

# There is No Standard Model of ZFC and ZFC<sub>2</sub>

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**Abstract:** In this article we proved so-called strong reflection principles corresponding to formal theories  $Th$  which has omega-models or nonstandard model with standard part. An possible generalization of Lob's theorem is considered. Main results are:

- (i)  $\neg Con(ZFC + \exists M_{st}^{ZFC})$ ,
- (ii)  $\neg Con(NF + \exists M_{st}^{NF})$ ,
- (iii)  $\neg Con(ZFC_2)$ , (iv) let  $k$  be an inaccessible cardinal then  $\neg Con(ZFC + \exists \kappa)$ .

*Keywords: Gödel encoding, Completion of ZFC, Russell's paradox,  $\omega$ -model, Henkin semantics, full second-order semantic, strongly inaccessible cardinal*

## Part I. There is no standard model of ZFC

1. Introduction.

1.1. Main results.

2. Derivation of the inconsistent definable set in set theory  $\overline{ZFC}_2^{Hs}$ ,  $ZFC_{st}$  and  $ZFC_{Nst}$ .

2.1. Derivation of the inconsistent definable set in set theory  $\overline{ZFC}_2^{Hs}$ .

2.2. Derivation of the inconsistent definable set in set theory  $ZFC_{st}$ .

2.3. Derivation of the inconsistent definable set in set theory  $ZFC_{Nst}$ .

3. Derivation of the inconsistent provably definable set in set theory  $\overline{ZFC}_2^{Hs}$ ,  $ZFC_{st}$  and  $ZFC_{Nst}$ .

3.1. Derivation of the inconsistent provably definable set in  $\overline{ZFC}_2^{Hs}$ .

3.2. Derivation of the inconsistent provably definable set in  $ZFC_{st}$ .

3.3. Derivation of the inconsistent provably definable set in  $ZFC_{Nst}$ .

3.4. Generalized Tarski's undefinability lemma.

3.5. Generalized Tarski's undefinability theorem.

3.6. Avoiding the contradictions from set theory  $\overline{ZFC}_2^{Hs}$ ,  $ZFC_{st}$  and set theory  $ZFC_{Nst}$  using Quinean approach.

3.6.1. Quinean set theory  $NF$ .

3.6.2. Set theory  $\overline{ZFC}_2^{Hs}$ ,  $ZFC_{st}$  and set theory  $ZFC_{Nst}$  with stratified axiom schema of replacement.

## Part II. Generalized Löb's Theorem.

1. Introduction.

2.1. Generalized Löb's Theorem.

2.2. Derivation inconsistent countable set in set theory  $ZFC_2^{Hs} + \exists M_{st}^{ZFC_2^{Hs}}$  and in set theory  $ZFC + \exists M_{st}^{ZFC}$  using Generalized Tarski's undefinability theorem.

2.3.Proof of the inconsistency of the set theory  $ZFC_2^{Hs} + \exists M^{ZFC_2^{Hs}}$  using Generalized Tarski's undefinability theorem.

## I.Introduction.

### 1.1.Main results.

Let us remind that accordingly to naive set theory, any definable collection is a set. Let  $R$  be the set of all sets that are not members of themselves. If  $R$  qualifies as a member of itself, it would contradict its own definition as a set containing all sets that are not members of themselves. On the other hand, if such a set is not a member of itself, it would qualify as a member of itself by the same definition. This contradiction is Russell's paradox. In 1908, two ways of avoiding the paradox were proposed, Russell's type theory and Zermelo set theory, the first constructed axiomatic set theory. Zermelo's axioms went well beyond Frege's axioms of extensionality and unlimited set abstraction, and evolved into the now-canonical Zermelo–Fraenkel set theory  $ZFC$ . *"But how do we know that  $ZFC$  is a consistent theory, free of contradictions? The short answer is that we don't; it is a matter of faith (or of skepticism)"*— E.Nelson wrote in his paper [1]. However, it is deemed unlikely that even  $ZFC_2$  which is significantly stronger than  $ZFC$  harbors an unsuspected contradiction; it is widely believed that if  $ZFC$  and  $ZFC_2$  were inconsistent, that fact would have been uncovered by now. This much is certain — $ZFC$  and  $ZFC_2$  is immune to the classic paradoxes of naive set theory: Russell's paradox, the Burali-Forti paradox, and Cantor's paradox.

**Remark 1.1.1.**Note that in this paper we view (i) the first order set theory  $ZFC$  under the

canonical first order semantics (ii) the second order set theory  $ZFC_2$  under the Henkin semantics [2],[3] and (iii) the second order set theory  $ZFC_2$  under the full second-order semantics [4],[5],[6].

**Remark 1.1.2.**Second-order logic essentially differs from the usual first-order predicate

calculus in that it has variables and quantifiers not only for individuals but also for subsets

of the universe and variables for  $n$ -ary relations as well [7],[8].The deductive calculus  $DED_2$  of second order logic is based on rules and axioms which guarantee that the quantifiers range at least over definable subsets [7]. As to the semantics, there are two types of models: (i) Suppose  $\mathbf{U}$  is an ordinary first-order structure and  $\mathbf{S}$  is a set of subsets of the domain  $A$  of  $\mathbf{U}$ . The main idea is that the set-variables range over  $\mathbf{S}$ , i.e.  $\langle \mathbf{U}, \mathbf{S} \rangle \models \exists X \Phi(X) \Leftrightarrow \exists S (S \in \mathbf{S}) [\langle \mathbf{U}, \mathbf{S} \rangle \models \Phi(S)]$ .

We call  $\langle \mathbf{U}, \mathbf{S} \rangle$  a Henkin model, if  $\langle \mathbf{U}, \mathbf{S} \rangle$  satisfies the axioms of  $DED_2$  and truth in  $\langle \mathbf{U}, \mathbf{S} \rangle$  is preserved by the rules of  $DED_2$ . We call this semantics of second-order logic the Henkin semantics and second-order logic with the Henkin semantics the Henkin second-order logic. There is a special class of Henkin models, namely those  $\langle \mathbf{U}, \mathbf{S} \rangle$  where  $\mathbf{S}$  is the set of all subsets of  $A$ . We call these full models. We call this semantics of second-order logic the full semantics and second-order logic with the full semantics the full second-order logic.

**Remark 1.1.3.**We emphasize that the following facts are the main features of

second-order logic:

**1. The Completeness Theorem:** A sentence is provable in  $\mathbf{DED}_2$  if and only if it holds in

all Henkin models [7].

**2. The Löwenheim-Skolem Theorem:** A sentence with an infinite Henkin model has a countable Henkin model.

**3. The Compactness Theorem:** A set of sentences, every finite subset of which has a Henkin model, has itself a Henkin model.

**4. The Incompleteness Theorem:** Neither  $\mathbf{DED}_2$  nor any other effectively given deductive calculus is complete for full models, that is, there are always sentences which are true in all full models but which are unprovable.

5. Failure of the Compactness Theorem for full models.

6. Failure of the Löwenheim-Skolem Theorem for full models.

7. There is a finite second-order axiom system  $\mathbb{Z}_2$  such that the semiring  $\mathbb{N}$  of natural numbers is the only full model (up to isomorphism) of  $\mathbb{Z}_2$ .

8. There is a finite second-order axiom system  $RCF_2$  such that the field  $\mathbb{R}$  of real numbers is the only (up to isomorphism) full model of  $RCF_2$ .

**Remark 1.1.4.** For let second-order  $ZFC$  be, as usual, the theory that results obtained from  $ZFC$  when the axiom schema of replacement is replaced by its second-order universal closure, i.e.

$$\forall X[Func(X) \Rightarrow \forall u \exists v \forall r[r \in v \Leftrightarrow \exists s(s \in u \wedge (s, r) \in X)]], \quad (1.1.1)$$

where  $X$  is a second-order variable, and where  $Func(X)$  abbreviates "  $X$  is a functional relation", see [7].

Thus we interpret the wff's of  $ZFC_2$  language with the full second-order semantics as required in [4],[5],[6],[7].

**Designation 1.1.1.** We will denote (i) by  $ZFC_2^{Hs}$  set theory  $ZFC_2$  with the Henkin semantics, (ii) by  $ZFC_2^{fss}$  set theory  $ZFC_2$  with the full second-order semantics, (iii) by  $\overline{ZFC}_2^{Hs}$  set theory  $ZFC_2^{Hs} + \exists M_{st}^{ZFC_2^{Hs}}$  and (iv) by  $ZFC_{st}$  set theory  $ZFC + \exists M_{st}^{ZFC}$ , where  $M_{st}^{Th}$  is a standard model of the theory  $Th$ .

**Remark 1.1.3.** There is no completeness theorem for second-order logic with the full second-order semantics. Nor do the axioms of  $ZFC_2^{fss}$  imply a reflection principle which ensures that if a sentence  $Z$  of second-order set theory is true, then it is true in some model  $M_{st}^{ZFC_2^{fss}}$  of  $ZFC_2^{fss}$  [5]. Let  $Z$  be the conjunction of all the axioms of  $ZFC_2^{fss}$ . We assume now that:  $Z$  is true, i.e.  $Con(ZFC_2^{fss})$ . It is known that the existence of a model for  $Z$  requires the existence of strongly inaccessible cardinals, i.e. under  $ZFC$  it can be shown that  $\kappa$  is a strongly inaccessible if and only if  $(H_\kappa, \in)$  is a model of  $ZFC_2^{fss}$ . Thus  $\neg Con(ZFC_2^{fss}) \Rightarrow \neg Con(ZFC + \exists \kappa)$ . In this paper we prove that:

(i)  $ZFC_{st} \triangleq ZFC + \exists M_{st}^{ZFC}$  (ii)  $\overline{ZFC}_2^{Hs} \triangleq ZFC_2^{Hs} + \exists M_{st}^{ZFC_2^{Hs}}$  and (iii)  $ZFC_2^{fss}$  is inconsistent, where  $M_{st}^{Th}$  is a standard model of the theory  $Th$ .

**Axiom  $\exists M^{ZFC}$ .** [8]. There is a set  $M^{ZFC}$  and a binary relation  $\epsilon \subseteq M^{ZFC} \times M^{ZFC}$  which makes  $M^{ZFC}$  a model for  $ZFC$ .

**Remark 1.1.3.** (i) We emphasize that it is well known that axiom  $\exists M^{ZFC}$  a single statement in  $ZFC$  see [7], Ch.II, section 7. We denote this statement through all this paper

by symbol  $Con(ZFC; M^{ZFC})$ . The completeness theorem says that  $\exists M^{ZFC} \Leftrightarrow Con(ZFC)$ .

(ii) Obviously there exists a single statement in  $ZFC_2^{Hs}$  such that  $\exists M^{ZFC_2^{Hs}} \Leftrightarrow Con(ZFC_2^{Hs})$ .

We denote this statement through all this paper by symbol  $Con(ZFC_2^{Hs}; M^{ZFC_2^{Hs}})$  and there

exists a single statement  $\exists M^{Z_2^{Hs}}$  in  $Z_2^{Hs}$ . We denote this statement through all this paper by

symbol  $Con(Z_2^{Hs}; M^{Z_2^{Hs}})$ .

**Axiom**  $\exists M_{st}^{ZFC}$ . [8]. There is a set  $M_{st}^{ZFC}$  such that if  $R$  is  $\{\langle x, y \rangle \mid x \in y \wedge x \in M_{st}^{ZFC} \wedge y \in M_{st}^{ZFC}\}$

then  $M_{st}^{ZFC}$  is a model for  $ZFC$  under the relation  $R$ .

**Definition 1.1.1.** [8]. The model  $M_{st}^{ZFC}$  is called a standard model since the relation  $\in$  used

is merely the standard  $\in$ - relation.

**Remark 1.1.4.** [8]. Note that axiom  $\exists M_{st}^{ZFC}$  doesn't imply axiom  $\exists M_{st}^{ZFC}$ .

**Remark 1.1.5.** We remind that in Henkin semantics, each sort of second-order variable has a particular domain of its own to range over, which may be a proper subset of all sets or functions of that sort. Leon Henkin (1950) defined these semantics and proved that Gödel's completeness theorem and compactness theorem, which hold for first-order logic, carry over to second-order logic with Henkin semantics. This is because Henkin semantics are almost identical to many-sorted first-order semantics, where additional sorts of variables are added to simulate the new variables of second-order logic.

Second-order logic with Henkin semantics is not more expressive than first-order logic. Henkin semantics are commonly used in the study of second-order arithmetic. Väänänen [6] argued that the choice between Henkin models and full models for second-order logic is analogous to the choice between  $ZFC$  and  $\mathbf{V}$  ( $\mathbf{V}$  is von Neumann universe), as a basis for set theory: "As with second-order logic, we cannot really choose whether we axiomatize mathematics using  $\mathbf{V}$  or  $ZFC$ . The result is the same in both cases, as  $ZFC$  is the best attempt so far to use  $\mathbf{V}$  as an axiomatization of mathematics."

**Remark 1.1.6.** Note that in order to deduce: (i)  $\sim Con(ZFC_2^{Hs})$  from  $Con(ZFC_2^{Hs})$ ,

(ii)  $\sim Con(ZFC)$  from  $Con(ZFC)$ , by using Gödel encoding, one needs something more than

the consistency of  $ZFC_2^{Hs}$ , e.g., that  $ZFC_2^{Hs}$  has an omega-model  $M_{\omega}^{ZFC_2^{Hs}}$  or an standard model  $M_{st}^{ZFC_2^{Hs}}$  i.e., a model in which the *integers are the standard integers*. To put it another way, why should we believe a statement just because there's a  $ZFC_2^{Hs}$ -proof of it? It's clear that if  $ZFC_2^{Hs}$  is inconsistent, then we won't believe  $ZFC_2^{Hs}$ -proofs. What's slightly more subtle is that the mere consistency of  $ZFC_2$  isn't quite enough to get us to believe arithmetical theorems of  $ZFC_2^{Hs}$ ; we must also believe that these arithmetical theorems are asserting something about the standard naturals. It is "conceivable" that  $ZFC_2^{Hs}$  might be consistent but that the only nonstandard models  $M_{Nst}^{ZFC_2^{Hs}}$  it has are those in which the integers are nonstandard, in which case we might not "believe" an arithmetical statement such as " $ZFC_2^{Hs}$  is inconsistent" even if there is a  $ZFC_2^{Hs}$ -proof of it.

**Remark 1.1.7.** However assumption  $\exists M_{st}^{ZFC_2^{Hs}}$  is not necessary. Note that in any nonstandard model  $M_{Nst}^{Z_2^{Hs}}$  of the second-order arithmetic  $Z_2^{Hs}$  the terms  $\bar{0}$ ,

$\bar{S}\bar{0} = \bar{1}, \bar{SS}\bar{0} = \bar{2}, \dots$  comprise the initial segment isomorphic to  $M_{st}^{Z_2^{Hs}} \subset M_{Nst}^{Z_2^{Hs}}$ . This initial segment is called the standard cut of the  $M_{Nst}^{Z_2^{Hs}}$ . The order type of any nonstandard

model of  $M_{Nst}^{Z_2^{Hs}}$  is equal to  $\mathbb{N} + A \times \mathbb{Z}$  for some linear order  $A$  [9]. Thus one can choose Gödel encoding inside  $M_{st}^{Z_2^{Hs}}$ .

**Remark 1.1.8.** However there is no any problem as mentioned above in second order set theory  $ZFC_2$  with the full second-order semantics because corresponding second order arithmetic  $Z_2^{fss}$  is categorical.

**Remark 1.1.9.** Note if we view second-order arithmetic  $Z_2$  as a theory in first-order predicate calculus. Thus a model  $M^{Z_2}$  of the language of second-order arithmetic  $Z_2$  consists of a set  $M$  (which forms the range of individual variables) together with a constant 0 (an element of  $M$ ), a function  $S$  from  $M$  to  $M$ , two binary operations  $+$  and  $\times$  on  $M$ , a binary relation  $<$  on  $M$ , and a collection  $D$  of subsets of  $M$ , which is the range of the set variables. When  $D$  is the full powerset of  $M$ , the model  $M^{Z_2}$  is called a full model. The use of full second-order semantics is equivalent to limiting the models of second-order arithmetic to the full models. In fact, the axioms of second-order arithmetic have only one full model. This follows from the fact that the axioms of Peano arithmetic with the second-order induction axiom have only one model under second-order semantics, i.e.  $Z_2$ , with the full semantics, is categorical by Dedekind's argument, so has only one model up to isomorphism. When  $M$  is the usual set of natural numbers with its usual operations,  $M^{Z_2}$  is called an  $\omega$ -model. In this case we may identify the model with  $D$ , its collection of sets of naturals, because this set is enough to completely determine an  $\omega$ -model. The unique full omega-model  $M_\omega^{Z_2^{fss}}$ , which is the usual set of natural numbers with its usual structure and all its subsets, is called the intended or standard model of second-order arithmetic.

## 2. Derivation of the inconsistent definable set in set theory $\overline{ZFC}_2^{Hs}$ and in set theory $ZFC_{st}$ .

### 2.1. Derivation of the inconsistent definable set in set theory $\overline{ZFC}_2^{Hs}$ .

**Designation 2.1.1.** Let  $\Gamma_X^{Hs}$  be the collection of the all 1-place open wff of the set theory  $\overline{ZFC}_2^{Hs}$ .

**Definition 2.1.1.** Let  $\Psi_1(X), \Psi_2(X)$  be 1-place open wff's of the set theory  $\overline{ZFC}_2^{Hs}$ .

(i) We define now the equivalence relation  $(\cdot \sim_X \cdot) \subset \Gamma_X^{Hs} \times \Gamma_X^{Hs}$  by

$$\Psi_1(X) \sim_X \Psi_2(X) \Leftrightarrow \forall X[\Psi_1(X) \Leftrightarrow \Psi_2(X)] \quad (2.1.1)$$

(ii) A subset  $\Lambda_X^{Hs}$  of  $\Gamma_X^{Hs}$  such that  $\Psi_1(X) \sim_X \Psi_2(X)$  holds for all  $\Psi_1(X)$  and  $\Psi_2(X)$  in  $\Lambda_X^{Hs}$ , and never for  $\Psi_1(X)$  in  $\Lambda_X^{Hs}$  and  $\Psi_2(X)$  outside  $\Lambda_X^{Hs}$ , is called an equivalence class of  $\Gamma_X^{Hs}$  by  $\sim_X$ .

