamental Lorentz transformation (LT) for two RFs, moving relative to each other, is that we start out with three sets of synchronizations:

1. All clocks on the first RF are synchronized; (so they are simultaneous in the perspective of this RF);
2. Similarly, all clocks on the second RF are synchronized;
3. The clocks at the origins of the two RFs: At time 0 these are at the same location, and they are then synchronized; this 'point of initiation' is a fundamental initial condition for the experiment. We refer to these two clocks as *basic clocks* (BCs), and this synchronization provides an example of *basic simultaneity* (see above).

Further, one implicitly assume that the clocks on each of the RFs *remain* synchronized. We will here argue that also the two BCs at the origins of the RFs – which we synchronized at time 0 - will remain synchronized. They move away from each other at constant speed, v; but there is a symmetric situation; so there is no way to claim that one of the two clocks goes faster than the other.

So our claim is that when the two 'basic clocks' at the origins of the two RFs show the same time, this corresponds (in some sense) to simultaneous events 'at a distance'. Actually we could consider this to be a consequence of the standard assumption of symmetry between the two RFs. We will find that this leads to a rather strong form of simultaneity, as all observers can agree on this. Further, in the above argument there is no need to restrict to consider just two RFs, so we can get simultaneity for any number of events.

In the paper we start out by introducing a two dimensional time vectors related to any event (*t,* x). The clock readings of the BC on the location of the event is an essential element of the vector, which proves useful for defining simultaneity 'at a distance'. Note that we restrict to consider just a single space parameter.

This paper gives a purely mathematical description of the phenomenon, investigating implications of the LT, and there is no attempt of a physical interpretation.

2 Time as a two-dimensional variable

We now introduce a two-dimensional state vector. We refer to this as a time vector, and we will utilize it to define simultaneity. We first introduce this time vector for a single RF.

2.1 Time vector of a single RF

We consider a RF, *K*. At virtually any position, there is a synchronized clock with a clock reading denoted, *t*. When at position, *x* there is a clock reading, *t,* we will simply refer to (*t*, *x*) as an event. Further, we introduce the parameter

$$w = x/t \tag{1}$$

which we of course can interpret as the velocity of an object that has moved from the origin at time 0 to the position, *x* at 'time' (clock reading), *t.*

We have previously (Hokstad (2017)) suggested the following time vector for this event

$$\vec{t}(t,x) = \begin{pmatrix} \sqrt{t^2-(x/c)^2} \\ x/c \end{pmatrix} \tag{2}$$

Using, $w = x/t,$ we can also write this time vector as

$$\vec{t}(t,w) = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} \sqrt{1-(w/c)^2} \\ w/c \end{pmatrix} t \tag{3}$$

Now both components of this vector has a specific interpretation. The first component equals

$$t^{BC} = t\sqrt{1-(w/c)^2} \tag{4}$$

Here we recognize the standard time dilation formula, (*cf.* App. A). We just imagine a RF, K_w moving relative to *K* at velocity *w*, and assume that at time 0 the origins of the two RFs where at the same location, and that the two clocks at this position where then synchronized. This synchronization implies that these clocks are of particular interest, and we call them Basic Clocks (BC). We also refer to this event of synchronization (at time, 0) as the 'point of initiation'.

Thus, we interpret t^{BC} of *eq.* (4) as the clock reading of the BC (located at the origin of K_w) at the moment when this clock has reached the position, $x = wt$ on *K,* (at an instant when the local clock on *K* reads *t*). We may say that t^{BC} defines the 'basic time' of the event (*t,* x) on *K*. So now we have the following alternative expression for our time vector:

$$\vec{t}(t,x) = \begin{pmatrix} t^{BC} \\ x/c \end{pmatrix} \tag{5}$$

To summarize, *eqs.* (2), (3) and (5) are all valid expression for the time vector of the event (*t*, *x*) on *K*:

$$\vec{t} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} \sqrt{t^2-(x/c)^2} \\ x/c \end{pmatrix} = \begin{pmatrix} \sqrt{1-(w/c)^2} \\ w/c \end{pmatrix} t = \begin{pmatrix} t^{BC} \\ x/c \end{pmatrix} \tag{6}$$

