Proof of Beal's conjecture and related examples

Kamal Barghout

PMU

kbarghout@pmu.edu.sa

ABSTRACT

In this article we prove Beal's conjecture by deductive reasoning by means of elementary algebraic methods. The main assertion in the proof stands upon that the LHS of Beal's conjecture represents the sum of two monomials of like terms. The monomial on the RHS of Beal's conjecture can be built by combining the two monomials on the LHS. By representing any number in exponential form of single power as having a unique base-unit it is to be proved that any number in exponential form to be added to it to yield a sum in exponential form of single power must have the same base-unit by virtue of the two numbers having a "block-form" with a building block of their common base-unit. By conversion of the addition process of the two exponential numbers to multiplication, the GCF of the two terms on the LHS of Beal's conjecture can be factored. Upon factorization of the GCF and making use of power rules, it must be combined with the sum of the two coefficients of the two terms to yield the monomial on the RHS of the conjecture, confirming the proposition that they must have a common and unique base-unit to successfully combine.

1. Introduction and conclusion

Beal's conjecture states that if $a^x + b^y = c^z$, where a, b, c, x, y and z are positive integers with x, y, z > 2, then a, b, and c have a common factor. The conjecture was made by math enthusiast Daniel Andrew Beal in 1997 [1]. So far it has been a challenge to the public as well as to mathematicians to prove the conjecture and no counterexample has been successfully presented to disprove it.

Because solutions to Beal's conjecture deal with the sum of two numbers in exponential form of single power we may represent the expression on the LHS as the sum of two monomials with common variable x. Since the values of the coefficients of the monomials have a special relationship with the monomials' variables, it is feasible to represent the sum according to this relationship which remains elementary and follows the rules of exponentiation. Since the two monomials share the same variable, a basic relationship then is that their variable constitutes a solution of common numerical factor. The key to the analogy is that a common factor of the monomial-variable of x^m can be taken as a common factor of the two monomial terms. The LHS of Beal's "equation" can be represented then by the expression $\alpha x^m + \beta x^m$. By factoring out the GCF the expression becomes $(\alpha + \beta)\, x^m$, where m is positive integer and the coefficients α, β and $(\alpha + \beta)$ are factors such that if multiplied by the GCF of the monomial variable term x^m it results in a product of two terms in exponential form which can be combined by the rules of exponentiation. Only when the two terms of the sum of $(\alpha + \beta)$ and the GCF term of x^m can be manipulated such that they either share the same power or the same base they can combine into a number expressed in exponential form of single power, representing the RHS of Beal's equation. The success of combining the GCF with all three terms implies that the CF provides the base-unit of the basic building block upon which all three terms in the conjecture build their exponential structure.

The following is an example,

Example 1: The equation $1 + 2^3 = 3^2$ represents a trivial solution to the equation $x^m + 2^3x^m = 3^2x^m$, where $m = 0$. If we wish to build higher power solutions from this basic equation we may employ a proper CF term of x^m with higher power, e.g. 3^3. The two terms on the LHS become $(1 + 2^3)3^3$ and can be simplified to $3^2 \cdot 3^3$ and further to 3^5 to yield the equation $3^3 + 6^3 = 3^5$ as implied by Beal's conjecture. The common factor allows us to build a higher power equation but constitutionalizes a rule that all three terms must build their structure on the base-unit introduced by the CF.

The solution in example 1 is an identity of the form $\delta x^m = \delta x^m$, where the two monomials that make the sum are the two terms on the LHS of Beal's conjecture that share the same variable term x^m of $x = 3$ and power $m = 3$. The common term x^m must be chosen such that it should combine with the three terms by use of exponential rules. In this example, m can be chosen as 3 and x must be any number from the geometric sequence of 3, 9, 27, 81, to allow the term to be reshaped to combine with both 3^2 and 2^3. Any solution of x must abide with the rules of exponentiation depending on the values of all three coefficients of the equation and the equation then has infinitely many solutions. In the case of the three terms of the equation having the same exponent, the identity has then infinite number of solutions corresponding to the positive integer values of x and m. This is clear in an identity equation of the form of that of Fermat's last theorem, where m must be 2 and x is positive integer. It is to be noted that the sum of the coefficients on the LHS of $(1 + 2^3)$ has terms that are elements in the Diophantine quadruple of $\{1,3,8,120\}$ which allows the sum to produce a square that perfectly combines with the carefully chosen CF.

In this article, with a deeper understanding of the mechanism of how numbers in exponential form behave and the fact that they must be constructed from a basic building block of their base-unit, we will prove Beal's conjecture. To be specific, any number in exponential form and single power must be built from a basic base-unit. In example 1, the base-unit of 2^3 is the number 2 and the base-unit of 3^2 is the number 3. When the trivial equation was upgraded to higher powers, the new terms had common basic units. Specifically, all three terms of the equation $3^3 + 6^3 = 3^5$ had the same basic base-unit of the number 3, introduced by the CF. By converting the sum of the two monomials on the LHS of the equation to the monomial on the RHS of the equation, asserting that an identity equation of a specific value of the variable term x^m has a special relationship with the coefficients of the two terms on the LHS of the equation, the process requires a common factor for a successful transition from addition operation to multiplication to yield a single term with all three terms expressible as a single power. The success of the conversion process of addition of the two terms on the LHS to multiplication and subsequently the successful combining of the resulting product into one term of exponential form implies that all three terms of Beal's conjecture must have a common factor of the same base-unit related to the common variable of the two monomials on the LHS, implying a proof of Beal's conjecture.

2. Assertion of mathematical operations and definitions

In this section we will analyze the mathematical operations involved in the proof of the conjecture and further define few mathematical terms for easier use in the theoretical assertion of the proof.