(iii) The collection of all possible equivalence classes of  $\Gamma_X^{Hs}$  by  $\sim_X$ , denoted  $\Gamma_X^{Hs} / \sim_X$

$$\Gamma_X^{Hs} / \sim_X \triangleq \{[\Psi(X)]_{Hs} | \Psi(X) \in \Gamma_X^{Hs}\}, \quad (2.1.2)$$

is the quotient set of  $\Gamma_X^{Hs}$  by  $\sim_X$ .

(iv) For any  $\Psi(X) \in \Gamma_X^{Hs}$  let  $[\Psi(X)]_{Hs} \triangleq \{\Phi(X) \in \Gamma_X^{Hs} | \Psi(X) \sim \Phi(X)\}$  denote the equivalence class to which  $\Psi(X)$  belongs. All elements of  $\Gamma_X^{Hs}$  equivalent to each other are also elements of the same equivalence class.

**Definition 2.1.2.**[9]. Let  $Th$  be any theory in the recursive language  $\mathcal{L}_{Th} \supset \mathcal{L}_{PA}$ , where  $\mathcal{L}_{PA}$  is a language of Peano arithmetic. We say that a number-theoretic relation  $R(x_1, \dots, x_n)$  of

$n$  arguments is expressible in  $Th$  if and only if there is a wff  $\widehat{R}(x_1, \dots, x_n)$  of  $Th$  with the free variables  $x_1, \dots, x_n$  such that, for any natural numbers  $k_1, \dots, k_n$ , the following hold:

(i) If  $R(k_1, \dots, k_n)$  is true, then  $\vdash_{Th} \widehat{R}(\bar{k}_1, \dots, \bar{k}_n)$ .

(ii) If  $R(k_1, \dots, k_n)$  is false, then  $\vdash_{Th} \neg \widehat{R}(\bar{k}_1, \dots, \bar{k}_n)$ .

**Designation 2.1.2.**(i) Let  $g_{ZFC_2^{Hs}}(u)$  be a Gödel number of given an expression  $u$  of the set theory  $\overline{ZFC}_2^{Hs} \triangleq ZFC_2^{Hs} + \exists M_{st}^{ZFC_2^{Hs}}$ .

(ii) Let  $\mathbf{Fr}_2^{Hs}(y, v)$  be the relation :  $y$  is the Gödel number of a wff of the set theory  $\overline{ZFC}_2^{Hs}$  that contains free occurrences of the variable  $X$  with Gödel number  $v$  [8]-[9].

(iii) Note that the relation  $\mathbf{Fr}_2^{Hs}(y, v)$  is expressible in  $\overline{ZFC}_2^{Hs}$  by a wff  $\widehat{\mathbf{Fr}}_2^{Hs}(y, v)$

(iv) Note that for any  $y, v \in \mathbb{N}$  by definition of the relation  $\mathbf{Fr}_2^{Hs}(y, v)$  follows that

$$\widehat{\mathbf{Fr}}_2^{Hs}(y, v) \Leftrightarrow \exists ! \Psi(X) \left[ \left( g_{\overline{ZFC}_2^{Hs}}(\Psi(X)) = y \right) \wedge \left( g_{\overline{ZFC}_2^{Hs}}(X) = v \right) \right], \quad (2.1.3)$$

where  $\Psi(X)$  is a unique wff of  $\overline{ZFC}_2^{Hs}$  which contains free occurrences of the variable  $X$  with Gödel number  $v$ . We denote a unique wff  $\Psi(X)$  defined by using equivalence

(1.2.3)

by symbol  $\Psi_{y,v}(X)$ , i.e.

$$\widehat{\mathbf{Fr}}_2^{Hs}(y, v) \Leftrightarrow \exists ! \Psi_{y,v}(X) \left[ \left( g_{\overline{ZFC}_2^{Hs}}(\Psi_{y,v}(X)) = y \right) \wedge \left( g_{\overline{ZFC}_2^{Hs}}(X) = v \right) \right], \quad (2.1.4)$$

(v) Let  $\wp_2^{Hs}(y, v, v_1)$  be a Gödel number of the following wff:  $\exists ! X[\Psi(X) \wedge Y = X]$ , where  $g_{\overline{ZFC}_2^{Hs}}(\Psi(X)) = y, g_{\overline{ZFC}_2^{Hs}}(X) = v, g_{\overline{ZFC}_2^{Hs}}(Y) = v_1$ .

(vi) Let  $\text{Pr}_{ZFC_2^{Hs}}(z)$  be a predicate asserting provability in  $\overline{ZFC}_2^{Hs}$ , which defined by formula

(2.6) in section 2, see Remark 2.2 and Designation 2.3, (see also [9]-[10]).

**Remark 2.1.0.** Note that this function  $g_{\overline{ZFC}_2^{Hs}}(\Psi_{y,v}(X)) = y$  is expressible in set theory  $\overline{ZFC}_2^{Hs}$

by a wff of the set theory  $\overline{ZFC}_2^{Hs}$  that contains free occurrences of the variable  $y \in \mathbb{N}$ . Note that formula  $\Psi_{y,v}(X)$  is given by an expression  $u_0 u_1 \dots u_j \dots u_r$ , i.e.

$\Psi_{y,v}(X) =: u_0 u_1 \dots u_j \dots u_r$ , where each  $u_j$  is a symbol of  $\overline{ZFC}_2^{Hs}$ . We introduce now a functions  $[\Psi_{y,v}(X); j] : \Psi_{y,v}(X) \rightarrow u_j, j = 0, 1, \dots$ , i.e.  $[\Psi_{y,v}(X); j] =: u_j$  and rewrite expression  $u_0 u_1 \dots u_j \dots u_r$  in the following equivalent form

$$[\Psi_{y,v}(X); 0][\Psi_{y,v}(X); 1] \dots [\Psi_{y,v}(X); j] \dots [\Psi_{y,v}(X); r].$$

By definitions we obtain that

$$g_{ZFC_{st}}(\Psi_{y,v}(X)) = y \Leftrightarrow y = 2^{g([\Psi_{y,v}(X); 0])} 3^{g([\Psi_{y,v}(X); 1])} \dots p_j^{g([\Psi_{y,v}(X); j])} \dots p_r^{g([\Psi_{y,v}(X); r])}.$$

Let us denote by  $(y)_j$  the exponent  $g([\Psi_{y,v}(X); j])$  in this factorization

$$y = 2^{g([\Psi_{y,v}(X); 0])} 3^{g([\Psi_{y,v}(X); 1])} \dots p_j^{g([\Psi_{y,v}(X); j])} \dots p_r^{g([\Psi_{y,v}(X); r])}.$$

If  $y = 1, (y)_j = 1$  for all  $j$ . If  $x = 0$ , we arbitrarily let  $(y)_j = 0$  for all  $j$ . Then the functions  $(y)_j, j = 0, 1, \dots$  is primitive recursive, since  $(y)_j = \mu_{z < y} (p_j^z | y \wedge \neg p_j^{z+1} | y)$ , [8]. Thus the

function

$(y)_j$  is expressible in set theory  $\overline{ZFC}_2^{Hs}$  by formula denoted below by  $\lambda_j(y, g([\Psi_{y,v}(X); j]))$ .

For  $y > 0$ , let  $lh(y)$  be the number of non-zero exponents in the factorization of  $y$  into powers of primes, or, equivalently, the number of distinct primes that divide  $y$ . Let  $lh(0) = 0$ , then  $lh(y)$  is primitive recursive. Thus function  $g_{\overline{ZFC}_2^{Hs}}(\Psi_{y,v}(X)) = y$  is

expressible

in set theory  $\overline{ZFC}_2^{Hs}$  by formula  $\Xi(\Psi_{y,v}(X), y)$

$$\Xi(\Psi_{y,v}(X), y) \Leftrightarrow \bigwedge_{j \leq lh(y)} \lambda_j(y, g([\Psi_{y,v}(X); j])).$$

**Definition 2.1.3.** Let  $\Gamma_X^{Hs}$  be the countable collection of the all 1-place open wff's of the set theory  $\overline{ZFC}_2^{Hs}$  that contains free occurrences of the variable  $X$ .

**Definition 2.1.4.** Let  $g_{\overline{ZFC}_2^{Hs}}(X) = v$ . Let  $\Gamma_v^{Hs}$  be a set of the all Gödel numbers of the 1-place open wff's of the set theory  $\overline{ZFC}_2^{Hs}$  that contains free occurrences of the variable  $X$  with Gödel number  $v$ , i.e.

$$\Gamma_v^{Hs} = \{y \in \mathbb{N} \mid \langle y, v \rangle \in \mathbf{Fr}_2^{Hs}(y, v)\}, \quad (2.1.5)$$

or in the following equivalent form:

$$\forall y (y \in \mathbb{N}) \left[ y \in \Gamma_v^{Hs} \Leftrightarrow (y \in \mathbb{N}) \wedge \widehat{\mathbf{Fr}}_2^{Hs}(y, v) \right]. \quad (2.1.6)$$

**Remark 2.1.1.** Note that from the axiom of separation it follows directly that  $\Gamma_v^{Hs}$  is a set in the sense of the set theory  $\overline{ZFC}_2^{Hs}$ .

**Definition 2.1.5.**(i) We define now the equivalence relation

$$(\cdot \sim_v \cdot) \subset \Gamma_v^{Hs} \times \Gamma_v^{Hs} \quad (2.1.7)$$

in the sense of the set theory  $\overline{ZFC}_2^{Hs}$  by

$$y_1 \sim_v y_2 \Leftrightarrow (\forall X [\Psi_{y_1, v}(X) \Leftrightarrow \Psi_{y_2, v}(X)]) \quad (2.1.8)$$

Note that from the axiom of separation it follows directly that the equivalence relation  $(\cdot \sim_v \cdot)$  is a relation in the sense of the set theory  $\overline{ZFC}_2^{Hs}$ .

(ii) A subset  $\Lambda_v^{Hs}$  of  $\Gamma_v^{Hs}$  such that  $y_1 \sim_v y_2$  holds for all  $y_1$  and  $y_2$  in  $\Lambda_v^{Hs}$ , and never for  $y_1$  in

$\Lambda_v^{Hs}$  and  $y_2$  outside  $\Lambda_v^{Hs}$ , is an equivalence class of  $\Gamma_v^{Hs}$ .

(iii) For any  $y \in \Gamma_v^{Hs}$  let  $[y]_{Hs} \triangleq \{z \in \Gamma_v^{Hs} \mid y \sim_v z\}$  denote the equivalence class to which  $y$  belongs. All elements of  $\Gamma_v^{Hs}$  equivalent to each other are also elements of the same equivalence class.

(iv) The collection of all possible equivalence classes of  $\Gamma_v^{Hs}$  by  $\sim_v$ , denoted  $\Gamma_v^{Hs} / \sim_v$

$$\Gamma_v^{Hs} / \sim_v \triangleq \{[y]_{Hs} \mid y \in \Gamma_v^{Hs}\}. \quad (2.1.9)$$

**Remark 2.1.2.** Note that from the axiom of separation it follows directly that  $\Gamma_v^{Hs} / \sim_v$  is a

set in the sense of the set theory  $\overline{ZFC}_2^{Hs}$ .

**Definition 2.1.6.** Let  $\mathfrak{S}_2^{Hs}$  be the countable collection of the all sets definable by 1-place open wff of the set theory  $\overline{ZFC}_2^{Hs}$ , i.e.

$$\forall Y \{Y \in \mathfrak{S}_2^{Hs} \Leftrightarrow \exists \Psi(X) [([\Psi(X)]_{Hs} \in \Gamma_X^{Hs} / \sim_X) \wedge [\exists! X [\Psi(X) \wedge Y = X]]]\}. \quad (2.1.10)$$

**Definition 2.1.7.** We rewrite now (2.1.10) in the following equivalent form

$$\forall Y \{ Y \in \mathfrak{S}_2^{Hs} \Leftrightarrow \exists \Psi(X) [([\Psi(X)]_{Hs} \in \Gamma_X^{*Hs} / \sim_X) \wedge (Y = X)] \}, \quad (2.1.11)$$

where the countable collection  $\Gamma_X^{*Hs} / \sim_X$  is defined by

$$\forall \Psi(X) \{ [\Psi(X)] \in \Gamma_X^{*Hs} / \sim_X \Leftrightarrow [([\Psi(X)] \in \Gamma_X^{Hs} / \sim_X) \wedge \exists ! X \Psi(X)] \} \quad (2.1.12)$$

**Definition 2.1.8.** Let  $\mathfrak{R}_2^{Hs}$  be the countable collection of the all sets such that

$$\forall X (X \in \mathfrak{S}_2^{Hs}) [X \in \mathfrak{R}_2^{Hs} \Leftrightarrow X \notin X]. \quad (2.1.13)$$

**Remark 2.1.3.** Note that  $\mathfrak{R}_2^{Hs} \in \mathfrak{S}_2^{Hs}$  since  $\mathfrak{R}_2^{Hs}$  is a collection definable by 1-place open wff

$$\Psi(Z, \mathfrak{S}_2^{Hs}) \triangleq \forall X (X \in \mathfrak{S}_2^{Hs}) [X \in Z \Leftrightarrow X \notin X].$$

From (2.1.13) one obtains

$$\mathfrak{R}_2^{Hs} \in \mathfrak{R}_2^{Hs} \Leftrightarrow \mathfrak{R}_2^{Hs} \notin \mathfrak{R}_2^{Hs}. \quad (2.1.14)$$

But (2.1.14) gives a contradiction

$$(\mathfrak{R}_2^{Hs} \in \mathfrak{R}_2^{Hs}) \wedge (\mathfrak{R}_2^{Hs} \notin \mathfrak{R}_2^{Hs}). \quad (2.1.15)$$

However contradiction (2.1.15) it is not a contradiction inside  $\overline{ZFC}_2^{Hs}$  for the reason that the countable collection  $\mathfrak{S}_2^{Hs}$  is not a set in the sense of the set theory  $\overline{ZFC}_2^{Hs}$ .

**In order to obtain a contradiction inside  $\overline{ZFC}_2^{Hs}$  we introduce the following definitions.**

**Definition 2.1.9.** We define now the countable set  $\Gamma_v^{*Hs} / \sim_v$  by

$$\forall y \left\{ [y]_{Hs} \in \Gamma_v^{*Hs} / \sim_v \Leftrightarrow ([y]_{Hs} \in \Gamma_v^{Hs} / \sim_v) \wedge \widehat{\mathbf{F}\mathbf{r}}_2^{Hs}(y, v) \wedge [\exists ! X \Psi_{y,v}(X)] \right\}. \quad (2.1.16)$$

**Remark 2.1.4.** Note that from the axiom of separation it follows directly that  $\Gamma_v^{*Hs} / \sim_v$  is a set in the sense of the set theory  $\overline{ZFC}_2^{Hs}$ .

**Definition 2.1.10.** We define now the countable set  $\mathfrak{S}_2^{*Hs}$  by formula

$$\forall Y \left\{ Y \in \mathfrak{S}_2^{*Hs} \Leftrightarrow \exists y [([y] \in \Gamma_v^{*Hs} / \sim_v) \wedge (g_{\overline{ZFC}_2^{Hs}}(X) = v) \wedge Y = X] \right\}. \quad (2.1.17)$$

Note that from the axiom schema of replacement (1.1.1) it follows directly that  $\mathfrak{S}_2^{*Hs}$  is a

set in the sense of the set theory  $\overline{ZFC}_2^{Hs}$ .

**Definition 2.1.12.** We define now the countable set  $\mathfrak{R}_2^{*Hs}$  by formula

$$\forall X (X \in \mathfrak{S}_2^{*Hs}) [X \in \mathfrak{R}_2^{*Hs} \Leftrightarrow X \notin X]. \quad (2.1.18)$$

Note that from the axiom schema of separation it follows directly that  $\mathfrak{R}_2^{*Hs}$  is a set in the

sense of the set theory  $\overline{ZFC}_2^{Hs}$ .

**Remark 2.1.5.** Note that  $\mathfrak{R}_2^{*Hs} \in \mathfrak{S}_2^{*Hs}$  since  $\mathfrak{R}_2^{*Hs}$  is a definable by the following formula

$$\Psi^*(Z) \triangleq \forall X (X \in \mathfrak{S}_2^{*Hs}) [X \in Z \Leftrightarrow X \notin X]. \quad (2.1.19)$$

**Theorem 2.1.1.** Set theory  $\overline{ZFC}_2^{Hs}$  is inconsistent.

Proof. From (2.1.18) and Remark 2.1.5 we obtain  $\mathfrak{R}_2^{*Hs} \in \mathfrak{R}_2^{*Hs} \Leftrightarrow \mathfrak{R}_2^{*Hs} \notin \mathfrak{R}_2^{*Hs}$  from which immediately one obtains a contradiction  $(\mathfrak{R}_2^{*Hs} \in \mathfrak{R}_2^{*Hs}) \wedge (\mathfrak{R}_2^{*Hs} \notin \mathfrak{R}_2^{*Hs})$ .

## 2.2. Derivation of the inconsistent definable set in set



## theory $ZFC_{st}$ .

**Designation 2.2.1.**(i) Let  $g_{ZFC_{st}}(u)$  be a Gödel number of given an expression  $u$  of the set theory  $ZFC_{st} \triangleq ZFC + \exists M_{st}^{ZFC}$ .

(ii) Let  $\mathbf{Fr}_{st}(y, v)$  be the relation :  $y$  is the Gödel number of a wff of the set theory  $ZFC_{st}$  that contains free occurrences of the variable  $X$  with Gödel number  $v$  [9].

