LEGENDRE 'S CONJECTURE PROOF.

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Abstract :
In this paper, we are going to give the proof of legendre conjecture by using the Chebotarev -Artin 's theorem ,Dirichlet arithmetic theorem and the principle inclusion-exclusion of Moivre

1 Introduction

Legendre 's conjecture ,proposed by Andrien -Marie Legendre ,states that there is a prime number between n^2 and $(n+1)^2$ for every positive integer n. The conjecture is one of Landau's problems (1912) on prime numbers ;as of 2017,the conjecture has neither been proved nor disproved . in this paper we are going to give the proof of this conjecture

1.1 Principle of the Demonstration

Let n an odd integer and denote by $\mathbb{C}_{2n-1}$ the set of the integers of $[1, 2n-1]$ and let f_n be the bijective mapping such that : $\begin{array}{rcl} f_n : \mathbb{C}_{2n-1} & \to & n^2 + \mathbb{C}_{2n-1} \\ m & \mapsto & n^2 + m \end{array}$

Denote by G_n the subsect of $n^2 + \mathbb{C}_{2n-1}$ consisting of prime numbers and G'_n that of composite numbers we have $n^2 + \mathbb{C}_{2n-1} = G_n \cup G'_n$ .Let $\mathcal{P}_{(n+1)^2}$ the set of prime numbers less than or equal to $(n+1)^2$. Let

$$\delta(n) = card(G_n)$$

,as $\delta(n)$ represents the prime numbers between n^2 and $(n+1)^2$ then $\Pi((n+1)^2) = \delta(n) + \Pi(n^2)$
,
Moreover $\mathbb{C}_{2n-1}$Observe that each integer $m \in \mathbb{C}_{2n-1}$ such that $m \geq 2$ has at least one prime divisor $p \leq \sqrt{2n-1}$.
Let $\mathcal{P}_{\leq\sqrt{2n-1}} = \{p_1, p_2, ..., p_r\}$ where $p_1 = 2$,$p_2 = 3$,...$p_r = \max(\mathcal{P}_{\leq\sqrt{2n-1}})$.
Moreover, remembering that

$$\mathbb{C}_{2n-1} = \bigcup_{p\in\mathcal{P}_{\leq\sqrt{n}}, p\geq 2} A_p \cup \{1\}$$

where

$$A_p = \{p, 2p, 3p, 4p, ...(\lfloor\frac{2n-1}{p}\rfloor)p\}$$

. We notice that A_p is an arithmetic sequence of first term p and reason p So

$$n^2 + \mathbb{C}_{2n-1} = f_n(\mathbb{C}_{2n-1}) = \bigcup_{p\in\mathcal{P}_{\leq\sqrt{2n-1}}, p\geq 2} f_n(A_p) \cup \{n^2+1\}$$

As

$$f_n(A_p) = \{n^2 + p, n^2 + 2p, n^2 + 3p, ...n^2 + \lfloor \frac{2n-1}{p} \rfloor p\}$$

is an arithmetic sequence of first term $n^2 + p$ and reason p .
We will evaluate the quantity of prime numbers in $\bigcup_{p \in \mathcal{P}_{\leq \sqrt{2n-1}}, p \geq 2} f_n(A_p)$
by applying the principle -exclusion of Moivre and Chébotarev -Artin theorem in each $f_n(A_p)$ in the case where $p \nmid n$

2 Chebotarev-Artin 's Theorem

Let $a, b > 0$ such that $\gcd(a, b) = 1$,$\Pi(X, a, b) = card(p \leq X, p \equiv a[b])$ then $\exists c > 0$ such that $\Pi(X, a, b) = \frac{L_i(X)}{\phi(b)} + \bigcirc(cXe^{-\sqrt{\ln X}})$
The prime number theorem states that $L_i(X) \sim_{+\infty} \Pi(X)$ so
$\Pi(X, a, b) = \frac{\Pi(X)}{\phi(b)} + \bigcirc(cXe^{-\sqrt{\ln X}})$

3 corollary

Let $a, b > 0$ such that $\gcd(a, b) = 1$,$\Pi(X, a, b) = card(p \leq X, p \equiv a[b])$ then $\exists c > 0$ such that

$$\frac{\Pi(X, a, b)}{\Pi(X)} = \frac{1}{\phi(b)} + \bigcirc(c \ln Xe^{-\sqrt{\ln X}})$$

.
From probabilistic point of view, the probability of prime numbers less than or equal to X in an arithmetic progression of reason b and of the first term has such that $\gcd(a, b) = 1$ is worth $\frac{1}{\phi(b)} + \bigcirc(c \ln Xe^{-\sqrt{\ln X}})$ for X large enough .In the following we will justify the application of Chebotein-Artin's theorem for sets $\bigcap_{j=1, p_{i_j} \in \mathcal{P}_{\leq \sqrt{2n-1}}}^{k} f_n(A_{2p_{i_j}})$
for $1 \leq i_1 < i_2 < ..... < i_k$