We note that the first component of the time vector is a valid expression only when $|x|/c < t,$ (that is $|w|<c$). So at a given position, *x* the time vector is only defined for clock readings, $t > |x|/c$. Here $|x|/c$ is the time required for a light flash occurred at the point of initiation to reach the position, *x*. Actually, in the limit, when |w|→c, the moving BC reaching x will read $t^{BC} = 0$ (*eq.* (4)), and so apparently no time has then elapsed, even if the local clock on K reads t. So when $v=c$ the clock reading remains constant, and there is purely a spatial expansion. We conclude that it is only after time $t > |x|/c$ that (6) defines the time vector at a fixed position, x.

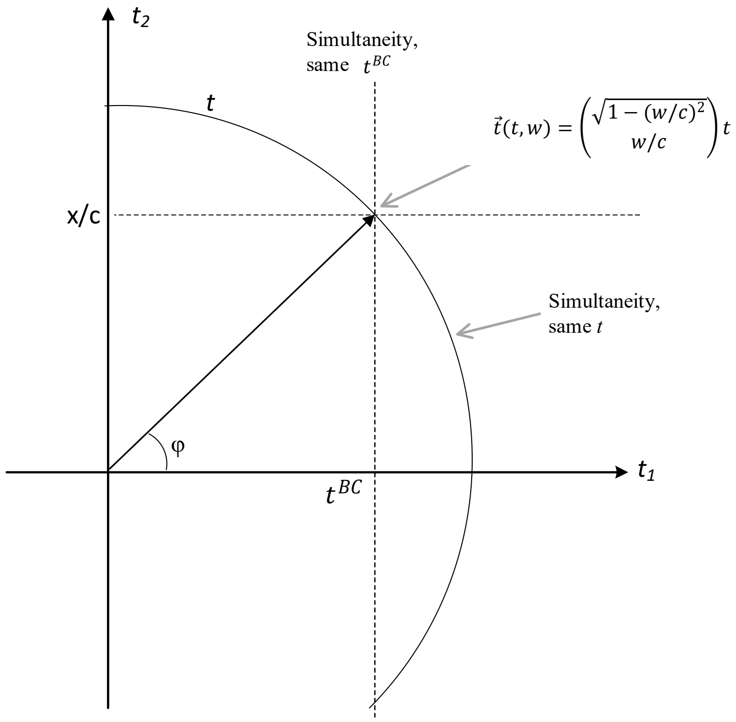

Figure 1 One specific time vector, $\vec{t}(t,w) = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} \sqrt{1-(w/c)^2} \\ w/c \end{pmatrix} t = \begin{pmatrix} t^{BC} \\ x/c \end{pmatrix}$ for a specific clock reading, *t* at the position, *x = wt*, when |*t*|>*x/c*; (here sin φ = w/c).

However, for $t > |x|/c$ we can present the time vector, $\vec{t} = \vec{t}(t,w) = \vec{t}(t,x)$ as a point on the semicircle with radius, t in the (t^{BC}, x/c) space; see Fig. 1. In summary, the components of the time vector have simple interpretations:

1. The first component, $t^{BC} = t\sqrt{1-(w/c)^2}$ equals the clock reading of the BC of the (possibly imagined) RF, K_w which has now reached the position $x = w\,t$ on K. We call this the 'basic time' (or 'BC reading') at this position.
2. The second component, x/c, equals the time required for a light flash to go to the distance, x from the origin of K (with its BC) to the given position. So this equals the distance in time between the BC at the origin of K and the BC at x.

Thus, both components refer to aspects of 'distance in time' from the 'point of initiation', ($x = t = 0$).

Further, the absolute value of the time vector equals the clock reading of the event itself:

$$\left|\vec{t}(t,w)\right| = t \tag{7}$$

Thus, events on K, with time vectors having the same absolute value, also have identical clock readings, and thus are simultaneous 'in the perspective' of K. The semicircle of Fig. 1 illustrates this.

We essentially see Fig. 1 as an illustration of the time vectors of one specific RF. Different semicircles (of radius, t) which we could draw, represent different 'clock times'. By specifying both a t and a position, x, we also obtain a corresponding t^{BC}. Both at position x and $-x$ there will be a BC which reads t^{BC}, (they have moved in opposite directions from the origin of K).

