

Beal's conjecture as sum of two vectors in polynomial vector space that defines one-variable polynomial identity-proof

Kamal Barghout

PMU

kbarghout@pmu.edu.sa

Abstract

Beal's equation is identified as polynomial identity. The central theme to prove Beal's conjecture here is identifying its numerical solution as a specific solution to the polynomial identity of $\alpha x^l + \beta x^l = \delta x^l$, where α, β, δ , and l are positive integers and x is the indeterminate. Beal's equation then may be represented by the sum of two same degree monomials representing two vectors in the vector space of polynomials in one variable with basis $1, x, x^2, x^3, x^4 \dots$ and coefficients in \mathbb{Z} of values zeros except for those of the two monomials. Accordingly, all three monomials of Beal's equation numerically produce terms of single power by following the rules of exponentiation considering vector space operations: addition of polynomials, and multiplication by integer scalars.

1. Introduction and conclusion

Beal's conjecture states that if $a^x + b^y = c^z$, where a, b, c, x, y and z are positive integers with $x, y, z > 2$, then a, b , and c have a common factor. The conjecture was made by math enthusiast Daniel Andrew Beal in 1997 [1]. So far it has been a challenge to the public as well as to mathematicians to prove the conjecture and no counterexample has been successfully presented to disprove it.

We identify Beal's equation $a^x + b^y \equiv c^z$ as polynomial identity of which the numerical solution is a specific solution to the general polynomial identity of $\alpha x^l + \beta x^l = \delta x^l$, where α, β, δ, l are positive integers, and x is the indeterminate whose value must combine with the coefficients of each term to produce a single power term following the rules of exponentiation, and $(\alpha + \beta) = \delta$. The LHS of the polynomial identity represents the expression of the sum of two monomial-vectors in a polynomial vector space.

To disprove the claim that Beal's numerical solution is a solution to a polynomial identity, a counter example must be presented. In this sense, the LHS of Beal's equation can be treated as the sum of two single variable monomials of the same degree that necessarily must produce the RHS of the equation. Following this claim, to produce a numerical solution to Beal's Diophantine equation one must combine the two LHS terms algebraically to produce the one-term expression on the RHS. A GCD representing the numerical value of the same indeterminate of the two monomials exists between the two LHS expressions. The GCD allows for the process of combining the two LHS expressions into one by the distributive property and exponential rules.

Suppose a solution to Beal's equation produces $c^z = 3^5$, which can in turn be split to $3^2 \cdot 3^3$ and the coefficient term 3^2 expands to $(2^3 + 1)$ producing the exponential Diophantine equation of,

$$3^3 + 6^3 = 3^5$$

The idea is to consider c^z as a vector in a polynomial ring over the positive integers, e.g. the numerical value of $c^z = 3^5$ is a polynomial of x^3 of indeterminate $x = 3$ by a coefficient of 3^2 in \mathbb{Z} . Beal's equation then may be represented by the sum of two same degree monomials representing two vectors in a vector space of polynomials in one variable (ring of polynomials) with basis $1, x, x^2, x^3, x^4 \dots$ and coefficients in \mathbb{Z} of values zeros except for those of the two monomials. The sum is an element in the ring of polynomials. In other words, we are adding two same degree variable monomials with different coefficients that produce a sum of the same degree monomial with the condition that the numerical evaluation leads to single power terms. In the above example, the monomial-equation is,

$$2^3 x^3 + x^3 = 3^2 x^3$$

In the identity equation (1) we are adding two same degree and one variable monomials by addition and multiplication rules of polynomials. Note that x^3 is just one of the set of the basis vectors of the vector space of the ring of polynomials and must be present. The numerical evaluation of the polynomial-equation dictates that the indeterminate x must have a value of 3 for the equation to have single power terms. This produces the specific solution of $3^3 + 6^3 = 3^5$. The existence of the common basis vector x^3 is a must since we are adding vectors in a vector space. Any other value of x satisfies the polynomial identity but does not comply with Beal's equation of single power terms.