(iii) Note that the relation  $\mathbf{Fr}_{st}(y, v)$  is expressible in  $ZFC_{st}$  by a wff  $\widehat{\mathbf{Fr}}_{st}(y, v)$

(iv) Note that for any  $y, v \in \mathbb{N}$  by definition of the relation  $\mathbf{Fr}_{st}(y, v)$  follows that

$$\widehat{\mathbf{Fr}}_{st}(y, v) \Leftrightarrow \exists ! \Psi(X) [(g_{ZFC_{st}}(\Psi(X)) = y) \wedge (g_{ZFC_{st}}(X) = v)], \quad (2.2.1)$$

where  $\Psi(X)$  is a unique wff of  $ZFC_{st}$  which contains free occurrences of the variable  $X$  with Gödel number  $v$ . We denote a unique wff  $\Psi(X)$  defined by using equivalence (2.2.1)

by symbol  $\Psi_{y,v}(X)$ , i.e.

$$\widehat{\mathbf{Fr}}_{st}(y, v) \Leftrightarrow \exists ! \Psi_{y,v}(X) [(g_{ZFC_{st}}(\Psi_{y,v}(X)) = y) \wedge (g_{ZFC_{st}}(X) = v)], \quad (2.2.2)$$

(v) Let  $\wp_{st}(y, v, v_1)$  be a Gödel number of the following wff:  $\exists ! X [\Psi(X) \wedge Y = X]$ , where  $g_{ZFC_{st}}(\Psi(X)) = y, g_{ZFC_{st}}(X) = v, g_{ZFC_{st}}(Y) = v_1$ .

(vi) Let  $\text{Pr}_{ZFC_{st}}(z)$  be a predicate asserting provability in  $ZFC_{st}$ , which defined by formula

(2.6) in section 2, see Remark 2.2 and Designation 2.3,(see also [8]-[9]).

**Remark 2.2.0.**Note that this function  $g_{ZFC_{st}}(\Psi_{y,v}(X)) = y$  is expressible in set theory  $ZFC_{st}$

by a wff of the set theory  $ZFC_{st}$  that contains free occurrences of the variable  $y \in \mathbb{N}$ .

Note that formula  $\Psi_{y,v}(X)$  is given by an expression  $u_0 u_1 \dots u_j \dots u_r$ , i.e.

$\Psi_{y,v}(X) =: u_0 u_1 \dots u_j \dots u_r$ , where each  $u_j$  is a symbol of  $ZFC_{st}$ . We introduce now a functions  $[\Psi_{y,v}(X); j] : \Psi_{y,v}(X) \rightarrow u_j, j = 0, 1, \dots$ , i.e.  $[\Psi_{y,v}(X); j] =: u_j$  and rewrite expression  $u_0 u_1 \dots u_j \dots u_r$  in the following equivalent form

$$[\Psi_{y,v}(X); 0][\Psi_{y,v}(X); 1] \dots [\Psi_{y,v}(X); j] \dots [\Psi_{y,v}(X); r].$$

By definitions we obtain that

$$g_{ZFC_{st}}(\Psi_{y,v}(X)) = y \Leftrightarrow y = 2^{g([\Psi_{y,v}(X); 0])} 3^{g([\Psi_{y,v}(X); 1])} \dots p_j^{g([\Psi_{y,v}(X); j])} \dots p_r^{g([\Psi_{y,v}(X); r])}.$$

Let us denote by  $(y)_j$  the exponent  $g([\Psi_{y,v}(X); j])$  in this factorization

$$y = 2^{g([\Psi_{y,v}(X); 0])} 3^{g([\Psi_{y,v}(X); 1])} \dots p_j^{g([\Psi_{y,v}(X); j])} \dots p_r^{g([\Psi_{y,v}(X); r])}.$$

If  $y = 1, (y)_j = 1$  for all  $j$ . If  $x = 0$ , we arbitrarily let  $(y)_j = 0$  for all  $j$ . Then the functions  $(y)_j, j = 0, 1, \dots$  is primitive recursive, since  $(y)_j = \mu_{z < y} (p_j^z | y \wedge \neg p_j^{z+1} | y)$ , [8]. Thus the function

$(y)_j$  is expressible in set theory  $ZFC_{st}$  by formula denoted below by  $\lambda_j(y, g([\Psi_{y,v}(X); j]))$ .

For  $y > 0$ , let  $lh(y)$  be the number of non-zero exponents in the factorization of  $y$  into powers of primes, or, equivalently, the number of distinct primes that divide  $y$ . Let  $lh(0) = 0$ , then  $lh(y)$  is primitive recursive. Thus function  $g_{ZFC_{st}}(\Psi_{y,v}(X)) = y$  is expressible

in set theory  $ZFC_{st}$  by formula  $\Xi(\Psi_{y,v}(X), y)$

$$\Xi(\Psi_{y,v}(X), y) \Leftrightarrow \bigwedge_{j \leq lh(y)} \lambda_j(y, g([\Psi_{y,v}(X); j])).$$

**Definition 2.2.1.** Let  $\Gamma_X^{st}$  be the countable collection of the all 1-place open wff's of the set theory  $ZFC_{st}$  that contains free occurrences of the variable  $X$ .

**Definition 2.2.2.** Let  $g_{ZFC_{st}}(X) = v$ . Let  $\Gamma_v^{st}$  be a set of the all Gödel numbers of the 1-place open wff's of the set theory  $ZFC_{st}$  that contains free occurrences of the variable  $X$  with Gödel number  $v$ , i.e.

$$\Gamma_v^{st} = \{y \in \mathbb{N} \mid \langle y, v \rangle \in \mathbf{Fr}_{st}(y, v)\}, \quad (2.2.3)$$

or in the following equivalent form:

$$\forall y (y \in \mathbb{N}) \left[ y \in \Gamma_v^{st} \Leftrightarrow (y \in \mathbb{N}) \wedge \widehat{\mathbf{Fr}}_{st}(y, v) \right].$$

**Remark 2.2.1.** Note that from the axiom of separation it follows directly that  $\Gamma_v^{st}$  is a set in the sense of the set theory  $ZFC_{st}$ .

**Definition 2.2.3.**(i) We define now the equivalence relation  $(\cdot \sim_X \cdot) \subset \Gamma_X^{st} \times \Gamma_X^{st}$  by

$$\Psi_1(X) \sim_X \Psi_2(X) \Leftrightarrow (\forall X [\Psi_1(X) \Leftrightarrow \Psi_2(X)]) \quad (2.2.4)$$

(ii) A subcollection  $\Lambda_X^{st}$  of  $\Gamma_X^{st}$  such that  $\Psi_1(X) \sim_X \Psi_2(X)$  holds for all  $\Psi_1(X)$  and  $\Psi_2(X)$  in  $\Lambda_X^{st}$ , and never for  $\Psi_1(X)$  in  $\Lambda_X^{st}$  and  $\Psi_2(X)$  outside  $\Lambda_X^{st}$ , is an equivalence class of  $\Gamma_X^{st}$ .

(iii) For any  $\Psi(X) \in \Gamma_X^{st}$  let  $[\Psi(X)]_{st} \triangleq \{\Phi(X) \in \Gamma_X^{st} \mid \Psi(X) \sim_X \Phi(X)\}$  denote the equivalence class to which  $\Psi(X)$  belongs. All elements of  $\Gamma_X^{st}$  equivalent to each other are also elements of the same equivalence class.

(iv) The collection of all possible equivalence classes of  $\Gamma_X^{st}$  by  $\sim_X$ , denoted  $\Gamma_X^{st} / \sim_X$

$$\Gamma_X^{st} / \sim_X \triangleq \{[\Psi(X)]_{st} \mid \Psi(X) \in \Gamma_X^{st}\}. \quad (2.2.5)$$

**Definition 2.2.4.**(i) We define now the equivalence relation  $(\cdot \sim_v \cdot) \subset \Gamma_v^{st} \times \Gamma_v^{st}$  in the sense of the set theory  $ZFC_{st}$  by

$$y_1 \sim_v y_2 \Leftrightarrow (\forall X [\Psi_{y_1,v}(X) \Leftrightarrow \Psi_{y_2,v}(X)]) \quad (2.2.6)$$

Note that from the axiom of separation it follows directly that the equivalence relation  $(\cdot \sim_v \cdot)$  is a relation in the sense of the set theory  $ZFC_{st}$ .

(ii) A subset  $\Lambda_v^{st}$  of  $\Gamma_v^{st}$  such that  $y_1 \sim_v y_2$  holds for all  $y_1$  and  $y_2$  in  $\Lambda_v^{st}$ , and never for  $y_1$  in  $\Lambda_v^{st}$  and  $y_2$  outside  $\Lambda_v^{st}$ , is an equivalence class of  $\Gamma_v^{st}$ .

(iii) For any  $y \in \Gamma_v^{st}$  let  $[y]_{st} \triangleq \{z \in \Gamma_v^{st} \mid y \sim_v z\}$  denote the equivalence class to which  $y$  belongs. All elements of  $\Gamma_v^{st}$  equivalent to each other are also elements of the same equivalence class.

(iv) The collection of all possible equivalence classes of  $\Gamma_v^{st}$  by  $\sim_v$ , denoted  $\Gamma_v^{st} / \sim_v$

$$\Gamma_v^{st} / \sim_v \triangleq \{[y]_{st} \mid y \in \Gamma_v^{st}\}. \quad (2.2.7)$$

**Remark 2.2.2.** Note that from the axiom of separation it follows directly that  $\Gamma_v^{st} / \sim_v$  is a

set in the sense of the set theory  $ZFC_{st}$ .

**Definition 2.2.5.** Let  $\mathfrak{S}_{st}$  be the countable collection of the all sets definable by 1-place open wff of the set theory  $ZFC_{st}$ , i.e.

$$\forall Y \{ Y \in \mathfrak{S}_{st} \Leftrightarrow \exists \Psi(X) [([\Psi(X)]_{st} \in \Gamma_X^{st} / \sim_X) \wedge [\exists! X [\Psi(X) \wedge Y = X]]] \}. \quad (2.2.8)$$

**Definition 2.2.6.** We rewrite now (2.2.8) in the following equivalent form

$$\forall Y \{ Y \in \mathfrak{S}_{st} \Leftrightarrow \exists \Psi(X) [([\Psi(X)]_{st} \in \Gamma_X^{*st} / \sim_X) \wedge (Y = X)] \}, \quad (2.2.9)$$

where the countable collection  $\Gamma_X^{*st} / \sim_X$  is defined by

$$\forall \Psi(X) \{ [\Psi(X)]_{st} \in \Gamma_X^{*st} / \sim_X \Leftrightarrow [([\Psi(X)]_{st} \in \Gamma_X^{st} / \sim_X) \wedge \exists! X \Psi(X)] \} \quad (2.2.10)$$

**Definition 2.2.7.** Let  $\mathfrak{R}_{st}$  be the countable collection of the all sets such that

$$\forall X (X \in \mathfrak{S}_{st}) [X \in \mathfrak{R}_{st} \Leftrightarrow X \notin X]. \quad (2.2.11)$$

**Remark 2.2.3.** Note that  $\mathfrak{R}_{st} \in \mathfrak{S}_{st}$  since  $\mathfrak{R}_{st}$  is a collection definable by 1-place open wff

$$\Psi(Z, \mathfrak{S}_{st}) \triangleq \forall X (X \in \mathfrak{S}_{st}) [X \in Z \Leftrightarrow X \notin X].$$

From (2.2.11) and Remark 2.2.3 one obtains directly

$$\mathfrak{R}_{st} \in \mathfrak{R}_{st} \Leftrightarrow \mathfrak{R}_{st} \notin \mathfrak{R}_{st}. \quad (2.2.12)$$

But (2.2.12) immediately gives a contradiction

$$(\mathfrak{R}_{st} \in \mathfrak{R}_{st}) \wedge (\mathfrak{R}_{st} \notin \mathfrak{R}_{st}). \quad (2.2.13)$$

However contradiction (2.2.13) it is not a true contradiction inside  $ZFC_{st}$  for the reason that the countable collection  $\mathfrak{S}_{st}$  is not a set in the sense of the set theory  $ZFC_{st}$ .

**In order to obtain a true contradiction inside  $ZFC_{st}$  we introduce the following definitions.**

**Definition 2.2.8.** We define now the countable set  $\Gamma_v^{*st} / \sim_v$  by formula

$$\forall y \left\{ [y]_{st} \in \Gamma_v^{*st} / \sim_v \Leftrightarrow ([y]_{st} \in \Gamma_v^{st} / \sim_v) \wedge \widehat{\mathbf{Fr}}_{st}(y, v) \wedge [\exists! X \Psi_{y,v}(X)] \right\}. \quad (2.2.14)$$

**Remark 2.2.4.** Note that from the axiom of separation it follows directly that  $\Gamma_v^{*st} / \sim_v$  is a set in the sense of the set theory  $ZFC_{st}$ .

**Definition 2.2.9.** We define now the countable set  $\mathfrak{S}_{st}^*$  by formula

$$\forall Y \{ Y \in \mathfrak{S}_{st}^* \Leftrightarrow \exists y [([y]_{st} \in \Gamma_v^{*st} / \sim_v) \wedge (g_{ZFC_{st}}(X) = v) \wedge Y = X] \}. \quad (2.2.15)$$

Note that from the axiom schema of replacement it follows directly that  $\mathfrak{S}_{st}^*$  is a set in the

sense of the set theory  $ZFC_{st}$ .

**Definition 2.2.10.** We define now the countable set  $\mathfrak{R}_{st}^*$  by formula

$$\forall X (X \in \mathfrak{S}_{st}^*) [X \in \mathfrak{R}_{st}^* \Leftrightarrow X \notin X]. \quad (2.2.16)$$

Note that from the axiom schema of separation it follows directly that  $\mathfrak{R}_{st}^*$  is a set in the sense of the set theory  $ZFC_{st}$ .

**Remark 2.2.5.** Note that  $\mathfrak{R}_{st}^* \in \mathfrak{S}_{st}^*$  since  $\mathfrak{R}_{st}^*$  is a definable by the following formula

$$\Psi^*(Z) \triangleq \forall X (X \in \mathfrak{S}_{st}^*) [X \in Z \Leftrightarrow X \notin X]. \quad (2.2.17)$$

**Theorem 2.2.1.** Set theory  $ZFC_{st}$  is inconsistent.

Proof. From (2.2.17) and Remark 2.2.5 we obtain  $\mathfrak{R}_{st}^* \in \mathfrak{R}_{st}^* \Leftrightarrow \mathfrak{R}_{st}^* \notin \mathfrak{R}_{st}^*$  from which immediately one obtains a contradiction  $(\mathfrak{R}_{st}^* \in \mathfrak{R}_{st}^*) \wedge (\mathfrak{R}_{st}^* \notin \mathfrak{R}_{st}^*)$ .

## 2.3. Derivation of the inconsistent definable set in $ZFC_{Nst}$ .

**Definition 2.3.1.** Let  $\overline{PA}$  be a first order theory which contain usual postulates of Peano

arithmetic [9] and recursive defining equations for every primitive recursive function as desired. So for any  $(n + 1)$ -place function  $f$  defined by primitive recursion over any  $n$ -place

base function  $g$  and  $(n + 2)$ -place iteration function  $h$  there would be the defining equations:

$$(i) f(0, y_1, \dots, y_n) = g(y_1, \dots, y_n), (ii) f(x + 1, y_1, \dots, y_n) = h(x, f(x, y_1, \dots, y_n), y_1, \dots, y_n).$$

**Designation 2.3.1.**(i) Let  $M_{Nst}^{ZFC}$  be a nonstandard model of  $ZFC$  and let  $M_{st}^{\overline{PA}}$  be a standard

model of  $\overline{PA}$ . We assume now that  $M_{st}^{\overline{PA}} \subset M_{Nst}^{ZFC}$  and denote such nonstandard model of the set theory  $ZFC$  by  $M_{Nst}^{ZFC}[\overline{PA}]$ . (ii) Let  $ZFC_{Nst}$  be the theory

$$ZFC_{Nst} = ZFC + M_{Nst}^{ZFC}[\overline{PA}].$$

**Designation 2.3.2.**(i) Let  $g_{ZFC_{Nst}}(u)$  be a Gödel number of given an expression  $u$  of the set theory  $ZFC_{Nst} \triangleq ZFC + \exists M_{Nst}^{ZFC}[\overline{PA}]$ .

(ii) Let  $\mathbf{Fr}_{Nst}(y, v)$  be the relation :  $y$  is the Gödel number of a wff of the set theory  $ZFC_{Nst}$

that contains free occurrences of the variable  $X$  with Gödel number  $v$  [9].

(iii) Note that the relation  $\mathbf{Fr}_{Nst}(y, v)$  is expressible in  $ZFC_{Nst}$  by a wff  $\widehat{\mathbf{Fr}}_{Nst}(y, v)$

(iv) Note that for any  $y, v \in \mathbb{N}$  by definition of the relation  $\mathbf{Fr}_{Nst}(y, v)$  follows that

$$\widehat{\mathbf{Fr}}_{Nst}(y, v) \Leftrightarrow \exists ! \Psi(X) [(g_{ZFC_{Nst}}(\Psi(X)) = y) \wedge (g_{ZFC_{Nst}}(X) = v)], \quad (2.3.1)$$

where  $\Psi(X)$  is a unique wff of  $ZFC_{st}$  which contains free occurrences of the variable  $X$  with Gödel number  $v$ . We denote a unique wff  $\Psi(X)$  defined by using equivalence (2.3.1)

by symbol  $\Psi_{y,v}(X)$ , i.e.

$$\widehat{\mathbf{Fr}}_{Nst}(y, v) \Leftrightarrow \exists ! \Psi_{y,v}(X) [(g_{ZFC_{Nst}}(\Psi_{y,v}(X)) = y) \wedge (g_{ZFC_{Nst}}(X) = v)], \quad (2.3.2)$$

(v) Let  $g_{Nst}(y, v, v_1)$  be a Gödel number of the following wff:  $\exists ! X[\Psi(X) \wedge Y = X]$ , where  $g_{ZFC_{Nst}}(\Psi(X)) = y, g_{ZFC_{Nst}}(X) = v, g_{ZFC_{Nst}}(Y) = v_1$ .

(vi) Let  $\text{Pr}_{ZFC_{Nst}}(z)$  be a predicate asserting provability in  $ZFC_{Nst}$ , which defined by formula

(2.6) in section 2, see Remark 2.2 and Designation 2.3,(see also [9]-[10]).

**Definition 2.3.2.** Let  $\Gamma_X^{Nst}$  be the countable collection of the all 1-place open wff's of the set theory  $ZFC_{Nst}$  that contains free occurrences of the variable  $X$ .