We observe that there is a strong link between the above approach and Minkowski's approach to space-time; *cf.* space-time distance given as $\sqrt{c^2t^2 - x^2 - y^2 - z^2}$ in his four-dimensional space, Minkowski (1909). As stated in Petkov (2012), Minkowski refers to our BC reading (see $\sqrt{t^2 - (x/c)^2}$ of *eq.* (2)), as 'proper time, and our t as 'coordinate time'. Below we apply this to pursue the concepts of simultaneity.

It seems we could generalize the present approach to hold also for a three-dimensional space with coordinates (x, y, z). We would then define w by $w = \sqrt{x^2 + y^2 + z^2}/t$.

2.2 Time formulated as a complex variable

We can of course formulate our time vector for the event (t, x) as a complex variable. In polar form, we can write the vector $\vec{t}(t, w)$ in (3) as:

$$\mathbf{t}(t, w) = te^{i\varphi}, \tag{8}$$

Here the argument, $\varphi \in (-\pi/2,\ \pi/2)$, is given by

$$\sin\varphi = w/c. \tag{9}$$

When $\varphi = 0$, we have $w = x = 0$. Then the corresponding event occurs at the origin of K, and the relevant BC is the one located on K itself. In this case only, the time variable becomes a real number.

Further, the magnitude, t; the real part, $\mathrm{Re}(\mathbf{t}(t, w)) = t\sqrt{1 - (w/c)^2} = t\cos\varphi = t^{BC}$; and the imaginary part, $\mathrm{Im}(\mathbf{t}(t, w)) = t \cdot (w/c) = t\sin\varphi = x/c$ all have interpretations as described in Section 2.1.

2.3 Relating time vectors of different RFs (Lorentz Transformation)

We now consider the relation between the time vectors of two different RFs, which move relative to each other. Thus, we have a K_v moving relative to a K_0 at a speed, v. As in the previous notation, an event on K_v is specified by the clock reading, t_v and the position, x_v, (and thus applies also for $v = 0$). We further let $w_v = x_v/t_v$, and define φ_v by

$$\sin\varphi_v = w_v/c, \text{ (for } w_v < c) \tag{10}$$

As seen above, (*eq.* (6)) there are various ways to write the time vector on K_v, one alternative being:

$$\vec{t} = \begin{pmatrix} t_v^{BC} \\ x_v/c \end{pmatrix}$$

In analogy with the formulation as a complex variable (*eqs.* (8) and (9)) we can also write it in the form

$$\vec{t}(t_v, w_v) = \begin{pmatrix} t_{v,1} \\ t_{v,2} \end{pmatrix} = \begin{pmatrix} \cos\varphi_v \\ \sin\varphi_v \end{pmatrix} t_v = \begin{pmatrix} 1 \\ \tan\varphi_v \end{pmatrix} t_v^{BC} \tag{11}$$

Now consider the case that $\vec{t}(t_v, w_v)$ and $\vec{t}(t_0, w_0)$ describe the same event, just expressed by the co-ordinates of K_v and K_0, respectively (*i.e.* 'basic simultaneity'). Then the Lorentz transformation (LT) will provide the relation between these vectors, *cf.* Appendix A. It is easily verified, (and rather well known *cf. eq.* (2)), that the first component $t_{v,1}$ is then invariant under the LT, and so in this case we have

$$t_{v,1} = t^{BC}, \text{ (independent of } v).$$

The point is simply that time vectors, $\vec{t}(t_v, w_v)$ refer to the same event, and thus experience the same BC reading, t^{BC}. Fig. 2 illustrates this. The time vector on K_0 (blue) and the on K_v (red) have identical first component, t^{BC}, and are related by the LT. Fig. 3 gives another illustration, where K^{BC} represents the RF of the BC being present at the event.