A similar situation occurs when we add two fractions. In this case we take the basis of the rational vector space as $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \dots$. While addition of fractions in the physical sense is obvious, the similar addition of numbers in exponential form is not so obvious. The following is an example,

$$\frac{5}{3} + \frac{7}{2} = \left(\frac{5}{3} + \frac{7}{2}\right) \frac{6}{6} = 10 \left(\frac{1}{6}\right) + 21 \left(\frac{1}{6}\right) = 31 \left(\frac{1}{6}\right)$$

In the example above, the identity equation represents the addition of two “rational vectors” in the rational vector space of numeral values as $10\left(\frac{1}{6}\right) + 21\left(\frac{1}{6}\right)$ of the general identity equation of $\alpha\left(\frac{1}{x}\right) + \beta\left(\frac{1}{x}\right) = \delta\left(\frac{1}{x}\right)$ that have components only in the vector basis $\frac{1}{6}$ and zero components for the rest of the basis vector elements. By the use of the rules of addition and multiplication of fractions we obtain a sum of single fraction term in the defined vector space of basis vector $\frac{1}{x} = \frac{1}{6}$, which if cancelled out from both sides of the equation, the equation reduces to the numerical trivial one of $10 + 21 = 31$ which represents the addition process of elements in \mathbb{Z} .

In the above two examples of fractional and exponential identity equations, the common basis vector is a requirement to express the vectors that the equations represent in their corresponding vector spaces and shows up as common GCD of the corresponding numerical evaluation of the equations.

To prove Beal’s conjecture then it is sufficient to show that the numerical solution of Beal’s equation is a specific solution to the general polynomial identity equation of $\alpha x^l + \beta x^l = \delta x^l$ in polynomial vector space.

2. Proof of Beal’s conjecture

Proposition 1. Let the set $\mathbb{Z}[x]$ of all polynomials with integer coefficients be a ring over \mathbb{Z} with the usual operations of addition and multiplication of polynomials. Further, let the general polynomial identity equation of $\alpha x^l + \beta x^l = \delta x^l$ represent addition of two polynomial vectors of same degree in the polynomial vector space of basis $1, x, x^2, x^3, x^4 \dots$ and coefficients in \mathbb{Z} of values zeros except for those of the two polynomials, and α, β, δ, l are positive integers, and x the indeterminate. Then the solution to the polynomial equation is every value of the indeterminate x .

Proof. The proof is obvious since the polynomial is identified as identity.

Corollary 1. Let Beal’s equation represent the solution of the general polynomial identity equation of $\alpha x^l + \beta x^l = \delta x^l$ with the numerical evaluation leads to single power terms. Then, there exists a specific solution to Beal’s equation where the coefficients of each of the

polynomial terms α, β, δ must combine with the numerical value of the basis vector x^l to produce a single power number.

Proof. The proof is straight forward by employing exponential rules since the polynomial identity defines terms of one-variable and the numerical solution requires combining the coefficients of the terms with x^l .

Corollary 2. Let Beal's equation represent the solution of the general polynomial identity equation of $\alpha x^l + \beta x^l = \delta x^l$ as in Corollary 1. Then the solution of the equation has a common factor with common base.

Proof. Since the general polynomial identity that represents Beal's equation must have a common factor of x^l as the polynomial's basis vector, it follows that the specific solution of Beal's equation must have a common base of the numerical value of the base x .

Proposition 1 and Corollaries 1 and 2 provide a numerical common factor of x^l . This means a, b , and c have a common factor in Beal's equation of $a^x + b^y = c^z$. Since each term of the equation must combine with its coefficient to produce single power number by the exponential rules, the power either remains the same after combination if the coefficient is made to have the same power as the power of the GCD (l in x^l), or becomes greater if they both have the same base. Since Beal's equation describes terms in exponential form restricted to scaling the involved basis vectors of powers > 2 in the polynomial vector space, then the two terms on the LHS always share. If one or more of the terms in the equation have power two, the terms must share the basis vector with the exception of special cases such as the trivial Pythagorean triples and Fermat-Catalan equation which may be described by adding two same degree monomials with basis vector of x^0 in monomial vector space. That is possible because we are adding two monomials only which may allow exceptions of powers 2 and the sum of the coefficients of the terms on the LHS must be multiples of the base of the common basis vector. This concludes the proof of Beal's conjecture for powers ≥ 2 .

The next section presents examples of numerical solutions of Beal's Diophantine equation. The examples show how we can take the GCD of x^l on the LHS of the equation and combine them by the power rules to produce the RHS.

3. Examples of Beal's identity solutions

Example 3.1 The equation $70^3 + 105^3 = 35^4$ complies with Beal's conjecture. Factoring the GCD of 35^4 from the LHS we obtain $(3^3 + 2^3) 35^3$. Simplifying we obtain $35 \cdot 35^3 = 35^4$, the RHS of the equation.