**Definition 2.3.3.** Let  $g_{ZFC_{Nst}}(X) = v$ . Let  $\Gamma_v^{Nst}$  be a set of the all Gödel numbers of the 1-place open wff's of the set theory  $ZFC_{Nst}$  that contains free occurrences of the variable  $X$

with Gödel number  $v$ , i.e.

$$\Gamma_v^{Nst} = \{y \in \mathbb{N} \mid \langle y, v \rangle \in \mathbf{Fr}_{Nst}(y, v)\}, \quad (2.3.3)$$

or in the following equivalent form:

$$\forall y(y \in \mathbb{N}) \left[ y \in \Gamma_v^{Nst} \Leftrightarrow (y \in \mathbb{N}) \wedge \widehat{\mathbf{Fr}}_{Nst}(y, v) \right].$$

**Remark 2.3.1.** Note that from the axiom of separation it follows directly that  $\Gamma_v^{st}$  is a set in the sense of the set theory  $ZFC_{Nst}$ .

**Definition 2.3.3.**(i) We define now the equivalence relation  $(\cdot \sim_X \cdot) \subset \Gamma_X^{Nst} \times \Gamma_X^{Nst}$  by

$$\Psi_1(X) \sim_X \Psi_2(X) \Leftrightarrow (\forall X[\Psi_1(X) \Leftrightarrow \Psi_2(X)]) \quad (2.3.4)$$

(ii) A subcollection  $\Lambda_X^{st}$  of  $\Gamma_X^{st}$  such that  $\Psi_1(X) \sim_X \Psi_2(X)$  holds for all  $\Psi_1(X)$  and  $\Psi_2(X)$  in

$\Lambda_X^{st}$ , and never for  $\Psi_1(X)$  in  $\Lambda_X^{st}$  and  $\Psi_2(X)$  outside  $\Lambda_X^{st}$ , is an equivalence class of  $\Gamma_X^{st}$ .

(iii) For any  $\Psi(X) \in \Gamma_X^{st}$  let  $[\Psi(X)]_{Nst} \triangleq \{\Phi(X) \in \Gamma_X^{st} | \Psi(X) \sim_X \Phi(X)\}$  denote the equivalence class to which  $\Psi(X)$  belongs. All elements of  $\Gamma_X^{st}$  equivalent to each other

are also elements of the same equivalence class.

(iv) The collection of all possible equivalence classes of  $\Gamma_X^{st}$  by  $\sim_X$ , denoted  $\Gamma_X^{st} / \sim_X$

$$\Gamma_X^{st} / \sim_X \triangleq \{[\Psi(X)]_{Nst} | \Psi(X) \in \Gamma_X^{st}\}. \quad (2.3.5)$$

**Definition 2.3.4.**(i) We define now the equivalence relation  $(\cdot \sim_v \cdot) \subset \Gamma_v^{Nst} \times \Gamma_v^{Nst}$  in the sense of the set theory  $ZFC_{Nst}$  by

$$y_1 \sim_v y_2 \Leftrightarrow (\forall X [\Psi_{y_1,v}(X) \Leftrightarrow \Psi_{y_2,v}(X)]) \quad (2.3.6)$$

Note that from the axiom of separation it follows directly that the equivalence relation  $(\cdot \sim_v \cdot)$  is a relation in the sense of the set theory  $ZFC_{Nst}$ .

(ii) A subset  $\Lambda_v^{Nst}$  of  $\Gamma_v^{Nst}$  such that  $y_1 \sim_v y_2$  holds for all  $y_1$  and  $y_2$  in  $\Lambda_v^{Nst}$ , and never for  $y_1$  in

$\Lambda_v^{Nst}$  and  $y_2$  outside  $\Lambda_v^{Nst}$ , is an equivalence class of  $\Gamma_v^{Nst}$ .

(iii) For any  $y \in \Gamma_v^{Nst}$  let  $[y]_{Nst} \triangleq \{z \in \Gamma_v^{Nst} | y \sim_v z\}$  denote the equivalence class to which  $y$

belongs. All elements of  $\Gamma_v^{Nst}$  equivalent to each other are also elements of the same equivalence class.

(iv) The collection of all possible equivalence classes of  $\Gamma_v^{Nst}$  by  $\sim_v$ , denoted  $\Gamma_v^{Nst} / \sim_v$

$$\Gamma_v^{Nst} / \sim_v \triangleq \{[y]_{Nst} | y \in \Gamma_v^{Nst}\}. \quad (2.3.7)$$

**Remark 2.3.2.** Note that from the axiom of separation it follows directly that  $\Gamma_v^{Nst} / \sim_v$  is a

set in the sense of the set theory  $ZFC_{Nst}$ .

**Definition 2.3.5.** Let  $\mathfrak{T}_{Nst}$  be the countable collection of the all sets definable by 1-place

open wff of the set theory  $ZFC_{Nst}$ , i.e.

$$\forall Y \{Y \in \mathfrak{T}_{Nst} \Leftrightarrow \exists \Psi(X) [([\Psi(X)]_{Nst} \in \Gamma_X^{st} / \sim_X) \wedge [\exists! X [\Psi(X) \wedge Y = X]]]\}. \quad (2.3.8)$$

**Definition 2.3.6.** We rewrite now (2.3.8) in the following equivalent form

$$\forall Y \{Y \in \mathfrak{T}_{Nst} \Leftrightarrow \exists \Psi(X) [([\Psi(X)]_{Nst} \in \Gamma_X^{*Nst} / \sim_X) \wedge (Y = X)]\}, \quad (2.3.9)$$

where the countable collection  $\Gamma_X^{*Nst} / \sim_X$  is defined by

$$\forall \Psi(X) \{[\Psi(X)]_{Nst} \in \Gamma_X^{*Nst} / \sim_X \Leftrightarrow [([\Psi(X)]_{Nst} \in \Gamma_X^{Nst} / \sim_X) \wedge \exists! X \Psi(X)]\} \quad (2.3.10)$$

**Definition 2.3.7.** Let  $\mathfrak{R}_{Nst}$  be the countable collection of the all sets such that

$$\forall X (X \in \mathfrak{R}_{Nst}) [X \in \mathfrak{R}_{Nst} \Leftrightarrow X \notin X]. \quad (2.3.11)$$

**Remark 2.3.3.** Note that  $\mathfrak{R}_{Nst} \in \mathfrak{T}_{Nst}$  since  $\mathfrak{R}_{Nst}$  is a collection definable by 1-place open wff

$$\Psi(Z, \mathfrak{T}_{Nst}) \triangleq \forall X (X \in \mathfrak{T}_{Nst}) [X \in Z \Leftrightarrow X \notin X].$$

From (2.3.11) one obtains

$$\mathfrak{R}_{Nst} \in \mathfrak{R}_{Nst} \Leftrightarrow \mathfrak{R}_{Nst} \notin \mathfrak{R}_{Nst}. \quad (2.3.12)$$

But (2.3.12) gives a contradiction

$$(\mathfrak{R}_{Nst} \in \mathfrak{R}_{Nst}) \wedge (\mathfrak{R}_{Nst} \notin \mathfrak{R}_{Nst}). \quad (2.3.13)$$

However a contradiction (2.3.13) it is not a true contradiction inside  $ZFC_{Nst}$  for the reason

that the countable collection  $\mathfrak{T}_{Nst}$  is not a set in the sense of the set theory  $ZFC_{Nst}$ .

**In order to obtain a true contradiction inside  $ZFC_{Nst}$  we introduce the following definitions.**

**Definition 2.3.8.** We define now the countable set  $\Gamma_v^{*Nst} / \sim_v$  by formula

$$\forall y \left\{ [y]_{Nst} \in \Gamma_v^{*Nst} / \sim_v \Leftrightarrow ([y]_{Nst} \in \Gamma_v^{Nst} / \sim_v) \wedge \widehat{\mathbf{Fr}}_{Nst}(y, v) \wedge [\exists! X \Psi_{y,v}(X)] \right\}. \quad (2.3.14)$$

**Remark 2.3.4.** Note that from the axiom of separation it follows directly that  $\Gamma_v^{*Nst} / \sim_v$  is

a set in the sense of the set theory  $ZFC_{st}$ .

**Definition 2.3.9.** We define now the countable set  $\mathfrak{T}_{Nst}^*$  by formula

$$\forall Y \{ Y \in \mathfrak{T}_{Nst}^* \Leftrightarrow \exists y ([y]_{Nst} \in \Gamma_v^{*Nst} / \sim_v) \wedge (g_{ZFC_{Nst}}(X) = v) \wedge Y = X \}. \quad (2.3.15)$$

Note that from the axiom schema of replacement it follows directly that  $\mathfrak{T}_{st}^*$  is a set in the

sense of the set theory  $ZFC_{Nst}$ .

**Definition 2.3.10.** We define now the countable set  $\mathfrak{R}_{Nst}^*$  by formula

$$\forall X (X \in \mathfrak{T}_{Nst}^*) [X \in \mathfrak{R}_{Nst}^* \Leftrightarrow X \notin X]. \quad (2.3.16)$$

Note that from the axiom schema of separation it follows directly that  $\mathfrak{R}_{Nst}^*$  is a set in the

sense of the set theory  $ZFC_{Nst}$ .

**Remark 2.3.5.** Note that  $\mathfrak{R}_{Nst}^* \in \mathfrak{T}_{Nst}^*$  since  $\mathfrak{R}_{Nst}^*$  is a definable by the following formula

$$\Psi^*(Z) \triangleq \forall X (X \in \mathfrak{T}_{Nst}^*) [X \in Z \Leftrightarrow X \notin X]. \quad (2.3.17)$$

**Theorem 2.3.1.** Set theory  $ZFC_{Nst}$  is inconsistent.

Proof. From (2.3.16) and Remark 2.3.5 we obtain  $\mathfrak{R}_{Nst}^* \in \mathfrak{R}_{Nst}^* \Leftrightarrow \mathfrak{R}_{Nst}^* \notin \mathfrak{R}_{Nst}^*$  from which one obtains a contradiction  $(\mathfrak{R}_{Nst}^* \in \mathfrak{R}_{Nst}^*) \wedge (\mathfrak{R}_{Nst}^* \notin \mathfrak{R}_{Nst}^*)$ .

### 3. Derivation of the inconsistent provably definable set in set theory $\overline{ZFC}_2^{Hs}$ , $ZFC_{st}$ and $ZFC_{Nst}$ .

#### 3.1. Derivation of the inconsistent provably definable set in set theory $\overline{ZFC}_2^{Hs}$ .

**Definition 3.1.1.** Let  $\widetilde{\mathfrak{T}}_2^{Hs}$  be the countable collection of all provable definable sets  $X$  such

that  $\overline{ZFC}_2^{Hs} \vdash \exists! X \Psi(X)$ , where  $\Psi(X)$  is a 1-place open wff i.e.,

$$\forall Y \left\{ Y \in \widetilde{\mathfrak{T}}_2^{Hs} \Leftrightarrow \overline{ZFC}_2^{Hs} \vdash \exists \Psi(X) [([\Psi(X)] \in \Gamma_X^{Hs} / \sim_X) \wedge [\exists! X [\Psi(X) \wedge Y = X]]] \right\}. \quad (3.1.1)$$

Let  $X \notin \vdash_{\overline{ZFC}_2^{Hs}} Y$  be a predicate such that  $X \notin \vdash_{\overline{ZFC}_2^{Hs}} Y \leftrightarrow \overline{ZFC}_2^{Hs} \vdash X \notin Y$ . Let  $\widetilde{\mathfrak{R}}_2^{Hs}$  be the countable collection of all sets such that

$$\forall X \left( X \in \widetilde{\mathfrak{S}}_2^{Hs} \right) \left[ X \in \widetilde{\mathfrak{R}}_2^{Hs} \leftrightarrow X \notin \vdash_{\overline{ZFC}_2^{Hs}} X \right]. \quad (3.1.2)$$

From (3.1.2) one obtains

$$\widetilde{\mathfrak{R}}_2^{Hs} \in \widetilde{\mathfrak{R}}_2^{Hs} \leftrightarrow \widetilde{\mathfrak{R}}_2^{Hs} \notin \vdash_{\overline{ZFC}_2^{Hs}} \widetilde{\mathfrak{R}}_2^{Hs}. \quad (3.1.3)$$

But obviously this is a contradiction. However contradiction (3.1.3) it is not a contradiction inside  $\overline{ZFC}_2^{Hs}$  for the reason that predicate  $X \notin \vdash_{\overline{ZFC}_2^{Hs}} Y$  is not a predicate of  $\overline{ZFC}_2^{Hs}$  and therefore countable collections  $\widetilde{\mathfrak{S}}_2^{Hs}$  and  $\widetilde{\mathfrak{R}}_2^{Hs}$  are not a sets of  $\overline{ZFC}_2^{Hs}$ . Nevertheless by using Gödel encoding the above stated contradiction can be shipped in special consistent extensions of  $\overline{ZFC}_2^{Hs}$ .

**Remark 3.1.1.** More formally I can to explain the gist of the contradictions derived in this

paper (see Proposition 2.5.(i)-(ii)) as follows.

Let  $M$  be Henkin model of  $ZFC_2^{Hs}$ . Let  $\widetilde{\mathfrak{R}}_2^{Hs}$  be the set of the all sets of  $M$  provably definable in  $\overline{ZFC}_2^{Hs}$ , and let  $\widetilde{\mathfrak{R}}_2^{Hs} = \left\{ x \in \widetilde{\mathfrak{S}}_2^{Hs} : \Box(x \notin x) \right\}$  where  $\Box A$  means ‘sentence  $A$  derivable in  $\overline{ZFC}_2^{Hs}$ ’, or some appropriate modification thereof. We replace now formula (3.1.1) by the following formula

$$\forall Y \left\{ Y \in \widetilde{\mathfrak{S}}_2^{Hs} \leftrightarrow \exists \Psi(X) \left[ ([\Psi(X)] \in \Gamma_X^{Hs} / \sim_X ) \wedge \Box \exists ! X [\Psi(X) \wedge Y = X] \right] \right\}. \quad (3.1.4)$$

and we replace formula (3.1.2) by the following formula

$$\forall X \left( X \in \widetilde{\mathfrak{S}}_2^{Hs} \right) \left[ X \in \widetilde{\mathfrak{R}}_2^{Hs} \Leftrightarrow \Box(X \notin X) \right]. \quad (3.1.5)$$

**Definition 3.1.2.** We rewrite now (3.1.4) in the following equivalent form

$$\forall Y \left\{ Y \in \widetilde{\mathfrak{S}}_2^{Hs} \Leftrightarrow \exists \Psi(X) \left[ ([\Psi(X)]_{Hs} \in \Gamma_X^{*Hs} / \sim_X ) \wedge (Y = X) \right] \right\}, \quad (3.1.6)$$

where the countable collection  $\Gamma_X^{*Hs} / \sim_X$  is defined by the following formula

$$\forall \Psi(X) \left\{ [\Psi(X)] \in \Gamma_X^{*Hs} / \sim_X \Leftrightarrow [([\Psi(X)]_{Hs} \in \Gamma_X^{Hs} / \sim_X ) \wedge \Box \exists ! X \Psi(X)] \right\} \quad (3.1.7)$$

**Definition 3.1.3.** Let  $\widetilde{\mathfrak{R}}_2^{Hs}$  be the countable collection of the all sets such that

$$\forall X \left( X \in \widetilde{\mathfrak{S}}_2^{Hs} \right) \left[ X \in \widetilde{\mathfrak{R}}_2^{Hs} \Leftrightarrow \Box X \notin X \right]. \quad (3.1.8)$$

**Remark 3.1.2.** Note that  $\widetilde{\mathfrak{R}}_2^{Hs} \in \widetilde{\mathfrak{S}}_2^{Hs}$  since  $\widetilde{\mathfrak{R}}_2^{Hs}$  is a collection definable by 1-place open wff

$$\Psi \left( Z, \widetilde{\mathfrak{R}}_2^{Hs} \right) \triangleq \forall X \left( X \in \widetilde{\mathfrak{S}}_2^{Hs} \right) \left[ X \in Z \Leftrightarrow \Box(X \notin X) \right].$$

From (3.1.8) one obtains

$$\widetilde{\mathfrak{R}}_2^{Hs} \in \widetilde{\mathfrak{R}}_2^{Hs} \Leftrightarrow \Box \left( \widetilde{\mathfrak{R}}_2^{Hs} \notin \widetilde{\mathfrak{R}}_2^{Hs} \right). \quad (3.1.9)$$

But (3.1.9) immediately gives a contradiction

$$\overline{ZFC}_2^{Hs} \vdash \left( \widetilde{\mathfrak{R}}_2^{Hs} \in \widetilde{\mathfrak{R}}_2^{Hs} \right) \wedge \left( \widetilde{\mathfrak{R}}_2^{Hs} \notin \widetilde{\mathfrak{R}}_2^{Hs} \right). \quad (3.1.10)$$

However contradiction (3.1.10) it is not a true contradiction inside  $\overline{ZFC}_2^{Hs}$  for the reason that the countable collection  $\widetilde{\mathfrak{S}}_2^{Hs}$  is not a set in the sense of the set theory  $\overline{ZFC}_2^{Hs}$ .

**In order to obtain a true contradiction inside  $\overline{ZFC}_2^{Hs}$  we introduce the following definitions.**

**Definition 3.1.4.** We define now the countable set  $\Gamma_v^{*Hs} / \sim_v$  by

$$\forall y \left\{ [y]_{Hs} \in \Gamma_v^{*Hs} / \sim_v \Leftrightarrow ([y]_{Hs} \in \Gamma_v^{Hs} / \sim_v) \wedge \widehat{\mathbf{Fr}}_2^{Hs}(y, v) \wedge [\Box \exists! X \Psi_{y,v}(X)] \right\}. \quad (3.1.11)$$

**Definition 3.1.5.** We choose now  $\Box A$  in the following form

$$\Box A \triangleq Bew_{\overline{ZFC}_2^{Hs}}(\#A) \wedge [Bew_{\overline{ZFC}_2^{Hs}}(\#A) \Rightarrow A]. \quad (3.1.12)$$

Here  $Bew_{\overline{ZFC}_2^{Hs}}(\#A)$  is a canonical Gödel formula which says to us that there exists proof in  $\overline{ZFC}_2^{Hs}$  of the formula  $A$  with Gödel number  $\#A$ .

**Remark 3.1.3.** Notice that the Definition 3.1.5 holds as definition of predicate really asserting provability in  $\overline{ZFC}_2^{Hs}$ .