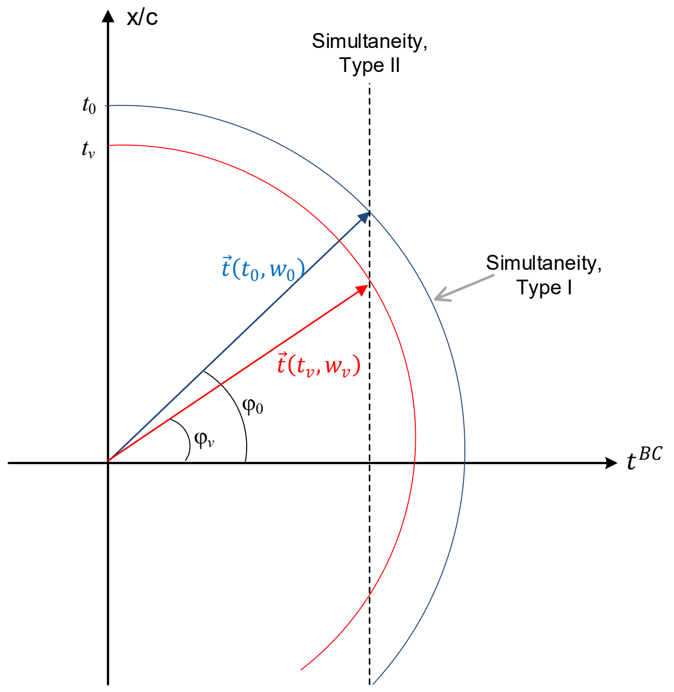

Figure 2 Time vector, $\vec{t}(t_0, w_0)$ on K_0 when its clocks read time t_0, (blue); and time vector, $\vec{t}(t_v, w_v)$ on K_v at the same position, (red). We actually consider the same event, described by two different RFs (and being related by the LT).

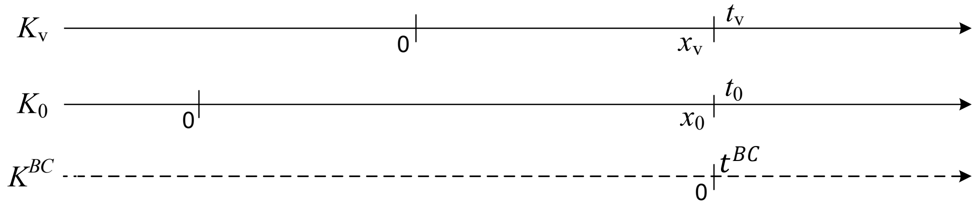

Figure 3 Two events (x_0, t_0) and (x_v, t_v) representing basic simultaneity, and the clock reading, t^{BC} of a BC at the same position. Origins of all RFs are marked with a zero, 0.

3 Simultaneity and the time vector

It should be quite clear by now that there is a rather close connection between the time vector and simultaneity. As indicated in Figs. 1 and 2 we consider two types of simultaneity. We will now refer to these as Type I and II, respectively, and now sum up the main features of these.

3.1 Simultaneity, Type I

The absolute value of the time vector is equal to the clock reading, t of the corresponding event. When we consider the events of one single RF, K, (but only then) we can use this as a measure of simultaneity; we say that events with identical, t are simultaneous '*in the perspective of K*'. We will refer to this as Simultaneity, Type I. So this occurs when the time vectors of a specific RF have the same *absolute value*; *cf.* vectors on the semicircles of Figs. 1 and 2.

As we know, however, this a very weak form of simultaneity. The various RFs will give different results regarding 'time', and will even disagree on the 'time' of a specific event, (what we have called basic simultaneity), *e.g.* see the two vectors of Fig.2. So the RFs disagree regarding simultaneity, and if we want to take a holistic view, it seems no way to give preference to the claims of one of them.

3.2 Simultaneity, Type II

We use our concept of Basic Clocks (BCs) to define Simultaneity, Type II. At the point of initiation (t=0 all RF) all BCs are located at the (common) origin and are moving relative to each other at various speeds. Thus for every event (t, x) on any RF (with $t > |x|/c$) there is a BC present. Events that have the same BC reading, t^{BC} are simultaneous in this sense (Type II). The BCs may of course be imagined; we are just stating what these clocks would read, *if* they were present at the location of an events.