Example 3.2 For the equation $7^6 + 7^7 = 98^3$, taking GCD of 7^3 from the LHS and add the coefficients we get $(7^3 + 7^4) 7^3$. The sum of the coefficient terms yields 2744 which can be shaped to 14^3 by taking the third root, which produces the RHS of the equation upon combining the terms by the power rule of the product of two numbers having the same power. If we factor out the GCD of 7^6 from the LHS of the equation, the expression becomes $(1 + 7)7^6$ and can further be expressed as $2^3 \cdot 7^6$. Simplifying we get $2^3 \cdot 49^3 = 98^3$, the RHS. This example works with two possible CF because the GCF of the basis vector x^l can be shaped to $x^l = x^{2n}$ representing $7^l = x^3$ and $x^n = 7^3$.

Example 3.3 For the equation $34^5 + 51^4 = 85^4$, the GCF is 17^4 . Factoring the GCF and combining the coefficients yields $(625)17^4 = 85^4$, the RHS.

Example 3.4 The LHS of the equation $760^3 + 456^3 = 152^4$ can be broken down to the base's primes and becomes $5^3 \cdot 2^9 \cdot 19^3 + 3^3 \cdot 2^9 \cdot 19^3$. The two terms now can be combined to yield $(3^3 + 5^3) 2^9 \cdot 19^3$, and by shaping 2^9 to 8^3 the expression becomes $(3^3 + 5^3) 8^3 \cdot 19^3$ with a GCD of 152^3 to yield $152 \cdot 152^3 = 152^4$, the RHS. The common base-factor of all three terms is 152 which have two distinct prime base-units made of 2 and 19.

Example 3.5 Let's consider the equation $27^4 + 162^3 = 9^7$. By factoring 27^4 we get $(1 + 8)27^4$, which becomes $3^2 \cdot 3^{12}$ and produces a final result of 3^{14} , which can be shaped to produce 9^7 , the RHS of the equation. It is important to make sure that the sum-term on the RHS of the equation has not been shaped differently before we judge whether the resulting equation is identical to the given one.

Example 3.6 Another example to beware of the end result as deemed different is the equation $33^5 + 66^5 = 1089^3$. The GCD on the LHS of the equation is 33^5 . Simplifying we get $(1 + 32)33^5 = 33^6$, which can easily be shaped to 1089^3 , the RHS of the equation. The same goes with the equation $8^3 + 8^3 = 4^5$, we get the sum as 2^{10} or 32^2 which can be shaped to 4^5 . The last example simply can be simplified by shaping the terms as $2^9 + 2^9 = 2^{10}$, which simplifies as $2 \cdot 2^9 = 2^{10}$. It is obvious here that both terms on the LHS have the same prime base-unit of 2.

Example 3.7 By factoring the GCD of 19^3 from the LHS of the equation $19^4 + 38^3 = 57^3$ we obtain $(19 + 8) 19^3$. Simplifying we get $27 \cdot 19^3$ which by shaping 27 becomes $3^3 \cdot 19^3$ and yields the RHS of the equation.

Example 3.8 By factoring out the GCD of 80^{12} from the LHS of the equation $80^{12} + 80^{13} = 1536000^4$ we obtain $(1 + 80) 80^{12}$. Simplifying we get $81 \cdot 80^{12}$ which becomes $3^4 \cdot 80^{12}$, and by shaping 80^{12} as 512000^4 we get the RHS of the equation.

Example 3.9 By factoring out the GCD of 28^3 from the LHS of the equation $84^3 + 28^3 = 28^4$ we obtain $(27 + 1) 28^3$. Simplifying, we get $28 \cdot 28^3$ which becomes the RHS.

Example 3.10 By factoring out the GCD of 1838^3 from the LHS of the equation $1838^3 + 97414^3 = 5514^4$ we obtain $(1 + 148877) 1838^3$. By borrowing 1838 factor from the coefficient term and simplifying we get $81 \cdot 1838^4$. The 81 can be shaped to 3^4 and the product yields the RHS.

Remark we have successfully evaluated every solution of Beal's equation we have checked by the process of factoring the GCD on the LHS of the equation and obtained the RHS verifying that all of the equations were identities. The common monomial of the GCD of x^l on the LHS of the general polynomial identity of $\alpha x^l + \beta x^l = \delta x^l$ is then the common basis vector in the polynomial vector space of basis $1, x, x^2, x^3, x^4, \dots$, whose value gives a common base of the equation in accordance with Beal's conjecture.

4. References

- [1] Daniel Mauldin, R. (1997) A Generalization of Fermat's Last Theorem: The Beal Conjecture and Prize Problem. Notices of the American Mathematical Society, **44**, 1436-1439