3.1 Remarks

It is obvious to note that , $\bigcap_{j=1, p_{i_j} \in \mathcal{P}_{\leq \sqrt{X}}}^{k} f_n(A_{p_{i_j}})$ is the set of multiples of $\prod_{j=1}^{k} p_{i_j}$ which allows us to write

$$\bigcap_{j=1, p_{i_j} \in \mathcal{P}_{\leq \sqrt{2n-1}}}^{k} f_n(A_{p_{i_j}}) = \{n^2 + m \prod_{j=2}^{k} p_{i_j} | 1 \leq m \leq \lfloor \frac{2n-1}{\prod_{j=2}^{k} p_{i_j}} \rfloor\}$$

This set is an arithmetic sequence of reason $\prod_{j=2}^{k} p_{i_j}$ and first term $n^2 + \lfloor \frac{2n-1}{\prod_{j=2}^{k} p_{i_j}} \rfloor \prod_{j=2}^{k} p_{i_j}$.
The hypothesis of application of Chebotarev-Artin's theorem will be justified if and only if $\gcd(\prod_{j=2}^{k} p_{i_j}, \prod_{j=2}^{k} p_{i_j} + n^2) = 1$ which is the case if $\prod_{j=2}^{k} p_{i_j} \nmid n$

4 DEMONSTRATION OF LEGENDRE CONJECTURE

4.1 THEOREM

Let n an integer be an odd integer arbitrarily large , $\psi_{\mathbb{P}_n}$ prime indicator . then

$$\Pi((n+1)^2) - \Pi(n^2) = \psi_{n^2+1} + \sum_{k=1}^{n-1} \psi_{n^2+2k}$$

4.2 Useful Lemma

Let $a_1, a_2, ...a_r$ be r numbers then

$$1-\sum_{i=1}^{r}\frac{1}{a_i}+\sum_{1\leq i<j\leq r}\frac{1}{a_ia_j}+.....+\frac{(-1)^r}{a_1a_2....a_r}=\prod_{i=1}^{r}\frac{a_i-1}{a_i}$$

4.3 Proof

Let us consider the polynomial :$P(X)=\prod_{i=1}^{r}(X-\frac{1}{a_i})$ from the coefficient-root relations

$$P(X)=X^r+\sum_{k=1}^{r}\sum_{1\leq i_1<i_2<....<i_k\leq r}\frac{(-1)^kX^{r-k}}{\prod_{j=1}^{k}a_{i_j}}$$

taking $X=1$, the lemma is thus proved.

4.4 Proof of Theorem 1

According to the p rinciple of inclusion -exclusion of Moivre we have :

$$\varrho(\bigcup_{p\in\mathcal{P}_{\leq\sqrt{2n-1}},p\geq 3,p\nmid n}f_n(A_p))=\sum_{k=2}^{r}(-1)^k\sum_{2\leq i_2<i_3<....<i_k\leq r}\varrho(\bigcap_{j=2,p_{i_j}\in\mathcal{P}_{\leq\sqrt{2n-1}},p_{i_j}\nmid n}^{k}f_n(A_{p_{i_j}}))$$

where ϱ represents the probability of prime numbers so

$$\varrho(n^2+\mathbb{C}_{2n-1}\backslash f(A_2))=\varrho(\bigcup_{p\in\mathcal{P}_{\leq\sqrt{2n-1}},p\geq 3,p\nmid n}f_n(A_p))$$

. According to Chebotarev's theorem -Artin : $\forall k\geq 1$

$$\varrho(\bigcap_{j=2,p_{i_j}\in\mathcal{P}_{\leq\sqrt{2n-1}},p_{i_j}\nmid n}^{k}f_n(A_{p_{i_j}}))=\frac{1}{\phi(\prod_{j=2}^{k}p_{i_j})}+h((n+1)^2)$$

with $h((n+1)^2)=\bigcirc(c\ln ne^{-\sqrt{\ln n}})$
Thus

$$\varrho(\bigcup_{p\in\mathcal{P}_{\leq\sqrt{2n-1}},p\geq 3,p\nmid n}f_n(A_p))=\sum_{k=2}^{r}\sum_{2\leq i_2<i_3<...<i_k\leq r}\frac{(-1)^k}{\prod_{j=2}^{k}(p_{i_j}-1),p_{i_j}\nmid n}+h((n+1)^2)$$

and applying the useful lemma, we have :

$$\varrho(\bigcup_{p\in\mathcal{P}_{\leq\sqrt{2n-1}},p\geq 3,p\nmid n}f_n(A_p))=h((n+1)^2)+[1-\prod_{i=2,p_i\nmid n}^{\sqrt{2n-1}}\frac{p_i-2}{p_i-1}]$$

As

$$\varrho(\bigcup_{p\in\mathcal{P}_{\leq\sqrt{2n-1}},p\geq 3,p\nmid n}f_n(A_p)\cup f(A_2))=\varrho(f(A_2))+\varrho(\bigcup_{p\in\mathcal{P}_{\leq\sqrt{2n-1}},p\geq 3,p\nmid n}f_n(A_p))-\varrho(B)$$