**Definition 3.1.7.** Using Definition 3.1.5, we replace now formula (3.1.7) by the following formula

$$\begin{aligned} \forall \Psi(X) \{ [\Psi(X)] \in \Gamma_X^{*Hs} / \sim_X \Leftrightarrow \exists \Psi(X) ([\Psi(X)] \in \Gamma_X^{Hs} / \sim_X) \wedge \\ \wedge [Bew_{\overline{ZFC}_2^{Hs}}(\# \exists! X [\Psi(X) \wedge Y = X])] \wedge \\ \wedge [Bew_{\overline{ZFC}_2^{Hs}}(\# \exists! X [\Psi(X) \wedge Y = X]) \Rightarrow \exists! X [\Psi(X) \wedge Y = X]] \}. \end{aligned} \quad (3.1.13)$$

**Definition 3.1.8.** Using Definition 3.1.5, we replace now formula (3.1.8) by the following formula

$$\begin{aligned} \forall X \left( X \in \widetilde{\mathfrak{S}}_2^{Hs} \right) \left[ X \in \widetilde{\mathfrak{R}}_2^{Hs} \Leftrightarrow [Bew_{\overline{ZFC}_2^{Hs}}(\#(X \notin X))] \wedge \right. \\ \left. \wedge [Bew_{\overline{ZFC}_2^{Hs}}(\#(X \notin X)) \Rightarrow X \notin X] \right]. \end{aligned} \quad (3.1.14)$$

**Definition 3.1.9.** Using Definition 3.1.5, we replace now formula (3.1.11) by the following formula

$$\begin{aligned} \forall y \{ [y]_{Hs} \in \Gamma_v^{*Hs} / \sim_v \Leftrightarrow \\ ([y]_{Hs} \in \Gamma_v^{Hs} / \sim_v) \wedge \widehat{\mathbf{Fr}}_2^{Hs}(y, v) \wedge [Bew_{\overline{ZFC}_2^{Hs}}(\# \exists! X [\Psi_{y,v}(X) \wedge Y = X])] \wedge \\ \wedge [Bew_{\overline{ZFC}_2^{Hs}}(\# \exists! X [\Psi_{y,v}(X) \wedge Y = X]) \Rightarrow \exists! X [\Psi_{y,v}(X) \wedge Y = X]] \}. \end{aligned} \quad (3.1.15)$$

**Definition 3.1.10.** Using Definitions 3.1.4-3.1.7, we define now the countable set  $\widetilde{\mathfrak{S}}_2^{*Hs}$  by formula

$$\forall Y \left\{ Y \in \widetilde{\mathfrak{S}}_2^{*Hs} \Leftrightarrow \exists y \left[ ([y]_{Hs} \in \Gamma_v^{*Hs} / \sim_v) \wedge (g_{\overline{ZFC}_2^{Hs}}(X) = v) \right] \right\}. \quad (3.1.16)$$

**Remark 3.1.4.** Note that from the axiom schema of replacement (1.1.1) it follows directly that  $\widetilde{\mathfrak{S}}_2^{*Hs}$  is a set in the sense of the set theory  $\overline{ZFC}_2^{Hs}$ .

**Definition 3.1.11.** Using Definition 3.1.8 we replace now formula (3.1.14) by the following formula



$$\forall X \left( X \in \widetilde{\mathfrak{R}}_2^{*Hs} \right) \quad (3.1.17)$$

$$\left[ X \in \widetilde{\mathfrak{R}}_2^{*Hs} \Leftrightarrow \left[ Bew_{\overline{ZFC}_2^{Hs}}(\#(X \notin X)) \right] \wedge \left[ Bew_{\overline{ZFC}_2^{Hs}}(\#(X \notin X)) \Rightarrow X \notin X \right] \right].$$

**Remark 3.1.5.** Notice that the expression (3.1.18)

$$\left[ Bew_{\overline{ZFC}_2^{Hs}}(\#(X \notin X)) \right] \wedge \left[ Bew_{\overline{ZFC}_2^{Hs}}(\#(X \notin X)) \Rightarrow X \notin X \right] \quad (3.1.18)$$

obviously is a well formed formula of  $\overline{ZFC}_2^{Hs}$  and therefore collection  $\widetilde{\mathfrak{R}}_2^{*Hs}$  is a set in the sense of  $\overline{ZFC}_2^{Hs}$ .

**Remark 3.1.6.** Note that  $\widetilde{\mathfrak{R}}_2^{*Hs} \in \widetilde{\mathfrak{S}}_2^{*Hs}$  since  $\widetilde{\mathfrak{R}}_2^{*Hs}$  is a collection definable by 1-place open wff

$$\Psi \left( Z, \widetilde{\mathfrak{R}}_2^{*Hs} \right) \triangleq$$

$$\forall X \left( X \in \widetilde{\mathfrak{S}}_2^{*Hs} \right) \left[ X \in Z \Leftrightarrow \right. \quad (3.1.19)$$

$$\left. \left[ Bew_{\overline{ZFC}_2^{Hs}}(\#(X \notin X)) \right] \wedge \left[ Bew_{\overline{ZFC}_2^{Hs}}(\#(X \notin X)) \Rightarrow X \notin X \right] \right].$$

**Theorem 3.1.1.** Set theory  $\overline{ZFC}_2^{Hs} \triangleq ZFC_2^{Hs} + \exists M_{st}^{\overline{ZFC}_2^{Hs}}$  is inconsistent.

Proof. From (3.1.17) we obtain

$$\widetilde{\mathfrak{R}}_2^{*Hs} \in \widetilde{\mathfrak{R}}_2^{*Hs} \Leftrightarrow \left[ Bew_{\overline{ZFC}_2^{Hs}} \left( \# \left( \widetilde{\mathfrak{R}}_2^{*Hs} \notin \widetilde{\mathfrak{R}}_2^{*Hs} \right) \right) \right] \wedge$$

$$\wedge \left[ Bew_{\overline{ZFC}_2^{Hs}} \left( \# \left( \widetilde{\mathfrak{R}}_2^{*Hs} \notin \widetilde{\mathfrak{R}}_2^{*Hs} \right) \right) \Rightarrow \widetilde{\mathfrak{R}}_2^{*Hs} \notin \widetilde{\mathfrak{R}}_2^{*Hs} \right]. \quad (3.1.20)$$

(a) Assume now that:

$$\widetilde{\mathfrak{R}}_2^{*Hs} \in \widetilde{\mathfrak{R}}_2^{*Hs}. \quad (3.1.21)$$

Then from (3.1.20) we obtain  $\vdash_{\overline{ZFC}_2^{Hs}} Bew_{\overline{ZFC}_2^{Hs}} \left( \# \left( \widetilde{\mathfrak{R}}_2^{*Hs} \notin \widetilde{\mathfrak{R}}_2^{*Hs} \right) \right)$  and

$\vdash_{\overline{ZFC}_2^{Hs}} Bew_{\overline{ZFC}_2^{Hs}} \left( \# \left( \widetilde{\mathfrak{R}}_2^{*Hs} \notin \widetilde{\mathfrak{R}}_2^{*Hs} \right) \right) \Rightarrow \widetilde{\mathfrak{R}}_2^{*Hs} \notin \widetilde{\mathfrak{R}}_2^{*Hs}$ , therefore  $\vdash_{\overline{ZFC}_2^{Hs}} \widetilde{\mathfrak{R}}_2^{*Hs} \notin \widetilde{\mathfrak{R}}_2^{*Hs}$  and

so

$$\vdash_{\overline{ZFC}_2^{Hs}} \widetilde{\mathfrak{R}}_2^{*Hs} \in \widetilde{\mathfrak{R}}_2^{*Hs} \Rightarrow \widetilde{\mathfrak{R}}_2^{*Hs} \notin \widetilde{\mathfrak{R}}_2^{*Hs}. \quad (3.1.22)$$

From (3.1.21)-(3.1.22) we obtain

$$\widetilde{\mathfrak{R}}_2^{*Hs} \in \widetilde{\mathfrak{R}}_2^{*Hs}, \widetilde{\mathfrak{R}}_2^{*Hs} \in \widetilde{\mathfrak{R}}_2^{*Hs} \Rightarrow \widetilde{\mathfrak{R}}_2^{*Hs} \notin \widetilde{\mathfrak{R}}_2^{*Hs} \vdash \widetilde{\mathfrak{R}}_2^{*Hs} \notin \widetilde{\mathfrak{R}}_2^{*Hs}$$

and thus  $\vdash_{\overline{ZFC}_2^{Hs}} \left( \widetilde{\mathfrak{R}}_2^{*Hs} \in \widetilde{\mathfrak{R}}_2^{*Hs} \right) \wedge \left( \widetilde{\mathfrak{R}}_2^{*Hs} \notin \widetilde{\mathfrak{R}}_2^{*Hs} \right)$ .

(b) Assume now that

$$\left[ Bew_{\overline{ZFC}_2^{Hs}} \left( \# \left( \widetilde{\mathfrak{R}}_2^{*Hs} \notin \widetilde{\mathfrak{R}}_2^{*Hs} \right) \right) \right] \wedge$$

$$\wedge \left[ Bew_{\overline{ZFC}_2^{Hs}} \left( \# \left( \widetilde{\mathfrak{R}}_2^{*Hs} \notin \widetilde{\mathfrak{R}}_2^{*Hs} \right) \right) \Rightarrow \widetilde{\mathfrak{R}}_2^{*Hs} \notin \widetilde{\mathfrak{R}}_2^{*Hs} \right]. \quad (3.1.23)$$

Then from (3.1.23) we obtain  $\vdash \widetilde{\mathfrak{R}}_2^{*Hs} \notin \widetilde{\mathfrak{R}}_2^{*Hs}$ . From (3.1.23) and (3.1.20) we obtain

$\vdash_{\overline{ZFC}_2^{Hs}} \widetilde{\mathfrak{R}}_2^{*Hs} \in \widetilde{\mathfrak{R}}_2^{*Hs}$ , so  $\vdash_{\overline{ZFC}_2^{Hs}} \widetilde{\mathfrak{R}}_2^{*Hs} \notin \widetilde{\mathfrak{R}}_2^{*Hs}, \widetilde{\mathfrak{R}}_2^{*Hs} \in \widetilde{\mathfrak{R}}_2^{*Hs}$  which immediately gives us a

contradiction  $\vdash_{\overline{ZFC}_2^{Hs}} \left( \widetilde{\mathfrak{R}}_2^{*Hs} \in \widetilde{\mathfrak{R}}_2^{*Hs} \right) \wedge \left( \widetilde{\mathfrak{R}}_2^{*Hs} \notin \widetilde{\mathfrak{R}}_2^{*Hs} \right)$ .

**Definition 3.1.12.** We choose now  $\square A$  in the following form

$$\square A \triangleq \overline{Bew}_{\overline{ZFC}_2^{Hs}}(\#A), \quad (3.1.24)$$

or in the following equivalent form

$$\square A \triangleq \overline{Bew}_{\overline{ZFC}_2^{Hs}}(\#A) \wedge \left[ \overline{Bew}_{\overline{ZFC}_2^{Hs}}(\#A) \Rightarrow A \right]$$

similar to (3.1.5). Here  $\overline{Bew}_{\overline{ZFC}_2^{Hs}}(\#A)$  is a Gödel formula (see Chapt. II section 2, Definition)

which really asserts provability in  $\overline{ZFC}_2^{Hs}$  of the formula  $A$  with Gödel number  $\#A$ .

**Remark 3.1.7.** Notice that the Definition 3.1.12 with formula (3.1.24) holds as definition

of predicate really asserting provability in  $\overline{ZFC}_2^{Hs}$ .

**Definition 3.1.13.** Using Definition 3.1.12 with formula (3.1.24), we replace now formula

(3.1.7) by the following formula

$$\begin{aligned} \forall \Psi(X) \{ & [\Psi(X)] \in \overline{\Gamma}_X^{*Hs} / \sim_X \Leftrightarrow \exists \Psi(X) ([\Psi(X)] \in \Gamma_X^{Hs} / \sim_X) \wedge \\ & \wedge \left[ \overline{Bew}_{\overline{ZFC}_2^{Hs}}(\#\exists! X[\Psi(X) \wedge Y = X]) \right] \}. \end{aligned} \quad (3.1.25)$$

**Definition 3.1.14.** Using Definition 3.1.12 with formula (3.1.24), we replace now formula

(3.1.8) by the following formula

$$\forall X \left( X \in \widetilde{\mathfrak{S}}_2^{Hs} \right) \left[ X \in \widetilde{\mathfrak{R}}_2^{Hs} \Leftrightarrow \left[ \overline{Bew}_{\overline{ZFC}_2^{Hs}}(\#(X \notin X)) \right] \right] \quad (3.1.26)$$

**Definition 3.1.15.** Using Definition 3.1.12 with formula (3.1.24), we replace now formula (3.1.11) by the following formula

$$\begin{aligned} \forall y \{ & [y]_{Hs} \in \Gamma_v^{*Hs} / \sim_v \Leftrightarrow \\ & ([y]_{Hs} \in \Gamma_v^{Hs} / \sim_v) \wedge \widehat{\mathbf{Fr}}_2^{Hs}(y, v) \wedge \left[ \overline{Bew}_{\overline{ZFC}_2^{Hs}}(\#\exists! X[\Psi_{y,v}(X) \wedge Y = X]) \right] \}. \end{aligned} \quad (3.1.27)$$

**Definition 3.1.16.** Using Definitions 3.1.13-3.1.17, we define now the countable set  $\widetilde{\mathfrak{S}}_2^{*Hs}$  by formula

$$\forall Y \left\{ Y \in \widetilde{\mathfrak{S}}_2^{*Hs} \Leftrightarrow \exists y \left[ ([y] \in \Gamma_v^{*Hs} / \sim_v) \wedge (g_{\overline{ZFC}_2^{Hs}}(X) = v) \right] \right\}. \quad (3.1.28)$$

**Remark 3.1.8.** Note that from the axiom schema of replacement (1.1.1) it follows directly that  $\widetilde{\mathfrak{S}}_2^{*Hs}$  is a set in the sense of the set theory  $\overline{ZFC}_2^{Hs}$ .

**Definition 3.1.17.** Using Definition 3.1.16 we replace now formula (3.1.26) by the following formula

$$\forall X \left( X \in \widetilde{\mathfrak{S}}_2^{*Hs} \right) \left[ X \in \widetilde{\mathfrak{R}}_2^{*Hs} \Leftrightarrow \left[ \overline{Bew}_{\overline{ZFC}_2^{Hs}}(\#(X \notin X)) \right] \right]. \quad (3.1.29)$$

**Remark 3.1.9.** Notice that the expressions (3.1.30)

$$\left[ \overline{Bew}_{\overline{ZFC}_2^{Hs}}(\#(X \notin X)) \right] \quad \text{and} \quad (3.1.30)$$

$$\left[ \overline{Bew}_{\overline{ZFC}_2^{Hs}}(\#(X \notin X)) \right] \wedge \left[ \overline{Bew}_{\overline{ZFC}_2^{Hs}}(\#(X \notin X)) \Rightarrow X \notin X \right]$$

obviously is a well formed formula of  $\overline{ZFC}_2^{Hs}$  and therefore collection  $\tilde{\mathfrak{R}}_2^{*Hs}$  is a set in the sense of  $\overline{ZFC}_2^{Hs}$ .

**Remark 3.1.10.** Note that  $\tilde{\mathfrak{R}}_2^{*Hs} \in \tilde{\mathfrak{T}}_2^{*Hs}$  since  $\tilde{\mathfrak{R}}_2^{*Hs}$  is a collection definable by 1-place open wff

$$\Psi(Z, \tilde{\mathfrak{R}}_2^{*Hs}) \triangleq \forall X (X \in \tilde{\mathfrak{T}}_2^{*Hs}) [X \in Z \Leftrightarrow \overline{Bew}_{\overline{ZFC}_2^{Hs}}(\#(X \notin X))]. \quad (3.1.31)$$

**Theorem 3.1.2.** Set theory  $\overline{ZFC}_2^{Hs} \triangleq ZFC_2^{Hs} + \exists M_{st}^{\overline{ZFC}_2^{Hs}}$  is inconsistent.

Proof. From (3.1.29) we obtain

$$\tilde{\mathfrak{R}}_2^{*Hs} \in \tilde{\mathfrak{R}}_2^{*Hs} \Leftrightarrow \left[ \overline{Bew}_{\overline{ZFC}_2^{Hs}} \left( \# \left( \tilde{\mathfrak{R}}_2^{*Hs} \notin \tilde{\mathfrak{R}}_2^{*Hs} \right) \right) \right]. \quad (3.1.32)$$

(a) Assume now that:

$$\tilde{\mathfrak{R}}_2^{*Hs} \in \tilde{\mathfrak{R}}_2^{*Hs}. \quad (3.1.33)$$

Then from (3.1.32) we obtain  $\vdash_{\overline{ZFC}_2^{Hs}} \overline{Bew}_{\overline{ZFC}_2^{Hs}} \left( \# \left( \tilde{\mathfrak{R}}_2^{*Hs} \notin \tilde{\mathfrak{R}}_2^{*Hs} \right) \right)$  and therefore

$$\vdash_{\overline{ZFC}_2^{Hs}} \tilde{\mathfrak{R}}_2^{*Hs} \notin \tilde{\mathfrak{R}}_2^{*Hs}$$

thus we obtain

$$\vdash_{\overline{ZFC}_2^{Hs}} \tilde{\mathfrak{R}}_2^{*Hs} \in \tilde{\mathfrak{R}}_2^{*Hs} \Rightarrow \tilde{\mathfrak{R}}_2^{*Hs} \notin \tilde{\mathfrak{R}}_2^{*Hs}. \quad (3.1.34)$$

From (3.1.33)-(3.1.34) we obtain  $\tilde{\mathfrak{R}}_2^{*Hs} \in \tilde{\mathfrak{R}}_2^{*Hs}$  and  $\tilde{\mathfrak{R}}_2^{*Hs} \in \tilde{\mathfrak{R}}_2^{*Hs} \Rightarrow \tilde{\mathfrak{R}}_2^{*Hs} \notin \tilde{\mathfrak{R}}_2^{*Hs}$  thus  $\vdash_{\overline{ZFC}_2^{Hs}} \tilde{\mathfrak{R}}_2^{*Hs} \notin \tilde{\mathfrak{R}}_2^{*Hs}$  and finally we obtain  $\vdash_{\overline{ZFC}_2^{Hs}} \left( \tilde{\mathfrak{R}}_2^{*Hs} \in \tilde{\mathfrak{R}}_2^{*Hs} \right) \wedge \left( \tilde{\mathfrak{R}}_2^{*Hs} \notin \tilde{\mathfrak{R}}_2^{*Hs} \right)$ .