2.1. Definitions

Block-number: It is a mathematical object defined here by any number a in exponential form of single power, $a = b^n$ that is made of specific number of base-units, where b and n are positive integers.

Block-elements: They comprise the block-number. The number of block-elements is the value of a block-number in the standard form of exponent 1.

Base: For a block-number b^n, b is the base. The base is the bundle that is made of base-units. The value of the base repeats to constitute the block-number.

Base-unit: It is an ensemble of elements that constitutes the basic building block that “branches” out to build a block-number. The base-unit of a block-number is a prime number or a composite number.

Block-bundle: For a block-number b^{n+r}, it is the ensemble of base-units of b^n that repeats upon branching when the block-number is increased by a factor of b^r, where r is a positive integer.

Exponent: For a block-number b^n, n is the exponent.

Exponent branching: It is the degree of branching of the block-bundle with a branching process based on the base’s value by increasing the exponent.

Base branching: It is the degree of branching of the block-bundle with a branching process based on the base’s value by increasing the base.

Size of a block-number: It is the total number of base-units of a specific block-number.

Factor of a block-number: When two block-numbers are added or subtracted from each other, a common block-number may be factored out.

Block equation: a block equation is an equation whose terms are block-numbers.

Size of a block equation: It is the size of the block-number on the RHS of the block equation.

Block-variable: For the block equation $lP^n + kP^n = (l + k)P^n$, P^n is the block-variable.

Block-Monomial: For the block equation $lP^n + kP^n = (l+k)P^n$, each term is a block-monomial comprised by the block-variable and its coefficient and share either the power or the base.

Remark: The definition of a number in exponential form of a positive integer base and power here as a block-number is an important characterization of the number as it describes the size of the mathematical object depending on the value of its base-unit.

2.2. Analysis of the multiplication process of two block-numbers

When two block-numbers are multiplied together both must have a specific relationship with each other for the multiplication process to yield a single block-number.

LEMMA 1. *Let $b^n \cdot k^l$ be the product of two block-numbers. The product yields a block-number if and only if $n = l$ or $b = k$ or both.*

Proof. If $n = l$, the product becomes $b^l \cdot k^l$. By the power rule of same exponent the product becomes $(bk)^l$, which constitutes a block-number. If $b = k$ the product becomes k^{n+l} by the power rule of same base, which is a block-number. The case of $n = l$ and $b = k$ is satisfied by one of the preceding processes.

Corollary 1. *The two processes of same power or same base to combine the product of two block-numbers by lemma 1 are equivalent*

Proof. Since the two processes keep the total number of base-units unchanged, it follows that they are equivalent.

Example 2: The equivalence of the two multiplication processes of same exponent or same base is obvious from the product of $3^4 \cdot 3^4$ which either equals 3^8 or 9^4 or by up-reshaping we obtain the standard form of 6561^1, all with positive integer bases and exponents and have a size of $6561/3 = 2176$ number of base-units.

Remark: For the product $b^n \cdot k^l$, when $n = l$, it means that the two terms have the same degree of branching but their bases multiply with each other upon combining and the result is a block-number of larger base, while for $b = k$ it means that the two terms keep the size of their bases unchanged but increase the degree of their branching (power) upon unification into a single block-number.

2.3 Reshaping a block-number

For a block-number to be expressed in a different form, it follows the exponent rules. Similar to multiplying a rational number by a common factor both in the numerator and the denominator to obtain the rational number in a different form, we may change the form of a block-number by changing its exponent value as well as its base value.

LEMMA 2. *If we divide or multiply the power of a block-number by a factor* n *we should raise the base to the nth power or take the nth root of the base respectively to keep the value of the block-number unchanged.*

Proof. The proof comes from the definition of the base and the exponent of a block-number. Let a block-number be k^l. Then, dividing the exponent by a factor *n* and raising the base to a power *n* leaves the block number unchanged as $k^{\frac{nl}{n}}$. The other case is accomplished by the same reasoning.

Example 3: The block-number 9^2 is equal to 3^4 by taking the square root of the base and multiplying the exponent by 2. Similarly, the block number of 27^2 is equal to 3^6 by taking the third root of the base and multiplying the exponent by 3. By repeating the process we obtain an exponent equals 1 and 3^6 will equal 729^1. No further up-reshaping can be made if we need to keep the number with positive integer base and power.

Remark: Reshaping a block-number does not change its size. It follows that the number of base-units of a block-number is constant at all times. The number of base-units in any block-number equals the number of elements divided by the base-unit.

Example 4: The block-number 6^5 has a composite base of $3 \cdot 2$ and its size is $7776/6 = 1296$.

Remark: To obtain the base-unit of a block-number we need to down-reshape it until we get the lowest possible integer base. The base is then the base-unit. In example 2 and from 3^8, the base-unit is 3 and the block-number has 6561 elements and a number of base-units of $6561/3 = 2187$, which is the size of the number.

Remark: When reshaping a block-number it is important to keep a positive integer base and power since that is a property of the block-number.

2.4. Increasing the size of a block-number

We can increase the size of a block-number by increasing its degree of branching (exponent) or increasing the value of the base by multiples. In both processes the block-number is increases by multiples of the base-units only.

Corollary 2. *For a block-number, it is only possible to increase its size by multiples of its block-bundle and consequently by multiples of its base-units.*

Proof. The corollary stems from the power rules of numbers. Since to increase the size of a block-number by a factor, the factor itself must be a block-number, and by lemma 1 it is only possible to perform the multiplication process of two block-numbers if they share either the base or the exponent. In both cases the new resulting block-number only increases by multiple of the base-unit. This is obvious since multiplying the block-bundle P^r of base P by a factor of k^r

yields $(kP)^r$ and multiplying P^l and P^r yields P^{r+l}, both processes yield a number with multiples of the block-bundle P^r and multiples of the block-base P as well as multiples of the base-units.