We distinguish between two cases:

i. We have just one event, described by two (or several) RFs, *cf.* Fig.2. So the BC reading, t^{BC} is the same, independent of the RF considered. This is the situation described by the LT, and the red and the blue time vector are two representations (in two different RFs) of the same event. Thus, they have the same t^{BC}. In this trivial case we apply just a single BC, and refer to Simultaneity *Type II, Local*.

ii. The second case is much more interesting; it provides a means to give an objective definition of 'simultaneity at a distance'. The main point is that the above argument applies for various BCs 'at a distance'. For instance, looking at Fig. 2 we immediately see that the event described with the two given time vectors are also simultaneous with the corresponding event on the negative x-axis; where there is another BC with the same t^{BC}. However, we are not limited to such a special case. There will be BCs at any speed (<c) available at any position, with the ability to specify simultaneity of events 'at a distance'. Further, the requirement that various events have identical BC readings is obviously equivalent to (see (6)) that

$$t_v^{BC} = \sqrt{t_v^2 - (x_v^2/c)^2} = \text{Const.} \tag{12}$$

cf. the Minkowski distance. As known, this expression is invariant under the LT, but more important is the case of using several BCs. We refer to this as Simultaneity, *Type II, At a distance.*

Simultaneity, Type II obviously disagrees with Simultaneity, Type I. Events that are simultaneous according to the corresponding BC readings, t^{BC}, are *not* simultaneous 'in the perspective of the relevant RF. In conclusion, Type II is a much more sensible definition, as it is consistent and applies across RFs. Further, Type II, Local is rather trivial and really not that interesting. Type II, At a distance, however provides useful new insight

We finally note the limitation of the given approach: We have defined simultaneity relative to a specific 'point of initiation', only. However, if we first identify simultaneous events 'at a distance' relative to a given 'point of initiation', it should also be possible to define further simultaneities based on these events, now treated as new 'common' points of initiation.

4 The travelling twin

The travelling twin paradox is frequently discussed, *e.g.* see Schuler and Robert (2014). As stated for instance in Mermin (2005) the paradox illustrates that two identical clocks, initially in the same place and reading the same time, can end up with different readings if they move apart from each other and then back together. We gave a lengthy discussion in Hokstad (2018), and now restrict to a comment on the simultaneity of events related to the actual arrival.

In this thought experiment we start out with two synchronized clocks at the origins of two reference frames: the RF of the earth, and the RF of the rocket of the travelling twin. We note that both clocks are located at the origin of their RFs, and so both are basic clocks (BCs) in our notation. This makes the case very well suited to illustrate the current approach. Actually it is sufficient to point out that both clocks are BCs. So if the travelling twin's clock shows 4 years by his arrival, this is simultaneous with the event that the clock on the earth also shows 4 years. This follows from our Simultaneity, *Type II, At a distance.*

However, to illustrate this further, we also consider the relevant time vectors. We use the numerical example of Mermin (2005). The distance from the earth to the star equals $x_0 = 3$ light years, *i.e.* $x_0/c =$ 3 years, and the velocity of the rocket is $v = 0.6c$, giving $\sqrt{1-(v/c)^2} = 0.8$. It follows that by the arrival of the travelling twin, the clock at the star belonging to the earthbound twin will read $x_0/v = 3/0.6$ = 5 years, (assuming that the he has a clock, located on the star. being synchronized with his own). At the same instant the clock of the travelling twin reads 5·0.8= 4 years (time dilation). In the literature one now often just points out that 'time' equals 5 years 'in the perspective' of the earthbound twin, and 4 years 'in the perspective' of the travelling twin.

Fig. 4 illustrates the time vectors related to the arrival at the star: Red time vector for the travelling twin; his clock showing 4 years. Blue vector for the earthbound twin; his clock showing 5 years. We note that since $x_0/c = 3$ years at this position, we also directly get $t^{BC} = \sqrt{5^2 - 3^2} = 4$ years, as already stated. The two semicircles represents times 'in the perspectives' of the two twins; obviously representing very conflicting views. We note that Fig. 4 is a special case of Fig. 2, as the RF of the BC equals that of the travelling twin, (red semicircle has radius t^{BC}).