(b) Assume now that

$$\left[ Bew_{\overline{ZFC}_2^{Hs}} \left( \# \left( \tilde{\mathfrak{R}}_2^{*Hs} \notin \tilde{\mathfrak{R}}_2^{*Hs} \right) \right) \right]. \quad (3.1.23)$$

Then from (3.1.35) we obtain  $\vdash_{\overline{ZFC}_2^{Hs}} \tilde{\mathfrak{R}}_2^{*Hs} \notin \tilde{\mathfrak{R}}_2^{*Hs}$ . From (3.1.35) and (3.1.32) we obtain

$$\vdash_{\overline{ZFC}_2^{Hs}} \tilde{\mathfrak{R}}_2^{*Hs} \in \tilde{\mathfrak{R}}_2^{*Hs}, \text{ thus } \vdash_{\overline{ZFC}_2^{Hs}} \tilde{\mathfrak{R}}_2^{*Hs} \notin \tilde{\mathfrak{R}}_2^{*Hs} \text{ and } \vdash_{\overline{ZFC}_2^{Hs}} \tilde{\mathfrak{R}}_2^{*Hs} \in \tilde{\mathfrak{R}}_2^{*Hs} \text{ which}$$

immediately gives us a contradiction  $\vdash_{\overline{ZFC}_2^{Hs}} \left( \tilde{\mathfrak{R}}_2^{*Hs} \in \tilde{\mathfrak{R}}_2^{*Hs} \right) \wedge \left( \tilde{\mathfrak{R}}_2^{*Hs} \notin \tilde{\mathfrak{R}}_2^{*Hs} \right)$ .

## 3.2. Derivation of the inconsistent provably definable set in $ZFC_{st}$ .

Let  $\mathfrak{T}_{st}$  be the countable collection of all sets  $X$  such that  $ZFC_{st} \vdash \exists! X \Psi(X)$ , where  $\Psi(X)$  is a 1-place open wff i.e.,

$$\forall Y \{ Y \in \mathfrak{T}_{st} \Leftrightarrow ZFC_{st} \vdash \exists \Psi(X) [([\Psi(X)] \in \Gamma_X^{st} / \sim_X) \wedge \exists! X [\Psi(X) \wedge Y = X]] \}. \quad (3.2.1)$$

Let  $X \notin_{\vdash ZFC_{st}} Y$  be a predicate such that  $X \notin_{\vdash ZFC_{st}} Y \Leftrightarrow ZFC_{st} \vdash X \notin Y$ . Let  $\mathfrak{R}$  be the countable collection of all sets such that

$$\forall X \left[ X \in \mathfrak{R}_{st} \Leftrightarrow (X \in \mathfrak{T}_{st}) \wedge (X \notin_{\perp ZFC_{st}} X) \right]. \quad (3.2.2)$$

From (3.2.1) one obtains

$$\mathfrak{R}_{st} \in \mathfrak{R}_{st} \Leftrightarrow \mathfrak{R}_{st} \notin_{\perp ZFC_{st}} \mathfrak{R}_{st}. \quad (3.2.3)$$

But (3.2.3) gives a contradiction

$$(\mathfrak{R}_{st} \in \mathfrak{R}_{st}) \wedge (\mathfrak{R}_{st} \notin \mathfrak{R}_{st}). \quad (3.2.4)$$

However contradiction (3.2.4) it is not a contradiction inside  $ZFC_{st}$  for the reason that predicate  $X \notin_{\perp ZFC_{st}} Y$  is not a predicate of  $ZFC_{st}$  and therefore countable collections  $\mathfrak{T}_{st}$  and  $\mathfrak{R}_{st}$  are not a sets of  $ZFC_{st}$ . Nevertheless by using Gödel encoding the above stated contradiction can be shipped in special consistent extensions of  $ZFC_{st}$ .

**Designation 3.2.1** (i) Let  $M_{st}^{ZFC}$  be a standard model of  $ZFC$  and

(ii) let  $ZFC_{st}$  be the theory  $ZFC_{st} = ZFC + \exists M_{st}^{ZFC}$ ,

(iii) let  $\mathfrak{T}_{st}$  be the set of the all sets of  $M_{st}^{ZFC}$  provably definable in  $ZFC_{st}$ , and let

$\mathfrak{R}_{st} = \{X \in \mathfrak{T}_{st} : \Box_{st}(X \notin X)\}$  where  $\Box_{st}A$  means: 'sentence  $A$  derivable in  $ZFC_{st}$ ', or

some

appropriate modification thereof.

We replace now (3.2.1) by formula

$$\forall Y \{Y \in \mathfrak{T}_{st} \leftrightarrow \Box_{st}[\exists \Psi(\cdot) \exists ! X [\Psi(X) \wedge Y = X]]\}, \quad (3.2.5)$$

and we replace (3.2.2) by formula

$$\forall X \left[ X \in \mathfrak{R}_{st} \leftrightarrow (X \in \mathfrak{T}_{st}) \wedge \Box_{st}(X \notin X) \right]. \quad (3.2.6)$$

Assume that  $ZFC_{st} \vdash \mathfrak{R}_{st} \in \mathfrak{T}_{st}$ . Then, we have that:  $\mathfrak{R}_{st} \in \mathfrak{R}_{st}$  if and only if  $\Box_{st}(\mathfrak{R}_{st} \notin \mathfrak{R}_{st})$ , which immediately gives us  $\mathfrak{R}_{st} \in \mathfrak{R}_{st}$  if and only if  $\mathfrak{R}_{st} \notin \mathfrak{R}_{st}$ . But this is a contradiction, i.e.,  $ZFC_{st} \vdash (\mathfrak{R}_{st} \in \mathfrak{R}_{st}) \wedge (\mathfrak{R}_{st} \notin \mathfrak{R}_{st})$ . We choose now  $\Box_{st}A$  in the following form

$$\Box_{st}A \triangleq Bew_{ZFC_{st}}(\#A) \wedge [Bew_{ZFC_{st}}(\#A) \Rightarrow A]. \quad (3.2.7)$$

Here  $Bew_{ZFC_{st}}(\#A)$  is a canonical Gödel formula which says to us that there exists proof in  $ZFC_{st}$  of the formula  $A$  with Gödel number  $\#A \in M_{st}^{PA}$ .

**Remark 3.2.1.** Notice that definition (3.2.7) holds as definition of predicate really asserting provability in  $ZFC_{st}$ .

**Definition 3.2.2.** We rewrite now (3.2.5) in the following equivalent form

$$\forall Y \left\{ Y \in \tilde{\mathfrak{T}}_{st} \Leftrightarrow \exists \Psi(X) [([\Psi(X)]_{st} \in \Gamma_X^{*st} / \sim_X) \wedge (Y = X)] \right\}, \quad (3.2.8)$$

where the countable collection  $\Gamma_X^{*st} / \sim_X$  is defined by the following formula

$$\forall \Psi(X) \{ [\Psi(X)]_{st} \in \Gamma_X^{*st} / \sim_X \Leftrightarrow [([\Psi(X)]_{st} \in \Gamma_X^{st} / \sim_X) \wedge \Box_{st} \exists ! X \Psi(X)] \} \quad (3.2.9)$$

**Definition 3.2.3.** Let  $\tilde{\mathfrak{R}}_{st}$  be the countable collection of the all sets such that

$$\forall X \left( X \in \tilde{\mathfrak{T}}_{st} \right) \left[ X \in \tilde{\mathfrak{R}}_{st} \Leftrightarrow \Box_{st}(X \notin X) \right]. \quad (3.2.10)$$

**Remark 3.2.2.** Note that  $\tilde{\mathfrak{R}}_2^{Hs} \in \tilde{\mathfrak{T}}_2^{Hs}$  since  $\tilde{\mathfrak{R}}_2^{Hs}$  is a collection definable by 1-place open wff

$$\Psi(Z, \tilde{\mathfrak{R}}_{st}) \triangleq \forall X \left( X \in \tilde{\mathfrak{T}}_{st} \right) [X \in Z \Leftrightarrow \Box_{st}(X \notin X)]. \quad (3.2.11)$$

**Definition 3.2.4.** By using formula (3.2.7) we rewrite now (3.2.8) in the following

equivalent form

$$\forall Y \left\{ Y \in \tilde{\mathfrak{F}}_{st} \Leftrightarrow \exists \Psi(X) [([\Psi(X)]_{st} \in \Gamma_X^{*st} / \sim_X) \wedge (Y = X)] \right\}, \quad (3.2.12)$$

where the countable collection  $\Gamma_X^{*Hs} / \sim_X$  is defined by the following formula

$$\begin{aligned} & \forall \Psi(X) \{ [\Psi(X)]_{st} \in \Gamma_X^{*st} / \sim_X \Leftrightarrow \\ & [([\Psi(X)]_{st} \in \Gamma_X^{st} / \sim_X) \wedge Bew_{ZFC_{st}}(\#\exists! X \Psi(X))] \wedge \\ & \wedge [Bew_{ZFC_{st}}(\#\exists! X \Psi(X)) \Rightarrow \exists! X \Psi(X)] \} \end{aligned} \quad (3.2.13)$$

**Definition 3.2.5.** Using formula (3.2.7), we replace now formula (3.2.10) by the following formula

$$\begin{aligned} \forall X \left( X \in \tilde{\mathfrak{F}}_{st} \right) & \left[ X \in \tilde{\mathfrak{R}}_{st} \Leftrightarrow [Bew_{ZFC_{st}}(\#(X \notin X))] \wedge \right. \\ & \left. \wedge [Bew_{ZFC_{st}}(\#(X \notin X))] \right]. \end{aligned} \quad (3.2.14)$$

**Definition 3.2.6.** Using Definition 1.3.5, we replace now formula (3.2.11) by the following formula

$$\begin{aligned} & \forall y \{ [y]_{st} \in \Gamma_v^{*st} / \sim_v \Leftrightarrow \\ & ([y]_{st} \in \Gamma_v^{st} / \sim_v) \wedge \widehat{\mathbf{Fr}}_{st}(y, v) \wedge [Bew_{ZFC_{st}}(\#\exists! X [\Psi_{y,v}(X) \wedge Y = X])] \wedge \\ & \wedge [Bew_{ZFC_{st}}(\#\exists! X [\Psi_{y,v}(X) \wedge Y = X]) \Rightarrow \exists! X [\Psi_{y,v}(X) \wedge Y = X]] \}. \end{aligned} \quad (3.2.15)$$

**Definition 3.2.7.** Using Definitions 3.2.4-3.2.6, we define now the countable set  $\tilde{\mathfrak{F}}_{st}^*$  by formula

$$\forall Y \left\{ Y \in \tilde{\mathfrak{F}}_{st}^* \Leftrightarrow \exists y [([y]_{st} \in \Gamma_v^{*st} / \sim_v) \wedge (g_{ZFC_{st}}(X) = v)] \right\}. \quad (3.2.16)$$

**Remark 3.2.3.** Note that from the axiom schema of replacement it follows directly that  $\tilde{\mathfrak{F}}_{st}^*$  is a set in the sense of the set theory  $ZFC_{st}$ .

**Definition 3.2.8.** Using Definition 3.2.7 we replace now formula (3.2.14) by the following formula

$$\begin{aligned} & \forall X \left( X \in \tilde{\mathfrak{F}}_{st}^* \right) \\ & \left[ X \in \tilde{\mathfrak{R}}_{st}^* \Leftrightarrow [Bew_{ZFC_{st}}(\#(X \notin X))] \wedge [Bew_{\overline{ZFC}_{st}}(\#(X \notin X)) \Rightarrow X \notin X] \right]. \end{aligned} \quad (3.2.17)$$

**Remark 3.2.4.** Notice that the expression (3.2.18)

$$[Bew_{ZFC_{st}}(\#(X \notin X))] \wedge [Bew_{ZFC_{st}}(\#(X \notin X)) \Rightarrow X \notin X] \quad (3.2.18)$$

obviously is a well formed formula of  $ZFC_{st}$  and therefore collection  $\tilde{\mathfrak{R}}_{st}^*$  is a set in the sense of  $\overline{ZFC}_2^{Hs}$ .

**Remark 3.2.5.** Note that  $\tilde{\mathfrak{R}}_{st}^* \in \tilde{\mathfrak{F}}_{st}^*$  since  $\tilde{\mathfrak{R}}_{st}^*$  is a collection definable by 1-place open wff

$$\begin{aligned} \Psi(Z, \tilde{\mathfrak{R}}_{st}^*) &\triangleq \\ \forall X (X \in \tilde{\mathfrak{S}}_{st}^*) [X \in Z &\Leftrightarrow \\ [Bew_{ZFC_{st}}(\#(X \notin X))] \wedge [Bew_{ZFC_{st}}(\#(X \notin X)) &\Rightarrow X \notin X]]. \end{aligned} \quad (3.2.19)$$

**Theorem 3.2.1.** Set theory  $ZFC_{st} \triangleq ZFC + \exists M_{st}^{ZFC}$  is inconsistent.

Proof. From (3.2.17) we obtain

$$\begin{aligned} \tilde{\mathfrak{R}}_{st}^* \in \tilde{\mathfrak{R}}_{st}^* &\Leftrightarrow [Bew_{ZFC_{st}}(\#(\tilde{\mathfrak{R}}_{st}^* \notin \tilde{\mathfrak{R}}_{st}^*))] \wedge \\ \wedge [Bew_{ZFC_{st}}(\#(\tilde{\mathfrak{R}}_{st}^* \notin \tilde{\mathfrak{R}}_{st}^*)) &\Rightarrow \tilde{\mathfrak{R}}_{st}^* \notin \tilde{\mathfrak{R}}_{st}^*]. \end{aligned} \quad (3.2.20)$$

(a) Assume now that:

$$\tilde{\mathfrak{R}}_{st}^* \in \tilde{\mathfrak{R}}_{st}^*. \quad (3.2.21)$$

Then from (3.2.20) we obtain  $\vdash Bew_{ZFC_{st}}(\#(\tilde{\mathfrak{R}}_{st}^* \notin \tilde{\mathfrak{R}}_{st}^*))$  and

$\vdash Bew_{ZFC_{st}}(\#(\tilde{\mathfrak{R}}_{st}^* \notin \tilde{\mathfrak{R}}_{st}^*)) \Rightarrow \tilde{\mathfrak{R}}_{st}^* \notin \tilde{\mathfrak{R}}_{st}^*$ , therefore  $\vdash \tilde{\mathfrak{R}}_{st}^* \notin \tilde{\mathfrak{R}}_{st}^*$  and so

$$\vdash_{ZFC_{st}} \tilde{\mathfrak{R}}_{st}^* \in \tilde{\mathfrak{R}}_{st}^* \Rightarrow \tilde{\mathfrak{R}}_{st}^* \notin \tilde{\mathfrak{R}}_{st}^*. \quad (3.2.22)$$

From (3.2.21)-(3.2.22) we obtain  $\tilde{\mathfrak{R}}_{st}^* \in \tilde{\mathfrak{R}}_{st}^*, \tilde{\mathfrak{R}}_{st}^* \in \tilde{\mathfrak{R}}_{st}^* \Rightarrow \tilde{\mathfrak{R}}_{st}^* \notin \tilde{\mathfrak{R}}_{st}^* \vdash \tilde{\mathfrak{R}}_{st}^* \notin \tilde{\mathfrak{R}}_{st}^*$  and therefore  $\vdash_{ZFC_{st}} (\tilde{\mathfrak{R}}_{st}^* \in \tilde{\mathfrak{R}}_{st}^*) \wedge (\tilde{\mathfrak{R}}_{st}^* \notin \tilde{\mathfrak{R}}_{st}^*)$ .

(b) Assume now that

$$\begin{aligned} [Bew_{ZFC_{st}}(\#(\tilde{\mathfrak{R}}_{st}^* \notin \tilde{\mathfrak{R}}_{st}^*))] \wedge \\ \wedge [Bew_{ZFC_{st}}(\#(\tilde{\mathfrak{R}}_{st}^* \notin \tilde{\mathfrak{R}}_{st}^*)) \Rightarrow \tilde{\mathfrak{R}}_{st}^* \notin \tilde{\mathfrak{R}}_{st}^*]. \end{aligned} \quad (3.2.23)$$

Then from (3.2.23) we obtain  $\vdash \tilde{\mathfrak{R}}_2^{*Hs} \notin \tilde{\mathfrak{R}}_2^{*Hs}$ . From (3.2.23) and (3.2.20) we obtain

$\vdash_{ZFC_2^{Hs}} \tilde{\mathfrak{R}}_2^{*Hs} \in \tilde{\mathfrak{R}}_2^{*Hs}$ , so  $\vdash_{ZFC_2^{Hs}} \tilde{\mathfrak{R}}_2^{*Hs} \notin \tilde{\mathfrak{R}}_2^{*Hs}, \tilde{\mathfrak{R}}_2^{*Hs} \in \tilde{\mathfrak{R}}_2^{*Hs}$  which immediately gives us a contradiction  $\vdash_{ZFC_2^{Hs}} (\tilde{\mathfrak{R}}_2^{*Hs} \in \tilde{\mathfrak{R}}_2^{*Hs}) \wedge (\tilde{\mathfrak{R}}_2^{*Hs} \notin \tilde{\mathfrak{R}}_2^{*Hs})$ .

### 3.3. Derivation of the inconsistent provably definable set in $ZFC_{Nst}$ .

**Designation 3.3.1.**(i) Let  $\overline{PA}$  be a first order theory which contain usual postulates of Peano arithmetic [8] and recursive defining equations for every primitive recursive function

as desired.

(ii) Let  $M_{Nst}^{ZFC}$  be a nonstandard model of  $ZFC$  and let  $M_{st}^{\overline{PA}}$  be a standard model of  $\overline{PA}$ . We

assume now that  $M_{st}^{\overline{PA}} \subset M_{Nst}^{ZFC}$  and denote such nonstandard model of  $ZFC$  by  $M_{Nst}^{ZFC}[\overline{PA}]$ .

(iii) Let  $ZFC_{Nst}$  be the theory  $ZFC_{Nst} = ZFC + M_{Nst}^{ZFC}[\overline{PA}]$ .