2.5. Increasing the size of a block-number by exponent branching process

Increasing the size of a block-number by increasing the value of its exponent can be done by adding multiples of the base-unit every time we increase the power.

Example 5: Fig. 1 illustrates the process of branching by increasing the base by base-units of 3.

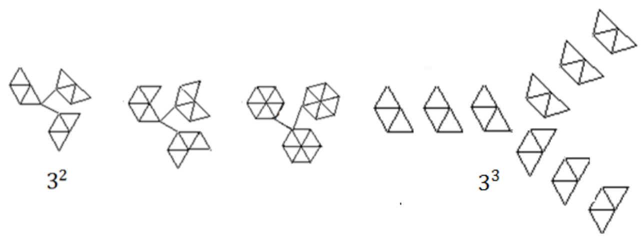

FIGURE 1

The left drawing of groups of block-elements made of triangles represents the block-number 3^2 as the block-bundle structure when increasing the exponent of the block-number by one. The right drawing represents the block-number 3^3 by adding three-multiples of the block-bundle structure of 3^2 since it is multiplied by the block-factor of 3^1, but the second from right drawing does not increase the power by one because it only doubles the block-bundle structure to become $2 \cdot 3^2$ instead of tripling it since the base-unit value is 3, three triangles in the figure, and the second from left increases it only by a fraction to become $4/3 \cdot 3^2$. Therefore, the two figures in the middle do not represent a block-number since there is no mechanism to combine the product into one block-term because the multiplication of the factor doesn't allow addition of unit-base multiples of the block-bundle structure. To increase the exponent by another factor to make it 3^4, we need to branch one more time by multiplying the whole structure (block-bundle) by a block-factor of 3^1, and the block-number becomes $3^1(3^3)$, which results in 3^4 by the power rules. The block-bundle structure now is 3^3. Increasing the size of a block-number here by repetition of its block-bundle ensures repetition of the base-unit as well.

2.6. Increasing the size of a block-number by base branching process

Increasing the size of a block-number by increasing the value of its base can be done by adding the correct number multiples of the base-units every time we increase the base by a factor.

Example 6: We can keep the branching value (exponent) of a block-number unchanged by multiplying the block-number by a factor whose exponent is the same as the block-number's exponent. The resulting block-number now has a base value equals the product of the two bases of the original block-number and the multiplied factor, hence increasing the base value by multiples of the base-units only. An example is increasing the block-number 3^2 by a factor of 2^2, which changes the block-number to 6^2 of base 6 but keeps the branching factor of 2 (See Fig. 2).

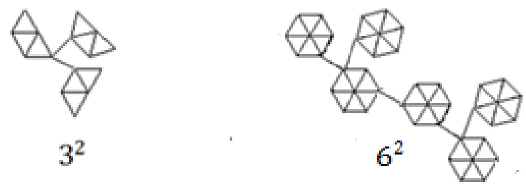

FIGURE 2

2.7. Addition of block-numbers

LEMMA 3. *A common factor is a necessary condition for a successful conversion of the addition process of two block-numbers into a product of two block-numbers.*

Proof. Lemma 1 above places restrictions on addition of block-numbers such that a proper common factor allows a common block-bundle as a building block to build the sum block-term.

Remark: Similar to the algebraic addition process of two numbers in the standard form by taking a common factor and adding the “residual” factors to convert the addition to multiplication, we can add block-numbers by factoring out the GCF and sum the residual factors of block-numbers to convert addition to multiplication. The residual factors must add up to a single block-number that must combine with GCF complying with power rules.

Remark It is asserted here that we can add two block-numbers to each other only if they share a base-unit. By lemma 1, the key to the success of the addition process of two block-numbers in yielding a block-number is that one of them must distribute its elements to the other in such a way that it adds up exactly as if the first block-number is being increased in size, which implies that the addition process must be successfully converted to a multiplication one to yield a single term of positive integer power and base, therefore they both must share a base-unit since increasing in size occurs only by multiples of the base-unit. In other words, when adding a block-number to another one it must distribute its building block base-units to the other number. Consequently, the block-bundle of one of the numbers should match the other’s block-bundle to successfully increase its size (See Fig. 1 and 2).

LEMMA 4. *If a GCF of exponent 0 is not a sufficient factor for a successful conversion of addition process into multiplication process of two block-numbers to yield a block-number of exponent larger than 1, higher exponents of GCF are necessary.*

Proof. Lemma 3 does not place restrictions on the value of the CF.

The GCF can easily be obtained by the process of breaking down the bases to their primes to obtain a greatest common block-factor whose base is the product of the shared primes. If the addition is "trivial", the common factor is unity and the addition is simple algebraic process in the standard form. The process of factoring the GCF out converts the addition process of two block-numbers to a product of two block-numbers. Consequently, if lemma 1 is not applicable a non-integer base or power may be obtained. This is a useful process to check whether the product of two block-numbers can be combined to yield a single block-number to conclude whether the addition process is successful in block-form. Since changing the size of a block-number retains its base-unit value, any successful addition of two block-numbers must ensure as a priori that the block-numbers share the base-unit since increasing the size of one by the other means increasing the size of the other by the first.

Example 7: We can add any two numbers by simply taking GCF as unity. First, up-reshape the two block-numbers to their standard forms. Then, add them together algebraically and down-reshape the sum to lowest integer base using lemma 2. For the sum of $3^3 + 6^3$, we can up-reshape each block-number to the standard form to get $27 + 216$. The sum is 243, which can be down-reshaped to 3^5. Alternatively, we take 3^3 as the GCF and sum the coefficients to get $9 \cdot 3^3$. By lemma 1 the product produces 3^5.