However, we also have another BC, namely that of the earthbound twin. Thus, we conclude that when these two time vectors have the same $t^{BC} = 4$ years, this represents Simultaneity Type II, at a distance.

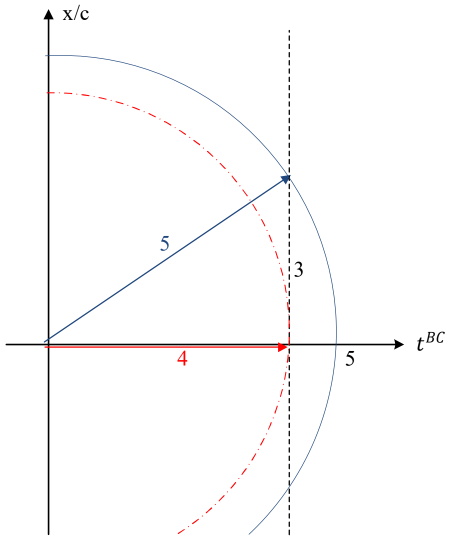

Figure 4. The time vectors at the star by the arrival: Red = RF of travelling twin. Blue = RF of earthbound twin. The relevant vectors are $\vec{t} = \binom{3}{4} = 5\binom{0{,}6}{0{,}8}$ and $\vec{t} = \binom{4}{0} = 4\binom{1}{0}$.

So the main finding here is that the arrival at the star is simultaneous with the event that clock on the earth shows 4 years. Actually 4 years is the only feasible result, considering the symmetry of the situation, *cf.* Hokstad (2018). However, we point out that one will actually arrive at the standard result of 10 and 8 years, for the ages (or rather clock readings) by the reunion of the twins on the earth. The return of the travelling twin requires a change of direction, and therefore the introduction of a third RF, and this will significantly affect the development, in total providing in the standard result.

5 Conclusions

The above presentation is based on one fundamental claim. We postulate an infinite set of (possibly imagined) reference frames (RFs) moving relative to each other at constant speeds. At the origins of these RFs there is a clock, and initially these are all synchronized. From symmetry, we conclude that they remain synchronized, and refer to them as basic clocks (BCs).

Then for any event (t, x) on any RF, we define a time vector in two dimensions:

1. the clock reading of the (imagined) basic clock BC currently at this position, (t^{BC})
2. the time required for a light flash to go from the origin of the RF (where there also is a BC) to the current position,(x/c).

So, both components (dimensions) represent a 'distance in time' from the 'point of initiation', when the BCs were synchronized. We find that the absolute value of this time vector equals the clock reading, t of the event, and see this as a measure for the overall distance in time from the 'point of initiation'.

This time vector provides a means to define various forms of simultaneity. Obviously, when time vectors on a specific RF have the same absolute value, t they will specify events that are simultaneous 'in the perspective' of this RF. This Simultaneity, Type I ,follows as t also equals the clock reading of the event.

However, the main result is that time vectors, which have identical first component (t^{BC}) correspond to simultaneous events. This represents simultaneity in a much stronger sense, as we can use it as a holistic definition, valid for all RFs. We denote it Simultaneity Type II. The definition can be used locally: When we consider a specific event, described by various RFs, the vectors of all RFs of course have the same t^{BC}, as for any event there is just one BC present.

However, the most useful application is to consider this simultaneity 'at a distance'. Events with the same t^{BC} will exhibit this form of simultaneity, also for distant events!

The travelling twin paradox represents a trivial application of this definition of simultaneity (Type II).

We note that the results apply for simultaneity relative to a common 'point of initiation'. Finally, observe that we can also formulate the time vector as a complex variable, and that there is a close link to the time-space of Minkowski.

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Appendix A The Lorentz transformation (LT) and time dilation

This Appendix reproduces some material from Hokstad (2018).