(iv) Let  $\mathfrak{S}_{Nst}$  be the set of the all sets of  $M_{St}^{ZFC}[\overline{PA}]$  provably definable in  $ZFC_{Nst}$ , and let  $\mathfrak{R}_{Nst} = \{X \in \mathfrak{S}_{Nst} : \Box_{Nst}(X \notin X)\}$  where  $\Box_{Nst}A$  means 'sentence  $A$  derivable in  $ZFC_{Nst}$ ', or

some appropriate modification thereof. We replace now (3.1.4) by formula

$$\forall Y\{Y \in \mathfrak{S}_{Nst} \leftrightarrow \Box_{Nst}[\exists \Psi(\cdot)\exists!X[\Psi(X) \wedge Y = X]]\}, \quad (3.3.1)$$

and we replace (3.1.5) by formula

$$\forall X[X \in \mathfrak{R}_{Nst} \leftrightarrow (X \in \mathfrak{S}_{Nst}) \wedge \Box_{Nst}(X \notin X)]. \quad (3.3.2)$$

Assume that  $ZFC_{Nst} \vdash \mathfrak{R}_{Nst} \in \mathfrak{S}_{Nst}$ . Then, we have that:  $\mathfrak{R}_{Nst} \in \mathfrak{R}_{Nst}$  if and only if  $\Box_{Nst}(\mathfrak{R}_{Nst} \notin \mathfrak{R}_{Nst})$ , which immediately gives us  $\mathfrak{R}_{Nst} \in \mathfrak{R}_{Nst}$  if and only if  $\mathfrak{R}_{Nst} \notin \mathfrak{R}_{Nst}$ . But this is a contradiction, i.e.,  $ZFC_{Nst} \vdash (\mathfrak{R}_{Nst} \in \mathfrak{R}_{Nst}) \wedge (\mathfrak{R}_{Nst} \notin \mathfrak{R}_{Nst})$ . We choose now  $\Box_{Nst}A$  in the following form

$$\Box_{Nst}A \triangleq Bew_{ZFC_{Nst}}(\#A) \wedge [Bew_{ZFC_{Nst}}(\#A) \Rightarrow A]. \quad (3.3.3)$$

Here  $Bew_{ZFC_{Nst}}(\#A)$  is a canonical Gödel formula which says to us that there exists proof

in  $ZFC_{Nst}$  of the formula  $A$  with Gödel number  $\#A \in M_{St}^{PA}$ .

**Remark 3.3.1.** Notice that definition (3.3.3) holds as definition of predicate really asserting provability in  $ZFC_{Nst}$ .

**Designation 3.3.2.**(i) Let  $g_{ZFC_{Nst}}(u)$  be a Gödel number of given an expression  $u$  of  $ZFC_{Nst}$ .

(ii) Let  $\mathbf{Fr}_{Nst}(y, v)$  be the relation :  $y$  is the Gödel number of a wff of  $ZFC_{Nst}$  that contains

free occurrences of the variable with Gödel number  $v$  [10].

(iii) Let  $\wp_{Nst}(y, v, v_1)$  be a Gödel number of the following wff:

$\exists!X[\Psi(X) \wedge Y = X]$ , where

$$g_{ZFC_{Nst}}(\Psi(X)) = y, g_{ZFC_{Nst}}(X) = v, g_{ZFC_{Nst}}(Y) = v_1.$$

(iv) Let  $\text{Pr}_{ZFC_{Nst}}(z)$  be a predicate asserting provability in  $ZFC_{Nst}$ , which defined by formula (2.6), see Chapt. II, section 2, Remark 2.2 and Designation 2.3,(see also [10]-[11]).

**Remark 3.3.2.** Let  $\mathfrak{S}_{Nst}$  be the countable collection of all sets  $X$  such that

$ZFC_{Nst} \vdash \exists!X\Psi(X)$ , where  $\Psi(X)$  is a 1-place open wff i.e.,

$$\forall Y\{Y \in \mathfrak{S}_{Nst} \Leftrightarrow ZFC_{Nst} \vdash \exists \Psi(X)\exists!X[\Psi(X) \wedge Y = X]\}. \quad (3.3.4)$$

We rewrite now (3.3.4) in the following form

$$\begin{aligned} & \forall Y\{Y \in \mathfrak{S}_{Nst}^* \Leftrightarrow \\ & (g_{ZFC_{Nst}}(Y) = v_1) \wedge \exists y \widehat{\mathbf{Fr}}_{Nst}(y, v) \wedge (g_{ZFC_{Nst}}(X) = v) \wedge [\text{Pr}_{ZFC_{Nst}}(\wp_{Nst}(y, v, v_1)) \wedge \\ & \wedge [\text{Pr}_{ZFC_{Nst}}(\wp_{Nst}(y, v, v_1)) \Rightarrow \exists!X[\Psi(X) \wedge Y = X]]]\} \end{aligned} \quad (3.3.5)$$

**Designation 3.3.3.** Let  $\wp_{Nst}(z)$  be a Gödel number of the following wff:  $Z \notin Z$ , where  $g_{ZFC_{Nst}}(Z) = z$ .

**Remark 3.3.3.** Let  $\mathfrak{R}_{Nst}$  above by formula (3.3.2), i.e.,

$$\forall Z[Z \in \mathfrak{R}_{Nst} \leftrightarrow (Z \in \mathfrak{S}_{Nst}) \wedge \Box_{Nst}(Z \notin Z)]. \quad (3.3.6)$$

We rewrite now (3.3.6) in the following form

$$\forall Z[Z \in \mathfrak{R}_{Nst}^* \leftrightarrow (Z \in \mathfrak{T}_{Nst}^*) \wedge g_{ZFC_{Nst}}(Z) = z \wedge \text{Pr}_{ZFC_{Nst}}(\wp_{Nst}(z))] \wedge \wedge [\text{Pr}_{ZFC_{Nst}}(\wp_{Nst}(z)) \Rightarrow Z \notin Z]. \quad (3.3.7)$$

**Theorem 3.3.1.**  $ZFC_{Nst} \vdash \mathfrak{R}_{Nst}^* \in \mathfrak{R}_{Nst}^* \wedge \mathfrak{R}_{Nst}^* \notin \mathfrak{R}_{Nst}^*$ .

### 3.4. Generalized Tarski's undefinability lemma.

**Remark 3.4.1.** Remind that: (i) if  $\mathbf{Th}$  is a theory, let  $T_{\mathbf{Th}}$  be the set of Gödel numbers of theorems of  $\mathbf{Th}$ , [10], (ii) the property  $x \in T_{\mathbf{Th}}$  is said to be expressible in  $\mathbf{Th}$  by wff  $\mathbf{True}(x_1)$  if the following properties are satisfied [10]:

(a) if  $n \in T_{\mathbf{Th}}$  then  $\mathbf{Th} \vdash \mathbf{True}(\bar{n})$ , (b) if  $n \notin T_{\mathbf{Th}}$  then  $\mathbf{Th} \vdash \neg \mathbf{True}(\bar{n})$ .

**Remark 3.4.2.** Notice it follows from (a)  $\wedge$  (b) that

$$\neg[(\mathbf{Th} \not\vdash \mathbf{True}(\bar{n})) \wedge (\mathbf{Th} \not\vdash \neg \mathbf{True}(\bar{n}))].$$

**Theorem 3.4.1.** (Tarski's undefinability Lemma) [10]. Let  $\mathbf{Th}$  be a consistent theory with

equality in the language  $\mathcal{L}$  in which the diagonal function  $D$  is representable and let  $g_{\mathbf{Th}}(u)$

be a Gödel number of given an expression  $u$  of  $\mathbf{Th}$ . Then the property  $x \in T_{\mathbf{Th}}$  is not expressible in  $\mathbf{Th}$ .

**Proof.** By the diagonalization lemma applied to  $\neg \mathbf{True}(x_1)$  there is a sentence  $\mathcal{F}$  such that: (c)  $\mathbf{Th} \vdash \mathcal{F} \Leftrightarrow \neg \mathbf{True}(\bar{q})$ , where  $q$  is the Gödel number of  $\mathcal{F}$ , i.e.  $g_{\mathbf{Th}}(\mathcal{F}) = q$ .

**Case 1.** Suppose that  $\mathbf{Th} \vdash \mathcal{F}$ , then  $q \in T_{\mathbf{Th}}$ . By (a),  $\mathbf{Th} \vdash \mathbf{True}(\bar{q})$ . But, from  $\mathbf{Th} \vdash \mathcal{F}$  and (c), by biconditional elimination, one obtains  $\mathbf{Th} \vdash \neg \mathbf{True}(\bar{q})$ . Hence  $\mathbf{Th}$  is inconsistent,

contradicting our hypothesis.

**Case 2.** Suppose that  $\mathbf{Th} \not\vdash \mathcal{F}$ . Then  $q \notin T_{\mathbf{Th}}$ . By (b),  $\mathbf{Th} \vdash \neg \mathbf{True}(\bar{q})$ . Hence, by (c) and

biconditional elimination,  $\mathbf{Th} \vdash \mathcal{F}$ . Thus, in either case a contradiction is reached.

**Definition 3.4.1.** If  $\mathbf{Th}$  is a theory, let  $T_{\mathbf{Th}}$  be the set of Gödel numbers of theorems of  $\mathbf{Th}$  and let  $g_{\mathbf{Th}}(u)$  be a Gödel number of given an expression  $u$  of  $\mathbf{Th}$ . The property  $x \in T_{\mathbf{Th}}$

is said to be strongly expressible in  $\mathbf{Th}$  by wff  $\mathbf{True}^*(x_1)$  if the following properties are

satisfied:

(a) if  $n \in T_{\mathbf{Th}}$  then  $\mathbf{Th} \vdash \mathbf{True}^*(\bar{n}) \wedge (\mathbf{True}^*(\bar{n}) \Rightarrow g_{\mathbf{Th}}^{-1}(n))$ ,

(b) if  $n \notin T_{\mathbf{Th}}$  then  $\mathbf{Th} \vdash \neg \mathbf{True}^*(\bar{n})$ .

**Theorem 3.4.2.** (Generalized Tarski's undefinability Lemma). Let  $\mathbf{Th}$  be a consistent theory

with equality in the language  $\mathcal{L}$  in which the diagonal function  $D$  is representable and let

$g_{\mathbf{Th}}(u)$  be a Gödel number of given an expression  $u$  of  $\mathbf{Th}$ . Then the property  $x \in T_{\mathbf{Th}}$  is not

strongly expressible in  $\mathbf{Th}$ .

**Proof.** By the diagonalization lemma applied to  $\neg \mathbf{True}^*(x_1)$  there is a sentence  $\mathcal{F}^*$  such

that: (c)  $\mathbf{Th} \vdash \mathcal{F}^* \Leftrightarrow \neg \mathbf{True}^*(\bar{q})$ , where  $q$  is the Gödel number of  $\mathcal{F}^*$ , i.e.  $g_{\mathbf{Th}}(\mathcal{F}^*) = q$ .

**Case 1.** Suppose that  $\mathbf{Th} \vdash \mathcal{F}^*$ , then  $q \in T_{\mathbf{Th}}$ . By (a),  $\mathbf{Th} \vdash \mathbf{True}^*(\bar{q})$ . But, from



$\mathbf{Th} \vdash \mathcal{F}^*$

and (c), by biconditional elimination, one obtains  $\mathbf{Th} \vdash \neg \mathbf{True}^*(\bar{q})$ . Hence  $\mathbf{Th}$  is inconsistent, contradicting our hypothesis.

**Case 2.** Suppose that  $\mathbf{Th} \not\vdash \mathcal{F}^*$ . Then  $q \notin T_{\mathbf{Th}}$ . By (b),  $\mathbf{Th} \vdash \neg \mathbf{True}^*(\bar{q})$ . Hence, by (c) and biconditional elimination,  $\mathbf{Th} \vdash \mathcal{F}^*$ . Thus, in either case a contradiction is reached.

**Remark 3.4.3.** Notice that it is widely believed on unnormal part of the mathematical community that Tarski's undefinability theorems 3.4.1-3.4.2 blocking any possible definitions of the sets  $\mathfrak{T}, \mathfrak{T}_{st}, \mathfrak{T}_{Nst}$ , mentioned in subsection 1.2 and therefore these theorems blocking definitions of the sets  $\mathfrak{R}, \mathfrak{R}_{st}, \mathfrak{R}_{Nst}$ , and correspondingly Tarski's undefinability theorem blocking the biconditionals

$$\begin{aligned} \mathfrak{R} \in \mathfrak{R} &\Leftrightarrow \mathfrak{R} \notin \mathfrak{R}, \mathfrak{R}_{st} \in \mathfrak{R}_{st} \Leftrightarrow \mathfrak{R}_{st} \notin \mathfrak{R}_{st}, \\ \mathfrak{R}_{Nst} \in \mathfrak{R}_{Nst} &\Leftrightarrow \mathfrak{R}_{Nst} \notin \mathfrak{R}_{Nst}. \end{aligned} \quad (3.4.1)$$

### 3.5. Generalized Tarski's undefinability theorem.

**Remark 3.5.1.(I)** Let  $\mathbf{Th}_1^\#$  be the theory  $\mathbf{Th}_1^\# \triangleq \overline{ZFC}_2^{Hs}$ .

In addition under assumption  $\widetilde{Con}(\mathbf{Th}_1^\#)$ , we establish a countable sequence of the consistent extensions of the theory  $\mathbf{Th}_1^\#$  such that:

- (i)  $\mathbf{Th}_1^\# \subsetneq \dots \subsetneq \mathbf{Th}_i^\# \subsetneq \mathbf{Th}_{i+1}^\# \subsetneq \dots \subsetneq \mathbf{Th}_\infty^\#$ , where
- (ii)  $\mathbf{Th}_{i+1}^\#$  is a finite consistent extension of  $\mathbf{Th}_i^\#$ ,
- (iii)  $\mathbf{Th}_\infty^\# = \bigcup_{i \in \mathbb{N}} \mathbf{Th}_i^\#$ ,
- (iv)  $\mathbf{Th}_\infty^\#$  proves the all sentences of  $\mathbf{Th}_1^\#$ , which valid in  $M$ , i.e.,  $M \models A \Rightarrow \mathbf{Th}_\infty^\# \vdash A$ , see Part II, section 2, Proposition 2.1.(i).

**(II)** Let  $\mathbf{Th}_{1,st}^\#$  be  $\mathbf{Th}_{1,st}^\# \triangleq ZFC_{st}$ .

In addition under assumption  $\widetilde{Con}(\mathbf{Th}_{1,st}^\#)$ , we establish a countable sequence of the consistent extensions of the theory  $\mathbf{Th}_{1,st}^\#$  such that:

- (i)  $\mathbf{Th}_{1,st}^\# \subsetneq \dots \subsetneq \mathbf{Th}_{i,st}^\# \subsetneq \mathbf{Th}_{i+1,st}^\# \subsetneq \dots \subsetneq \mathbf{Th}_{\infty,st}^\#$ , where
- (ii)  $\mathbf{Th}_{i+1,st}^\#$  is a finite consistent extension of  $\mathbf{Th}_{i,st}^\#$ ,
- (iii)  $\mathbf{Th}_{\infty,st}^\# = \bigcup_{i \in \mathbb{N}} \mathbf{Th}_{i,st}^\#$ ,
- (iv)  $\mathbf{Th}_{\infty,st}^\#$  proves the all sentences of  $\mathbf{Th}_{1,st}^\#$ , which valid in  $M_{st}^{ZFC}$ , i.e.,  $M_{st}^{ZFC} \models A \Rightarrow \mathbf{Th}_{\infty,st}^\# \vdash A$ ,

see Part II, section 2, Proposition 2.1.(ii).

**(III)** Let  $\mathbf{Th}_{1,Nst}^\#$  be  $\mathbf{Th}_{1,Nst}^\# \triangleq ZFC_{Nst}$ .

In addition under assumption  $\widetilde{Con}(\mathbf{Th}_{1,Nst}^\#)$ , we establish a countable sequence of the consistent extensions of the theory  $\mathbf{Th}_{1,Nst}^\#$  such that:

- (i)  $\mathbf{Th}_{1,Nst}^\# \subsetneq \dots \subsetneq \mathbf{Th}_{i,Nst}^\# \subsetneq \mathbf{Th}_{i+1,Nst}^\# \subsetneq \dots \subsetneq \mathbf{Th}_{\infty,Nst}^\#$ , where
- (ii)  $\mathbf{Th}_{i+1,Nst}^\#$  is a finite consistent extension of  $\mathbf{Th}_{i,Nst}^\#$ ,
- (iii)  $\mathbf{Th}_{\infty,Nst}^\# = \bigcup_{i \in \mathbb{N}} \mathbf{Th}_{i,Nst}^\#$
- (iv)  $\mathbf{Th}_{\infty,Nst}^\#$  proves the all sentences of  $\mathbf{Th}_{1,Nst}^\#$ , which valid in  $M_{Nst}^{ZFC}[PA]$ , i.e.,  $M_{Nst}^{ZFC}[PA] \models A \Rightarrow \mathbf{Th}_{\infty,Nst}^\# \vdash A$ ,

see Part II, section 2, Proposition 2.1.(iii).

**Remark 3.5.2.(I)** Let  $\mathfrak{T}_i, i = 1, 2, \dots$  be the set of the all sets of  $M$  provably definable in  $\mathbf{Th}_i^\#$ ,

$$\forall Y \{ Y \in \mathfrak{T}_i \leftrightarrow \square_i \exists \Psi(\cdot) \exists ! X [\Psi(X) \wedge Y = X] \}. \quad (3.5.1)$$

and let  $\mathfrak{R}_i = \{ x \in \mathfrak{T}_i : \square_i(x \notin x) \}$  where  $\square_i A$  means sentence  $A$  derivable in  $\mathbf{Th}_i^\#$ . Then

we have that  $\mathfrak{R}_i \in \mathfrak{R}_i$  if and only if  $\Box_i(\mathfrak{R}_i \notin \mathfrak{R}_i)$ , which immediately gives us  $\mathfrak{R}_i \in \mathfrak{R}_i$  if and only if  $\mathfrak{R}_i \notin \mathfrak{R}_i$ . We choose now  $\Box_i A, i = 1, 2, \dots$  in the following form

$$\Box_i A \triangleq Bew_i(\#A) \wedge [Bew_i(\#A) \Rightarrow A]. \quad (3.5.2)$$

Here  $Bew_i(\#A), i = 1, 2, \dots$  is a canonical Gödel formulae which says to us that there exist

proof in  $\mathbf{Th}_i^\#, i = 1, 2, \dots$  of the formula  $A$  with Gödel number  $\#A$ .