. where $B = f(A_2) \cap \bigcup_{p\in\mathcal{P}_{\leq\sqrt{2n-1}}, p\geq 3, p\nmid n} f_n(A_p)$ As

$$\varrho(B) = \sum_{k=2}^{r}(-1)^k \sum_{2\leq i_2<i_3<....<i_k\leq r} \varrho(\bigcap_{j=2, p_{i_j}\in\mathcal{P}_{\leq\sqrt{2n-1}}, p_{i_j}\nmid n}^{k} f_n(A_{p_{i_j}}) \cap f(A_2)$$

.Let

$$S_{2p} = f(A_2)) \cap f_n(A_p) = f(A_2 \cap A_p)$$

then S_{2p} is an arithmetic sequence of first term $n^2 + 2p$ and reason 2p definie in the same manner than the previous part so we have

$$\varrho(B) = \sum_{k=2}^{r}(-1)^k \sum_{2\leq i_2<i_3<....<i_k\leq r} \varrho(\bigcap_{j=2, p_{i_j}\in\mathcal{P}_{\leq\sqrt{2n-1}}, p_{i_j}\nmid n}^{k} f_n(S_{2p_{i_j}}))$$

then

$$\varrho(B) = h((n+1)^2) + [1 - \prod_{i=2, p_i\nmid n}^{\sqrt{2n-1}} \frac{p_i - 2}{p_i - 1}] = \varrho(\bigcup_{p\in\mathcal{P}_{\leq\sqrt{2n-1}}, p\geq 3, p\nmid n} f_n(A_p))$$

thus

$$\varrho(\bigcup_{p\in\mathcal{P}_{\leq\sqrt{2n-1}}, p\geq 3, p\nmid n} f_n(A_p) \cup f(A_2)) = \varrho(f(A_2))$$

.As

$$\varrho(\bigcup_{p\in\mathcal{P}_{\leq\sqrt{2n-1}}, p\geq 3, p\nmid n} f_n(A_p) \cup f(A_2)) = \frac{\Pi((n+1)^2) - \Pi(n^2)}{\Pi((n+1)^2)} = \varrho(f(A_2))$$

Noting that $\varrho(f(A_2)) = \frac{\sum_{k=1}^{n-1}\psi_{n^2+2k}}{\Pi((n+1)^2)}$ so

$$\Pi((n+1)^2) - \Pi(n^2) = \sum_{k=1}^{n-1}\psi_{n^2+2k}$$

the theorem holds

5 Theorem 2

Let n an integer be an even integer arbitrarily large , $\psi_{\mathbb{P}_n}$ prime indicator . then

$$\Pi((n+1)^2) - \Pi(n^2) = \sum_{k=1}^{n}\psi_{(n+1)^2-2k}$$

5.1 Proof of Theorem 2

Let $\begin{array}{rcl} f_n : \mathbb{C}_{2n} & \to & (n+1)^2 - \mathbb{C}_{2n} \\ m & \mapsto & (n+1)^2 - m \end{array}$

$$(n+1)^2 - \mathbb{C}_{2n} = f_n(\mathbb{C}_{2n}) = \bigcup_{p\in\mathcal{P}_{\leq\sqrt{2n-1}}, p\geq 2} f_n(A_p) \cup \{n^2 + 2n\}$$

As

$$f_n(A_p) = \{(n+1)^2 - \lfloor\frac{2n}{p}\rfloor p, ...(n+1)^2 - 3p, (n+1)^2 - 2p, n^2 + 3p, (n+1)^2 - p\}$$

is an arithmetic sequence of first term $(n+1)^2 - \lfloor\frac{2n}{p}\rfloor p$ and reason p and applying the same ideas the theorem holds, we can also prove it in obvious manner

5.2 corollary

$\exists N \in \mathbb{N}\ \forall n \geq N\ ,\Pi((n+1)^2) - \Pi(n^2) \geq 1$

5.3 Proof

According to the theorem 1 and 2 we can write

$$\Pi((n+1)^2) - \Pi(n^2) = \sum_{k=1}^{n} \psi_{(n+1)^2-2k} + \sum_{k=1}^{n-1} \psi_{n^2+2k}$$

As n+1 and n have not the same parity then $\sum_{k=1}^{n} \psi_{(n+1)^2-2k}$ or $\sum_{k=1}^{n-1} \psi_{n^2+2k}$ represents the arithmetic progression of reason 2 and first term n^2+1 or n^2+2 in the intervall of length at most $2n+1$ when n goes to $+\infty$ there is at least one prime according to Dirichlet arithmetic progression theorem then $\exists N \in \mathbb{N}$ such that

$$\forall n \geq N, \Pi((n+1)^2) - \Pi(n^2) = \sum_{k=1}^{n} \psi_{(n+1)^2-2k} + \sum_{k=1}^{n-1} \psi_{n^2+2k} \geq 1$$

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Références

[1] Not always buried deep selection from analytic and combunatorial number theory 2003,2004 Paul POLLACK

[2] Elementary number theory a revision by Jim Heferon,st Michael's college 2003-Dec

[3] generatingfunctionology ,Herbert S.Wilf

[4] Lecture on NX(p) Jean pierre serre