A.1 Alternative formulation of the LT

The LT represents the fundament for our discussions. In our notation the LT takes the form

$$t_v = \frac{t_0-(v/c^2)x_0}{\sqrt{1-(v/c)^2}} \tag{A1}$$

$$x_v = \frac{x_0-vt_0}{\sqrt{1-(v/c)^2}} \tag{A2}$$

We prefer a modified version of the LT. At any time, t_v and position, x_v we introduce w_v equal to $w_v = x_v/t_v$, (and therefore also $w_0 = x_0/t_0$). Then we insert $x_0 = w_0t_0$, and (A1) directly gives that the clock reading on the RF, K_v at this position equals:

$$t_v = t_v(w_0) = \frac{1- vw_0/c^2}{\sqrt{1-(v/c)^2}}t_0 \tag{A3}$$

Note that we here also write $t_v = t_v(w_0)$ to stress the dependence on w_0.

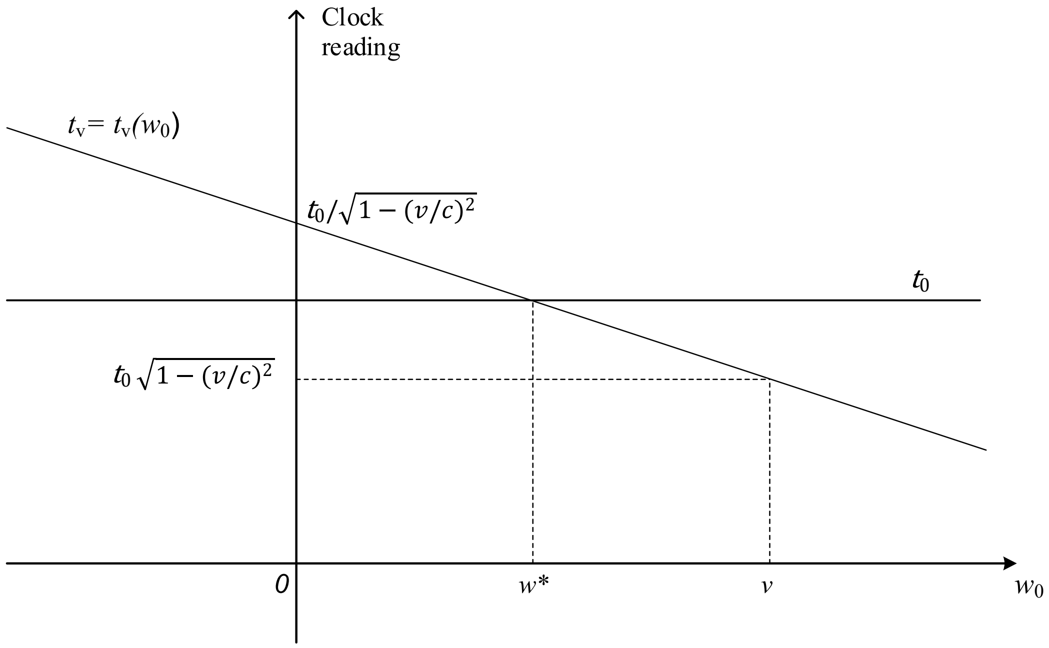

Figure A1. Clock readings in the perspective of K_0. Thus, 'time' all over K_0 equals t_0, while clock readings, $t_v(w_0)$ on the other RF is given as a function of w_0, where $w_0 = x_0/t_0$ provides the 'position' on K_0; *cf.* (3).

Further, by also inserting $x_0 = w_0 t_0$ and $x_v = w_v \cdot t_v$, we obtain

$$w_v = \frac{x_v}{t_v(w_0)} = \frac{w_0 - v}{1 - \frac{w_0}{c}\frac{v}{c}} \tag{A4}$$

So equations (A3), (A4) express the LT by parameters (t, w) rather than (t, x). We observe that clock readings, t_0 and t_v enters (A3) only!

Fig.A1 provides an illustration of the time dilation formula, (A3). This gives the clock reading both on K_0 and K_v *in the perspective of* K_0; (*i.e.* all clocks on K_0 having the same clock reading). Therefore, the figure illustrates an instant when clocks read t_0 all over this RF. The horizontal axis gives the 'position' $w_0 = x_0/t_0$ on K_0 at which the clock measurements are carried out. The vertical axis gives the actual clock readings. So as clocks on K_0 reads t_0 at any 'position', w_0, the clock readings on K_v at this instant, $t_v = t_v(w_0)$, is a linear function of w_0, see (A3).