Example 8: A simple example of the addition of two block-numbers comprised of different prime-bases is the sum of $2^3 + 3^3$. By trying the multiplicative identity 1 as GCF we get a sum in the standard form of 17^1 of power 1. This number cannot be further down-reshaped since 17 is a prime and power of 1 is all what we get. Therefore, if higher powers are desired, a GCF of higher power than 0 may be used. If we multiply both terms by a common factor of 35^3 that necessarily complies with lemma 1, the expression becomes $70^3 + 105^3$ which has a sum of 35^4. This is only possible because the new block-numbers share a common block-factor with common base-value of 35. The factor 35^3 is then the GCF which represents the block-bundle of the sum that repeats by base-unit value of 35. In other words, the number 35 then is the base-unit that repeats and constructs the block-bundle which repeats to construct all three terms of 70^3, 105^3 and 35^4 complying with lemma 1. Note that after multiplication of a proper CF all three terms have the base of the CF as a common unit upon which both can build the sum by adding their own base-units, and the size of the resulting sum of 35^4 equals the sum of the sizes of the two added numbers of 70^3 and 105^3, which is 42875. That is the total number of base-units. Note also that the base-unit of 35 is a composite number made of the product of the prime numbers 7 and 5.

3. Block-equation

A block-equation is an equation whose terms are block-numbers. It can be represented as,

$$\alpha x^l + \beta x^l = \delta x^l$$

Where $\delta = \alpha + \beta$, x^l is a GCF and each term constitutes a block-number.

Theorem 1. *Let the LHS of the block-equation $a^x + b^y = c^z$, where a, b, c, x, y and z are positive integers, represent the sum of two block-monomials with common variable P. Further, let the equation be made in the three unique forms,*

Case 1: *Both terms have the same powers but different bases,*

$$l^n P^n + k^n P^n = q^j P^n$$

Case 2: *Both terms have the same bases but different powers,*

$$P^r P^n + P^s P^n = q^j P^n$$

Case 3: *One term has the same base and different powers, the other one has the same power and different bases,*

$$P^r P^n + l^n P^n = q^j P^n$$

where P^n is the GCF obtained from the expression $a^x + b^y$; $(l^n + k^n)$, $(P^r + P^s)$, $(P^r + l^n)$ are the sums of the coefficient factors of the two block-terms, and P, l, k, n, r, s are positive integers. Then there exists a unique method to combine the two terms into a single block-term such that a unique solution $q^j P^n$ equals c^z exists and shares the common block-number P^n, where q^j equals one of the sums of the coefficient factors and q and j are positive integer variables.

Proof. Building on the facts established by lemmas 1-4, all three forms of Beal's equations are satisfied. The monomial-variable P^n constitutes the block-bundle that repeats to form all three terms of Beal's equation. Factoring the two terms on the LHS of Beal's equation of its block-bundle P^n leaves it as the product of P^n and one of the three sums of the terms' coefficient factors. For positive integer variables in the expressions, each of the coefficient sums must yield the single block-term q^j for the equality to produce a block-term. The sum of the two terms on LHS then becomes $q^j P^n$. After reshaping q^j and P^n by lemma 2 and combining factors extracted from one to the other if needed, the two terms must combine to a single block-term by lemma 1 to yield the sum.

Remark. The least value of q^j is 2 if each of the coefficient terms yields 1. If it is not possible to extract two terms from any product of two block-numbers and reshape them such that they have either a common positive integer base or a common positive integer exponent, then it is not possible to obtain a single block-number as the result of the multiplication process and the addition process fails.

Corollary 3. *If the sum of any number of block-monomials constitutes part of a block-equation, the equation must abide with theorem 1.*

Proof. Theorem 1 predicts that any solution to any block-equation of any number of terms must conform to the power rules of lemma 1.

Example 9: For the block-equation of the sum of three block-terms of $6^2 + 6^2 + 3^2 = 3^4$, theorem 1 states that there exists a GCF in the form of P^n. Let's work with the LHS of the equation to produce the RHS. The expression $6^2 + 6^2 + 3^2$ has a GCF of 3^2, leaving out the expression as $(2^2 + 2^2 + 1)3^2$. The expression can be reduced to $3^2 \cdot 3^2$, and by theorem 1 the sum yields the RHS of the block-equation.

3.1. Steps to generate the sum of two block-numbers

If two block-numbers can be added to produce a block-number, the following steps must be taken,

1. Check if 1 is a valid GCF by checking if the bases are coprime and a trivial sum is desired.
2. Obtain a GCF different than 1 of the two block-numbers. If one of the block-numbers is the GCF, a simple check for the GCF is to divide the larger block-term by the smaller one. We can also break down the bases of the two terms to their primes and calculate the GCF.
3. Factor out the GCF of the two block-numbers to convert the addition process to multiplication.
4. Use the power rules to reduce the product of the two numbers to a single term. Specifically look for how the two terms in the product of the two block-numbers can end up having the same base or the same power.
5. Combine the two terms by either of the two power rules in step 3 to a single term.

Remark: One of the main functions of a successful CF is to convert the terms of an equation to block-numbers. If the equation is already in block-form, the solution is then trivial and the GCF is just unity.

Example 10: In example 9 the coefficients constitute the block-equation $2^2 + 2^2 + 2^0 = 3^2$ where the sum is primitive and the GCF is unity of 2^0. We can choose another proper CF of 3^2, or any block-number of power 2, to generate another valid block-equation as in example 9.

