A proof of the falsity of the axiom of choice.

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Abstract

We observe two things in this paper : namely that the Banach Tarski paradox is false and that the correct part of the proof leads to a violation of the axiomof choice.

1 Proof.

The standard argument behind the Banach Tarski paradox goes as follows; one constructs two rotations a, b around an angle $r2\pi$ with r irrational around the x and z axis respectively. One considers the free group F_2 constructed by a, b which is split into five disjoint parts $S(a), S(a^{-1}), S(b), S(b^{-1}), e$ where $S(a)$ contains all irreducible words starting with the letter a. Clearly, $S(a) \sim S(b)$ geometrically and equally so when inverses are taken. The *axiom of choice* allows one to substract a set M containing one representant of each F_2 orbit on the two sphere. Consider the sets

$$A = S(a)M, B = S(a^{-1})M, C = S(b)M, D = S(b^{-1})M, M$$

and show that $b^n D \subset b^{n+m} D$ for $n, m > 0$ and that $\lim_{n\to\infty} b^n D = S^2$. However, if D were to miss points, then the above formula could not be true because a continuous mapping cannot fill in holes and therefore we reach the stronger conclusion that $D = S^2$. This cannot be given that generically, for any value of r, there exists a countable number of orbits such that $S(a), S(a^{-1}), S(b), S(b^{-1})$ determine disjoint suborbits (for a real infinite number of r, almost all orbits apply). Hence, M does not exist which proves the falsity of the axiom of choice.

To conclude the proof; consider the z axis v_z. $b^n(v_z) = v_z$ and $a^m(v_z) = \cos(rm2\pi)v_z - \sin(rm2\pi)v_y$. v_y is rotated by a^m in v_y and v_z whereas the action of b^n results in v_y and v_x. In general, we arrive at the conclusion that for any series $n_1, \ldots, n_k, m_1, \ldots, m_k$ a formula of the form $\sum_l \pm \prod_{j=1}^k g_j^l f_j^l = 1$ whereby the sum is finite and $f_j^l = \sin(rm_j 2\pi)$, $f_j^l = \cos(rm_j 2\pi)$ or $f_j^l = 1$ as well as $g_j^l = \sin(rn_j 2\pi)$, $g_j^l = \cos(rn_j 2\pi)$ or $g_j^l = 1$. Given that there exists a second countable number of words results in the fact that an uncountable number of real r gives rise to a free algebra.

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