A.2 Two standard special cases (observational principles)

Two special cases are of particular interest. Recall that the first clock comparison is carried out at the origins $x_v = x_0 = 0$ when $t_v = t_0 = 0$. We specify two choices for the second comparison of clock readings.

First we compare the clock located at $x_v = 0$ on K_v (with the passing clocks on K_0 showing t_0). Thus, also $w_v = 0$, and (A4) implies $w_0 = v$, and (3) gives the relation between the two clock readings at this position, *cf.* Fig. A1:

$$t_v = t_v(v) = t_0\sqrt{1-(v/c)^2} \tag{A5}$$

This equals the standard 'time dilation formula'. Secondly, we can compare the clock located at $x_0 = 0$ on K_0 with a passing clock on K_v. For $x_0 = w_0 = 0$, *i.e.* following the basic clock at the origin of K_0, *eq.* (A3) gives the following relation, (again see Fig. A1):

$$t_v = t_v(0) = t_0/\sqrt{1-(v/c)^2} \tag{A6}$$

Apparently, the relations, (A5), (A6) are contradictory; *eq.* (A5) tells that the clock on K_v goes slower, and (A6) tells that the clock on K_0 goes slower; *cf.* the Dingle's question, (McCausland 2008, 2012). Thus, the time dilation is not a feature of the RF, but follows from which single clock we choose to follow when we perform the second clock comparisons. Therefore, we prefer to formulate the time dilation formulas (A5) and (A6) in compact form as

$$t^{BC} = t^{MC}\sqrt{1-(v/c)^2} \tag{A7}$$

Here we have introduced the notation regarding the second clock comparison.

t^{BC} = The clock reading of a basic clock (BC)[1]. Thus, on this RF the same clock is used in the second clock comparison.

t^{MC} = The clock reading at the same location, but on the other RF. Therefore, this is the clock reading on the RF which use multiple clocks (MC) for the clock comparison; (*i.e.* it uses another clock in the second comparison).

Thus, both RFs can apply a BC for a certain clock comparison, and then conclude that 'time goes slower' on the RF which use BC. However, the same RF could also apply two clocks (MC) for a clock comparison with a BC on the other RF; and we would then conclude that 'time goes slower' on this other RF. Therefore, it is the *observational principle*, *i.e.* choice of clocks for the clock comparisons that matters; *cf.* discussion in Hokstad (2018). This is a well-known result. However, this duality has perhaps not received the attention it deserves in standard literature.

A.3 The symmetric case

There is another interesting special case of the LT, (A3), (A4). We can ask which value of w_0 (and thus w_v) will result in $t_v = t_0$. We easily find that this equality is obtained by choosing $w_0 = w^*$, where

$$w^* = \frac{c^2}{v}\left(1-\sqrt{1-(v/c)^2}\right) = \frac{v}{1+\sqrt{1-(v/c)^2}} \tag{A8}$$

Further, by this choice of w_0 we also get $w_v = -w^*$. This means that if we consistently consider the positions where simultaneously $x_0 = w^* t_0$ and $x_v = -w^* t_v = -w^* t_0$, then no time dilation will be observed at these positions. In other words (*cf.* Fig. A1):

$$t_v(w^*) = t_0$$

At this position we also find $x_v = -x_0$, thus, providing a nice symmetry. Note that when we choose the observational principle, (A8), then absolutely everything is symmetric, and it should be no surprise that we get $t_v = t_0$.

Here we note that the result (A8) has a direct interpretation related to velocities. According to standard results of TSR, the velocities v_1 and v_2 sums up to v, given by the formula

$$v = v_1 \oplus v_2 \stackrel{\text{def}}{=} \frac{v_1+v_2}{1+\frac{v_1}{c}\frac{v_2}{c}} \tag{A9}$$

This gives a definition of the operator $\oplus$ for adding velocities in TSR. Now it is easily verified that when w^* is given by (A8), then from (A9) we get $w^* \oplus w^* = v$. So this confirms that when our point of observation 'moves' with velocity w^* relative to K_0 and velocity $-w^*$, relative to K_v, it corresponds exactly to the case that the relative speed between K_0 and K_v equals v.

[1] We have previously used Single Clock (SC) to denote this clock reading