Example 11: For the equation $3^9 + 54^3 = 3^{11}$, theorem 1 states that there exists a GCF in the form of P^n. Let's work with the LHS of the equation to produce the RHS. The expression $3^9 + 54^3$ has a GCF of 3^9, leaving out the expression as $(1 + 8)3^9$. The expression can be reduced to $3^2 \cdot 3^9$. Lastly, the two parts of the expression can be combined by the power rules to 3^{11} since the two terms in the expression have the same base. The block-number 3^9 then is the GCF for the three terms in the equation and 3 is the common base-factor among the three terms. Only when one of the two methods by applying the power rules to combine the product terms is satisfied the combining process succeeds in producing a sum and therefore a common relative base-factor.

Example 12: For the equation $7^6 + 7^7 = 98^3$, one possible common block-factor is 7^3. In this case 7^3 is another block-bundle besides the GCF. The sum of the coefficients yields 2744 which can be shaped down to 14^3, which produces the RHS of the equation upon combining the terms by the power rule of the product of two numbers having the same power. If we factor out the GCF of 7^6 from the LHS of the equation, the expression becomes $(1+7)7^6$ and can further be expressed as $2^3 \cdot 7^6$. Notice that none of the terms in the original expression of $1+7$ had a block-form. With the introduction of a CF, the two terms took a block-form and so did their sum. To combine the resulting two terms we divide the power of the second term by 2, to make use of one of the methods to combine two block-terms of same power, and square its base to get $2^3 \cdot 49^3$. This expression yields the same term on the RHS of the equation. The three terms then share a GCF of 7^6 and therefore a common base of 7.

Example 13: For the block-equation $34^5 + 51^4 = 85^4$, an obvious prime common factor of the bases of the two block-numbers on the LHS of the equation is 17. The two numbers are then reduced to $17^5 \cdot 2^5 + 17^4 \cdot 3^4$, which can be further reduced to $(17 \cdot 2^5 + 3^4)17^4$, where 17^4 is the GCF term. The expression becomes then $(625)17^4$ and can be reduced to $5^4 \cdot 17^4$ by fully down-reshaping 625 to the number 5^4. We can see that the original sum was $17 \cdot 2^5 + 3^4$ with the first term expressed in non-block form. The CF transformed the first term into a block-term. The sum becomes 85^4. We conclude that the block-number 17^4 is the GCF-block-bundle of the terms on LHS of the equation and 17 is a common factor of their bases.

Example 14: The LHS of the block-equation $760^3 + 456^3 = 152^4$ can be broken down to the base's primes and becomes $5^3 \cdot 2^9 . 19^3 + 3^3 \cdot 2^9 . 19^3$. The two terms now can be combined to yield $(3^3 + 5^3)\, 2^9 . 19^3$, and by reshaping 2^9 to 8^3 the expression becomes $(3^3 + 5^3)\, 8^3 . 19^3$ with a GCF of 152^3 to yield $152 \cdot 152^3$, which yields the RHS of the equation.

Remark. The process to evaluate a block-equation as described by the theory should always naturally succeed to satisfy the block-equality if we pick the common factor as the GCF. Block-numbers of prime bases grow in size by branching out by the base-unit. Since block-numbers of different prime-bases grow in size at different branching rates, the only possible way for a successful addition is to build a base-unit made of a composite number of the primes.

Remark. To evaluate the block-equation we need to solve for the block-monomials' variable P. In example 14, P is 152.

3.2. Uniqueness of the sum term

Remark. The sum on the RHS of Beal's equation is unique but it may be expressed in a different form by the process of reshaping.

Example 15: Let's consider the block-equation $27^4 + 162^3 = 9^7$. By factoring out 27^4, or 3^{12} if we reshape it to its lowest base, as the GCF from the LHS of the equation we get $(1 + 8)27^4$, which becomes $3^2 \cdot 3^{12}$ and produces a final result of 3^{14}, which can be reshaped to produce 9^7, the RHS of the equation. It is important to make sure that the sum-term on the RHS of the block-equation has not been reshaped before we judge whether the resulting equation is identical to the given one.

Example 16: Another example to beware of the end result as deemed different is the equation $33^5 + 66^5 = 1089^3$. By following the steps above we find the sum of the two terms on the LHS of the equation is 33^6, which can easily be reshaped to 1089^3. The same goes with the equation $8^3 + 8^3 = 4^5$. By following the same steps we may get a solution 2^{10} or 32^2 which can be reshaped to 4^5.

For further exploration we will list more examples and the steps considered to obtain the solution to block-equations.

Example 17: By factoring out the GCF of 19^3 from the LHS of the block-equation $19^4 + 38^3 = 57^3$ we obtain $(19 + 8)\,19^3$. Simplifying we get $27 \cdot 19^3$ which by reshaping 27 becomes $3^3 \cdot 19^3$ and yields the RHS of the equation. The RHS monomial then is 3^3x^3, with variable value of 19, degree 3 and coefficient 3^3.

Example 18: By factoring out the GCF of 80^{12} from the LHS of the block-equation $80^{12} + 80^{13} = 1536000^4$ we obtain $(1 + 80)\,80^{12}$. Simplifying we get $81 \cdot 80^{12}$ which becomes $3^4 \cdot 80^{12}$, and by reshaping 80^{12} as 512000^4, the RHS monomial then becomes 3^4x^{12}, with a variable value 80, coefficient 3^4 and degree 12.

Example 19: By factoring out the GCF of 28^3 from the LHS of the block-equation $84^3 + 28^3 = 28^4$ we obtain $(27 + 1)\,28^3$. Simplifying, we get $28 \cdot 28^3$ and the RHS monomial then is $28\,x^3$, where the variable value and the coefficient are 28 and the degree is 3.

Example 20: By factoring out the GCF of $1838^3$from the LHS of the block-equation $1838^3 + 97414^3 = 5514^4$ we obtain $(1 + 148877)\,1838^3$. By borrowing 1838 factor from the coefficient term and simplifying we get $81 \cdot 1838^4$. The 81 can be reshaped as 3^4 and the product yields the RHS.

Remark In examples 7-20 we have produced the RHS of the given equations without a prior consideration of its final form just by applying theorem 1. Also, it is clear from the examples that the role of a common factor of the two terms on LHS of Beal's conjecture solution is to provide the basic base-unit to build the sum onto.

4. Beal's equation

Beal's equation is a "block-equation" with the LHS representing the sum of two block-monomials with common variable x. Following properties of addition of block-numbers, the expression on the LHS of the equation deals with addition of two block-numbers followed by multiplication operation to simplify the resulting product to a single block-term. In principle we need to combine the two terms on the LHS of Beal's equation in one term simply by making use of the product rules of exponents as ordained by lemma 1. Let's first proceed to prove that Beal's solution must have a common block-factor.

LEMMA 5. *The two terms on the LHS of Beal's solution are intrinsically factorable by a GCF.*

Proof. Since the two terms on the LHS of Beal's equation are in block-form and their sum is equalized with a block-number, it follows by corollary 2 that the two terms must share a block-bundle and therefore a GCF to successfully combine into a block-number.

LEMMA 6. *The factorization of the two terms on the LHS of Beal's equation* $a^x + b^y = c^z$ *must yield a product expression* $d^m \cdot g^n$*, where either* $d = g$ *or* $m = n$ *or both.*

Proof. It follows from lemma 1.

Remark The process to combine the two terms on the LHS of Beal's equation is then straight forward. We factor out the GCF and sum the coefficient terms of the block-numbers to convert the addition process to multiplication of two new block-numbers which can further be reduced by the power rules to a single block-number.

5. Beal's conjecture connection with Fermat's last theorem

In this section we will extrapolate the connection between Beal's conjecture and Fermat's last theorem. Specifically, we will relate power 2 of the terms of Beal's equation as restricted to Fermat's last theorem having no solutions for $n > 2$ of positive integers *A*, *B*, and *C*.

5.1. Beal's conjecture is a generalization of Fermat's Last Theorem

Fermat's Last Theorem states that no three positive integers *a*, *b*, and *c* satisfy the equation $a^n + b^n = c^n$ for any integer of *n* greater than 2. The theorem was rigorously proven by Andrew Wiles [2]. The cases *n* = 1 and *n* = 2 have been known to have infinitely many solutions.

While Pythagorean triples can be generated, for example by Euclid formula, a connection to Beal's conjecture of non-primitive triples has not been made to our knowledge. In the language of block-numbers of monomials, Fermat's Last Theorem says that a successful block-solution of the sum of two block-monomials of same degree to yield same degree monomial is that of degree 2 or 1. Or, no block-equation can be reshaped to the same power for all terms except for powers 1 and 2.

Corollary 4. *Fermat's equation keeps its form of its squared terms only if we multiply it by a CF that complies with same exponent rules of lemma 1.*

Proof. If a CF of power different than 0 is desired to evaluate Fermat's equation, only a CF of power 2 may be used, since if it is multiplied by Fermat's equation it combines successfully with all three terms of the equation to yield an exponent of 2, retaining its power of 2 and its block-form. This confirms Euclid's general formula to generate all Pythagorean triples (a, b, c) uniquely:

$$a = k \cdot (a^2 - b^2), b = k \cdot 2mn, c = k \cdot (a^2 + b^2),$$

where m, n, and k are positive integers with $m > n$, and with m and n coprime and not both odd. Here, k is the scaling factor of the GCF.

Example 21: Pythagorean triple of $(3, 4, 5)$ complies with Fermat's equation and are coprime with GCF of unity. We can associate it to another proper GCF of 2^2 and retain Fermat's form. This GCF is carefully chosen such that it combines with the terms on the LHS of the Fermat's equation $3^2 + 4^2 = 5^2$ by the same exponent rule; see lemma 1. Multiplying the proposed GCF of 2^2 converts the triple to non-primitive equation of $6^2 + 8^2 = 10^2$. Any common factor of x^2, where x is integer produces an infinite number of solutions to the non-primitive Pythagorean triples characterizing the block-equation $3^2x^2 + 4^2x^2 = 5^2x^2$.

Remark: For Fermat's equations: All Pythagorean triples can be generated from their primitive forms by multiplying the equation by a CF of power 2. Since a primitive Pythagorean equation cannot have both of the two terms on LHS with the same base, to generate Pythagorean triples from their primitive forms the only way is to multiply by a CF of power 2.

Example 22: In example 21, the terms of Pythagorean triple of (3, 4, 5) on the LHS of Fermat's equation $3^2 + 4^2 = 5^2$ can be reshaped to $3^2 + 2^4 = 5^2$, which is a "quasi-Beal's equation" Also, if multiplied by a factor of 25^2, the resulting Fermat's equation becomes $75^2 + 100^2 = 125^2$, and reshaping the terms yields a quasi-Beal's equation of $75^2 + 10^4 = 5^6$. A quasi-Beal's equation is described by Fermat–Catalan conjecture.

5.2. The sum of the two terms on the LHS of Fermat's equation

Corollary 5. *A successful summation of the two-terms on the LHS of Fermat's block-equation is only possible if the exponents are 2 or less.*

Proof. The proof is fulfilled by Fermat's Last Theorem.

Corollary 6. *The solution of a three-term block-equation of same power whose terms have exponents greater than 2 is intrinsically unsuccessful.*

Proof. It follows from Fermat's Last Theorem that no coprime terms of powers greater than 2 exist of Fermat's equation. By theorem 1, a proper CF of power n cannot be admitted to scale up the terms on the LHS of the equation to obtain a sum of power n because multiplying a CF of any power $n > 2$ to the equation of terms of power n will necessarily raise the power of the sum to $m > n$ and therefore any Fermat's equation fails for powers greater than 2. To see this, let's consider the general form of a block-equation of terms sharing a GCF of P^n, where $P > 1$ and $s > 2$,

$$l^s P^n + k^s P^n = q^j P^n$$

Case 1: By lemma 1, the terms on the LHS of the equation share the same exponent as the GCF and the equation will be,

$$l^s P^s + k^s P^s = q^j P^s$$

$$a^s + b^s = c^z$$

The expression on the LHS may be reduced to $(l^s + k^s)P^s$ where $j \neq s$ by Fermat's last theorem. If $q = P$, the expression becomes P^{j+s} which yields $z > s$ or $z < s$ but not $z = s$, by Fermat's last theorem. Since $s > 2$, the resulting equation is either Beal's equation if $z > s$ or quasi-Beal's equation if $z < s$.

Case 2: By lemma 1, the terms on the LHS of the equation may share the same base and the same exponent as the GCF. The then expression becomes,

$$k^s k^s + k^s k^s = c^z$$

$$2k^{2s} = c^z$$

If the term on the LHS to be in block-form, then $k = 2$ and the equation reduces to

$$2^{2s+1} = c^z$$

The last equation is Beal's equation since $z = 2s + 1 > 2s$.

Case 3: By lemma 1, the terms on the LHS of the equation may share the same bases but different powers and will be,

$$k^s k^n + k^s k^n = c^z$$

which produces

$$2k^{s+n} = c^z$$

If the term on the LHS to be in block-form, then $k = 2$ and the equation reduces to

$$2^{s+n+1} = c^z$$

The last equation is Beal's equation since $z = s + n + 1 > s + n$.

Corollary 7. *Fermat's Last Theorem as a solution to a block-equation and Beal's equation as an equation of a block-form suggest that any same power $n > 2$ of the two terms on the LHS of a three term block-equation must yield a sum of power greater than n.*

Proof. The proof is implied by Fermat's Last Theorem and Corollaries 4 and 6.

Proposition 1. *Failed Fermat's equation of exponents greater than 2 is an incomplete Beal's equation.*

It is proposed here that all failing Fermat's equations of powers greater than 2 are incomplete Beal's equations, missing a proper CF. We can consider any successful Fermat's equation of either a primitive Pythagorean coprime triples or a scaled one by some factor as a successful quasi-Beal's equation.

Example 23: For the failing Fermat's sum of $6^3 + 21^3$ we have $2^3 \cdot 3^3 + 7^3 \cdot 3^3$. By factoring out the GCF of 3^3 from the two terms we obtain $(351)\ 3^3$. Simplifying the coefficient term to its prime bases we get the expression $(13 \cdot 3^3)3^3$. This expression cannot be combined to a block-form. The wrong GCF has been used. Since the base-units 7 and 2 of the two terms of the original sum of $2^3 + 7^3$ branch at different rates and the application of a GCF of 1 does not yield a block-number, a GCF with composite number of the two terms is necessary to build the block-bundle that builds the sum. A proper one is 351^3, since 351 is the sum of the two terms, and the corresponding proper Beal's block-equation for the sum $2^3 + 7^3$ is $702^3 + 2457^3 = 351^4$. The GCF is the block-bundle and has been chosen carefully such that it combines with all three terms transforming them to block-from, abiding with lemma 1.

Example 24: From example 8, the sum of $3^3 + 2^3$ in the equation $3^3 + 2^3 = 35$ is a failed Fermat's equation but it can be made a successful Beal's equation by applying a proper CF of the sum of the two terms to the power 3 so it can combine with all three terms of the equation. Multiplying the equation by 35^3 we get the block equation as $105^3 + 70^3 = 35^4$. So, multiplying by a proper CF of a failed Fermat's equation of power greater than 2 always yields a valid Beal's equation.

Proposition 2. *Failed Fermat's equation of power 2 is an incomplete quasi-Beal's equation.*

It is proposed here that if you multiply a failed Fermat's equation by a proper CF it becomes a quasi-Beal's equation.

Example 25: The equation $2^2 + 3^2 = 13$ is a failed Fermat's equation of power 2. If we multiply the equation by a CF of 13^2 we obtain the quasi-Beal's equation of $26^2 + 39^2 = 13^3$. See also examples 21 and 22.

Remark We can see that to increase the power of the terms of any block-equation we can only increase the size of the terms of the LHS of the block-equation by corollary 2. Correspondingly, by theorem 1, the RHS of the block-equation increases by the same size as the two terms on the LHS for the equality to hold.

5.3. n terms of Fermat-like equations

Fermat-like n terms block-equation in powers higher than 2 can be achieved by increasing the number of terms in the equation. For integer power $n = 3$, a successful Fermat-like block-equation may be stated as $a^n + b^n + c^n = d^n$, where the bases are positive integers.

Conjecture 1: *A successful solution of same power of block-equations in the form of* $a^n + b^n + c^n + \cdot + \cdot = d^n$ *is only possible if n is equal to or less the number of terms.*

Remark: The conjecture was inspired by Fermat's last theorem.

Example 26: The conjecture is satisfied for $n = 2$ by example 9, where the sum 3^4 can be up-shaped to 9^2.

Example 27: A successful Fermat-like four term block-equation is $3^3 + 4^3 + 5^3 = 6^3$.

Corollary 8. *Fermat-like four term block-equation in the form* $a^n + b^n + c^n = d^n$ *and Beal's equation as equations of block-form suggest that for any successful power* n *greater than 3 the resulting equation must be Beal's four-term equation.*

Proof. The proof is similar to that of Corollary 7. By Corollary 6, the common factor that keeps the form of Fermat's equation of four terms unchanged must be of exponent 3 for $n = 3$. For $n > 3$ we choose a GCF of exponent n to combine the LHS of terms of power n converting the equation to quasi-Beal equation of four terms since the expression $a^n + b^n + c^n$ must yield the sum d^z, where $z \neq n$.

6. The proof of Beal's conjecture

Beal's conjecture proof stems from that a block-solution to the summation of any two or more block-numbers is only successful if the block-numbers share a block-bundle to build the sum upon and necessarily share a base-unit. In other words, the main function of the application of a CF to an equation is to provide it with a block-bundle via transforming it to an equation in block-form as Beal's conjecture requires. The sum of the two terms on the LHS of Beal's equation that results in a block-number fulfills the description of the sum of two block-monomials abiding with theorem 1. Deductively, the success of representing Beal's equation of powers greater than 2 as a block-equation is a sufficient proof of the validity of Beal's conjecture as a theory. Further, by Fermat's Last Theorem, it was proved here that the process of either reshaping the terms of Fermat's equation or multiplying it by a CF that brings one or more of the exponents of the equation's terms to a value higher than 2 converts Fermat's equation to a quasi-Beal equation. This distinguishes Fermat's equation from Beal's equation for exponents 2 and excludes Fermat's equation for exponents 3 and larger and describes it as only a valid Beal's block-equation by Corollaries 6 and 7. In other words, Beal's block-equation with terms of powers larger than 2 necessarily can only be brought about by the admission of a common factor since Fermat's LHS cannot be admitted by itself as the sum of the coefficients on the LHS of Beal's equation of $(a^n + b^n)$ for $n > 2$. Therefore Beal's block-equation satisfies the condition of $n > 2$ to generate a solution fulfilling the conjecture. Further, Beal's equation as a block-equation is not bound to a lower limit of power as described here by a quasi-Beal's equation.

The block-equation form of Beal's equation of exponents higher than 2 ensures a common block-factor of same base-unit among the three terms of the equation which constitutes a sufficient background to prove the conjecture.

To elaborate more, the conjecture requires that the two terms on the LHS to be in block-form, which follows the power rules upon executing multiplication operation to combine and form another block-number. For the summation of the two terms on the LHS of the solution of Beal's equation to yield a block-number they both must add their base-units to each other such that the elements of both numbers add up exactly as if one of them just increases in size.

The process of converting the summation of block-numbers to a product of two block-numbers and the successful process of combining the two terms of the product to yield a single block-number ensures that the total number of base-units is equal on both sides of a block-equation and therefore the sum ensures equal block-sizes on both sides of the equation. The base-unit that all terms share in any block-equation is just the basic-element in the case of trivial summation of block-numbers sharing unity as GCF and must be in block-form as a priori, therefore, in essence they don't need a CF other than 1. The following conditions are essential when evaluating a block-equation,

1. The number of block-elements on LHS of a block-equation should be equal to that on the RHS.

2. The number of unique same base-units on LHS of a block-equation should equal that on the RHS.

The following table lists possible sums and their description of the expressions on the LHS of the solutions of Beal's equation starting from $1 + 1$, some with no solution.

Eqn.	lP^n	kP^n	$(l + k)$	P^n	$q^s P^n$	c^z Block-solution	Block-solution of power > 2	Description
1	1	1	$1 + 1$	1	$2 \cdot 1$	2^1	No	Conjecture unsatisfied
2	1	2	$1 + 2$	1	$3 \cdot 1$	3^1	No	Conjecture

								unsatisfied
3	2^1	2^1	$1+1$	2^1	$2^1 \cdot 2^1$	2^2	No	Conjecture unsatisfied
4	2^2 4	2^2 4	$1+1$	2^2	$2^1 \cdot 2^2$	2^3	yes	Conjecture unsatisfied
5	5	5	$1+1$	5^1	$2^1 \cdot 5^1$	10^1	NO	Conjecture unsatisfied
6	2^3 8	7^1 7	2^3+7^1	1	$15 \cdot 1$	15^1	NO	Conjecture unsatisfied
7	2^3 8	2^3 8	$1+1$	2^3	$2^1 \cdot 2^3$	2^4	yes	Conjecture satisfied
8	2^4 16	2^4 16	$1+1$	2^4	$2^1 \cdot 2^4$	2^5	yes	Conjecture satisfied
9	3^3 27	3^5 243	$1+9$	3^3	$10 \cdot 3^3$	$-$	$-$	Unsuccessful block-simplification

The key to which of the next block-equation in the table would satisfy Beal's conjecture depends on whether each term of the block-equation $lP^n + kP^n = (l+k)P^n$ individually can be shaped in a block-form such that it satisfies Beal's restrictions which can be checked by lemma 1. In other words, solutions of Beal's equation are satisfied if all three terms can be expressed in a block-form, which can be checked by an iterative algorithm. It follows that only a specific choice of GCF succeeds to combine two block-terms to produce a block-equation.

The process as described in this article to obtain the RHS from the LHS describes an identity block-equation. This is confirmed by the evaluation of the sum on the LHS of Beal's equation as a stand-alone expression which justifies representing it as the sum of two block-monomials of common block-variable which constitutes a common base of the GCF and suggests a proof of Beal's conjecture since a solution is implied regardless of the value of the powers of the terms in any block-equation including a CF of power 0.

By the methods adopted here, any block-equation has a GCF; the least of which is 1 and produces the trivial solution. The theme for the proof of Beal's conjecture is that we must increase the size of the term on the RHS of Beal's solution to upgrade the solution to a higher size. By Corollary 2 this is done by multiplying it with a proper block-factor, and for the equality to hold, the block factor must be used to upgrade the terms on the LHS of the equation as well, which necessitates that all terms must have a common. Therefore, Beal's equation is satisfied for powers greater than 2 since Fermat's Last Theorem ordains that a three-term block-equation cannot have same power terms for powers greater than 2.

References

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