(II) Let  $\mathfrak{T}_{i,st}, i = 1, 2, \dots$  be the set of the all sets of  $M_{st}^{ZFC}$  provably definable in  $\mathbf{Th}_{i,st}^\#$ ,

$$\forall Y \{Y \in \mathfrak{T}_{i,st} \leftrightarrow \Box_{i,st} \exists \Psi(\cdot) \exists! X [\Psi(X) \wedge Y = X]\}. \quad (3.5.3)$$

and let  $\mathfrak{R}_{i,st} = \{x \in \mathfrak{T}_{i,st} : \Box_{i,st}(x \notin x)\}$  where  $\Box_{i,st} A$  means sentence  $A$  derivable in  $\mathbf{Th}_{i,st}^\#$ .

Then we have that  $\mathfrak{R}_{i,st} \in \mathfrak{R}_{i,st}$  if and only if  $\Box_{i,st}(\mathfrak{R}_{i,st} \notin \mathfrak{R}_{i,st})$ , which immediately gives us

$\mathfrak{R}_{i,st} \in \mathfrak{R}_{i,st}$  if and only if  $\mathfrak{R}_{i,st} \notin \mathfrak{R}_{i,st}$ . We choose now  $\Box_{i,st} A, i = 1, 2, \dots$  in the following form

$$\Box_{i,st} A \triangleq Bew_{i,st}(\#A) \wedge [Bew_{i,st}(\#A) \Rightarrow A]. \quad (3.5.4)$$

Here  $Bew_{i,st}(\#A), i = 1, 2, \dots$  is a canonical Gödel formulae which says to us that there exist proof in  $\mathbf{Th}_{i,st}^\#, i = 1, 2, \dots$  of the formula  $A$  with Gödel number  $\#A$ .

(III) Let  $\mathfrak{T}_{i,Nst}, i = 1, 2, \dots$  be the set of the all sets of  $M_{Nst}^{ZFC}[PA]$  provably definable in  $\mathbf{Th}_{i,Nst}^\#$ ,

$$\forall Y \{Y \in \mathfrak{T}_{i,Nst} \leftrightarrow \Box_{i,Nst} \exists \Psi(\cdot) \exists! X [\Psi(X) \wedge Y = X]\}. \quad (3.5.5)$$

and let  $\mathfrak{R}_{i,Nst} = \{x \in \mathfrak{T}_{i,Nst} : \Box_{i,Nst}(x \notin x)\}$  where  $\Box_{i,Nst} A$  means sentence  $A$  derivable in  $\mathbf{Th}_{i,Nst}^\#$ . Then we have that  $\mathfrak{R}_{i,Nst} \in \mathfrak{R}_{i,Nst}$  if and only if  $\Box_{i,Nst}(\mathfrak{R}_{i,Nst} \notin \mathfrak{R}_{i,Nst})$ , which immediately gives us  $\mathfrak{R}_{i,Nst} \in \mathfrak{R}_{i,Nst}$  if and only if  $\mathfrak{R}_{i,Nst} \notin \mathfrak{R}_{i,Nst}$ .

We choose now  $\Box_{i,Nst} A, i = 1, 2, \dots$  in the following form

$$\Box_{i,Nst} A \triangleq Bew_{i,Nst}(\#A) \wedge [Bew_{i,Nst}(\#A) \Rightarrow A]. \quad (3.5.6)$$

Here  $Bew_{i,Nst}(\#A), i = 1, 2, \dots$  is a canonical Gödel formulae which says to us that there exist proof in  $\mathbf{Th}_{i,Nst}^\#, i = 1, 2, \dots$  of the formula  $A$  with Gödel number  $\#A$ .

**Remark 3.5.3** Notice that definitions (3.5.2), (3.5.4) and (3.5.6) hold as definitions of predicates really asserting provability in  $\mathbf{Th}_i^\#, \mathbf{Th}_{i,st}^\#$  and  $\mathbf{Th}_{i,Nst}^\#, i = 1, 2, \dots$

correspondingly.

**Remark 3.5.4.** Of course the all theories  $\mathbf{Th}_i^\#, \mathbf{Th}_{i,st}^\#, \mathbf{Th}_{i,Nst}^\#, i = 1, 2, \dots$  are inconsistent, see

Part II, Proposition 2.10.(i)-(iii).

**Remark 3.5.5.(I)** Let  $\mathfrak{T}_\infty$  be the set of the all sets of  $M$  provably definable in  $\mathbf{Th}_\infty^\#$ ,

$$\forall Y \{Y \in \mathfrak{T}_\infty \leftrightarrow \Box_\infty \exists \Psi(\cdot) \exists! X [\Psi(X) \wedge Y = X]\}. \quad (3.5.7)$$

and let  $\mathfrak{R}_\infty = \{x \in \mathfrak{T}_\infty : \Box_\infty(x \notin x)\}$  where  $\Box_\infty A$  means 'sentence  $A$  derivable in  $\mathbf{Th}_\infty^\#$ . Then, we have that  $\mathfrak{R}_\infty \in \mathfrak{R}_\infty$  if and only if  $\Box_\infty(\mathfrak{R}_\infty \notin \mathfrak{R}_\infty)$ , which immediately gives us  $\mathfrak{R}_\infty \in \mathfrak{R}_\infty$  if and only if  $\mathfrak{R}_\infty \notin \mathfrak{R}_\infty$ . We choose now  $\Box_\infty A, i = 1, 2, \dots$  in the following form

$$\Box_\infty A \triangleq \exists i [Bew_i(\#A) \wedge [Bew_i(\#A) \Rightarrow A]]. \quad (3.5.8)$$

(II) Let  $\mathfrak{T}_{\infty,st}$  be the set of the all sets of  $M_{st}^{ZFC}$  provably definable in  $\mathbf{Th}_{\infty,st}^\#$ ,

$$\forall Y \{Y \in \mathfrak{T}_{\infty,st} \leftrightarrow \Box_{\infty,st} \exists \Psi(\cdot) \exists! X [\Psi(X) \wedge Y = X]\}. \quad (3.5.9)$$

and let  $\mathfrak{R}_{\infty, st}$  be the set  $\mathfrak{R}_{\infty, st} = \{x \in \mathfrak{T}_{\infty, st} : \Box_{\infty, st}(x \notin x)\}$ , where  $\Box_{\infty, st}A$  means ‘sentence  $A$  derivable in  $\mathbf{Th}_{\infty, st}^\#$ . Then, we have that  $\mathfrak{R}_{\infty, st} \in \mathfrak{R}_{\infty, st}$  if and only if  $\Box_{\infty, st}(\mathfrak{R}_{\infty, st} \notin \mathfrak{R}_{\infty, st})$ , which immediately gives us  $\mathfrak{R}_{\infty, st} \in \mathfrak{R}_{\infty, st}$  if and only if  $\mathfrak{R}_{\infty, st} \notin \mathfrak{R}_{\infty, st}$ . We choose now  $\Box_{\infty, st}A, i = 1, 2, \dots$  in the following form

$$\Box_{\infty, st}A \triangleq \exists i[Bew_{i, st}(\#A) \wedge [Bew_{i, st}(\#A) \Rightarrow A]]. \quad (3.5.10)$$

(III) Let  $\mathfrak{T}_{\infty, Nst}$  be the set of the all sets of  $M_{Nst}^{ZFC}[PA]$  provably definable in  $\mathbf{Th}_{\infty, Nst}^\#$ ,

$$\forall Y\{Y \in \mathfrak{T}_{\infty, Nst} \leftrightarrow \Box_{\infty, Nst}\exists\Psi(\cdot)\exists!X[\Psi(X) \wedge Y = X]\}. \quad (3.5.11)$$

and let  $\mathfrak{R}_{\infty, Nst}$  be the set  $\mathfrak{R}_{\infty, Nst} = \{x \in \mathfrak{T}_{\infty, Nst} : \Box_{\infty, Nst}(x \notin x)\}$  where  $\Box_{\infty, Nst}A$  means ‘sentence  $A$  derivable in  $\mathbf{Th}_{\infty, Nst}^\#$ . Then, we have that  $\mathfrak{R}_{\infty, Nst} \in \mathfrak{R}_{\infty, Nst}$  if and only if  $\Box_{\infty, Nst}(\mathfrak{R}_{\infty, Nst} \notin \mathfrak{R}_{\infty, Nst})$ , which immediately gives us  $\mathfrak{R}_{\infty, Nst} \in \mathfrak{R}_{\infty, Nst}$  if and only if  $\mathfrak{R}_{\infty, Nst} \notin \mathfrak{R}_{\infty, Nst}$ . We choose now  $\Box_{\infty, Nst}A, i = 1, 2, \dots$  in the following form

$$\Box_{\infty, Nst}A \triangleq \exists i[Bew_{i, Nst}(\#A) \wedge [Bew_{i, Nst}(\#A) \Rightarrow A]]. \quad (3.5.12)$$

**Remark 3.5.6.** Notice that definitions (3.5.8), (3.5.10) and (3.5.12) holds as definitions of a

predicate really asserting provability in  $\mathbf{Th}_\infty^\#, \mathbf{Th}_{\infty, st}^\#$  and  $\mathbf{Th}_{\infty, Nst}^\#$  correspondingly.

**Remark 3.5.7.** Of course all the theories  $\mathbf{Th}_\infty^\#, \mathbf{Th}_{\infty, st}^\#$  and  $\mathbf{Th}_{\infty, Nst}^\#$  are inconsistent, see Part II, Proposition 2.14.(i)-(iii).

**Remark 3.5.8.** Notice that under naive consideration the set  $\mathfrak{T}_\infty$  and  $\mathfrak{R}_\infty$  can be defined directly using a truth predicate, which of course is not available in the language of  $ZFC_2^{Hs}$  (but iff  $ZFC_2^{Hs}$  is consistent) by well-known Tarski’s undefinability theorem [10].

**Theorem 3.5.1. Tarski’s undefinability theorem: (I)** Let  $\mathbf{Th}_\mathcal{L}$  be first order theory with

formal language  $\mathcal{L}$ , which includes negation and has a Gödel numbering  $g(\circ)$  such that for

every  $\mathcal{L}$ -formula  $A(x)$  there is a formula  $B$  such that  $B \leftrightarrow A(g(B))$  holds. Assume that

$\mathbf{Th}_\mathcal{L}$

has a standard model  $M_{st}^{\mathbf{Th}_\mathcal{L}}$  and  $Con(\mathbf{Th}_{\mathcal{L}, st})$  where

$$\mathbf{Th}_{\mathcal{L}, st} \triangleq \mathbf{Th}_\mathcal{L} + \exists M_{st}^{\mathbf{Th}_\mathcal{L}}. \quad (3.5.13)$$

Let  $T^*$  be the set of Gödel numbers of  $\mathcal{L}$ -sentences true in  $M_{st}^{\mathbf{Th}_\mathcal{L}}$ . Then there is no  $\mathcal{L}$ -formula  $\mathbf{True}(n)$  (truth predicate) which defines  $T^*$ . That is, there is no  $\mathcal{L}$ -formula  $\mathbf{True}(n)$  such that for every  $\mathcal{L}$ -formula  $A$ ,

$$\mathbf{True}(g(A)) \leftrightarrow A \quad (3.5.14)$$

holds.

(II) Let  $\mathbf{Th}_\mathcal{L}^{Hs}$  be second order theory with Henkin semantics and formal language  $\mathcal{L}$ , which

includes negation and has a Gödel numbering

$g(\circ)$  such that for every  $\mathcal{L}$ -formula  $A(x)$  there is a formula  $B$  such that  $B \leftrightarrow A(g(B))$

holds.

Assume that  $\mathbf{Th}_\mathcal{L}^{Hs}$  has a standard model  $M_{st}^{\mathbf{Th}_\mathcal{L}^{Hs}}$  and  $Con(\mathbf{Th}_{\mathcal{L}, st}^{Hs})$ , where

$$\mathbf{Th}_{\mathcal{L}, st}^{Hs} \triangleq \mathbf{Th}_\mathcal{L}^{Hs} + \exists M_{st}^{\mathbf{Th}_\mathcal{L}^{Hs}} \quad (3.5.15)$$

Let  $T^*$  be the set of Gödel numbers of the all  $\mathcal{L}$ -sentences true in  $M$ . Then there is no  $\mathcal{L}$ -formula  $\mathbf{True}(n)$  (truth predicate) which defines  $T^*$ . That is, there is no  $\mathcal{L}$ -formula

$\mathbf{True}(n)$  such that for every  $\mathcal{L}$ -formula  $A$ ,

$$\mathbf{True}(g(A)) \Leftrightarrow A \quad (3.5.16)$$

holds.

**Remark 3.5.9.** Notice that the proof of Tarski's undefinability theorem in this form is again by simple reductio ad absurdum. Suppose that an  $\mathcal{L}$ -formula  $\mathbf{True}(n)$  defines  $T^*$ . In particular, if  $A$  is a sentence of  $\mathbf{Th}_{\mathcal{L}}$  then  $\mathbf{True}(g(A))$  holds in  $\mathbb{N}$  if and only if  $A$  is true in  $M_{st}^{\mathbf{Th}_{\mathcal{L}}}$ . Hence for all  $A$ , the Tarski  $T$ -sentence  $\mathbf{True}(g(A)) \Leftrightarrow A$  is true in  $M_{st}^{\mathbf{Th}_{\mathcal{L}}}$ . But the diagonal lemma yields a counterexample to this equivalence, by giving a "Liar" sentence  $S$  such that  $S \Leftrightarrow \neg \mathbf{True}(g(S))$  holds in  $M_{st}^{\mathbf{Th}_{\mathcal{L}}}$ . Thus no  $\mathcal{L}$ -formula  $\mathbf{True}(n)$  can define  $T^*$ .

**Remark 3.5.10.** Notice that the formal machinery of this proof is wholly elementary except for the diagonalization that the diagonal lemma requires. The proof of the diagonal lemma is likewise surprisingly simple; for example, it does not invoke recursive functions in any way. The proof does assume that every  $\mathcal{L}$ -formula has a Gödel number, but the specifics of a coding method are not required.

**Remark 3.5.11.** The undefinability theorem does not prevent truth in one consistent theory from being defined in a stronger theory. For example, the set of (codes for) formulas of first-order Peano arithmetic that are true in  $\mathbb{N}$  is definable by a formula in second order arithmetic. Similarly, the set of true formulas of the standard model of second order arithmetic (or  $n$ -th order arithmetic for any  $n$ ) can be defined by a formula in first-order  $ZFC$ .

**Remark 1.3.5.12.** Notice that it is widely believed on an unnormal part of mathematical community that Tarski's undefinability theorem blocking any possible definition of the sets

$\mathfrak{T}_{i \in \mathbb{N}}, \mathfrak{T}_{\infty}, \mathfrak{T}_{i, st}, \mathfrak{T}_{i, st}, \mathfrak{T}_{\infty, st}, \mathfrak{T}_{\infty, Nst}$ , and the sets  $\mathfrak{R}_{\infty}, \mathfrak{R}_{\infty, st}$ . Correspondingly Tarski's undefinability theorem blocking the biconditionals

$$\begin{aligned} \mathfrak{R}_i \in \mathfrak{R}_i &\Leftrightarrow \mathfrak{R}_i \notin \mathfrak{R}_i, i \in \mathbb{N}, \\ \mathfrak{R}_{\infty} \in \mathfrak{R}_{\infty} &\Leftrightarrow \mathfrak{R}_{\infty} \notin \mathfrak{R}_{\infty}, \text{etc.} \end{aligned} \quad (3.5.17)$$

Thus in contrast with naive definition of the sets  $\mathfrak{T}_{\infty}$  and  $\mathfrak{R}_{\infty}$  there is no any problem which arises from Tarski's undefinability theorem.

**Remark 3.5.13.(I)** We define again the set  $\mathfrak{T}_{\infty}$  but now by using generalized truth predicate  $\mathbf{True}_{\infty}^{\#}(g(A), A)$  such that

$$\begin{aligned} \mathbf{True}_{\infty}(g(A), A) &\Leftrightarrow \exists i [Bew_i(\#A) \wedge [Bew_i(\#A) \Rightarrow A]] \Leftrightarrow \\ \mathbf{True}_{\infty}(g(A)) \wedge [\mathbf{True}_{\infty}(g(A)) \Rightarrow A] &\Leftrightarrow A, \\ \mathbf{True}_{\infty}(g(A)) &\Leftrightarrow \exists i Bew_i(\#A). \end{aligned} \quad (3.5.18)$$

holds.

**(II)** We define the set  $\mathfrak{T}_{\infty, st}$  using generalized truth predicate  $\mathbf{True}_{\infty, st}^{\#}(g(A), A)$  such that

$$\begin{aligned} \mathbf{True}_{\infty, st}(g(A), A) &\Leftrightarrow \exists i [Bew_{i, st}(\#A) \wedge [Bew_{i, st}(\#A) \Rightarrow A]] \Leftrightarrow \\ \mathbf{True}_{\infty, st}(g(A)) \wedge [\mathbf{True}_{\infty, st}(g(A)) \Rightarrow A] &\Leftrightarrow A, \\ \mathbf{True}_{\infty, st}(g(A)) &\Leftrightarrow \exists i Bew_{i, st}(\#A) \end{aligned} \quad (3.5.19)$$

holds. Thus in contrast with naive definition of the sets  $\mathfrak{T}_{\infty}$  and  $\mathfrak{R}_{\infty}$  there is no any

problem

which arises from Tarski's undefinability theorem.

(III) We define the set  $\mathfrak{T}_{\infty, Nst}$  using generalized truth predicate  $\mathbf{True}_{\infty, Nst}^{\#}(g(A), A)$  such that

$$\begin{aligned} \mathbf{True}_{\infty, Nst}(g(A), A) &\Leftrightarrow \exists i [Bew_{i, Nst}(\#A) \wedge [Bew_{i, Nst}(\#A) \Rightarrow A]] \Leftrightarrow \\ &\mathbf{True}_{\infty, Nst}(g(A)) \wedge [\mathbf{True}_{\infty, Nst}(g(A)) \Rightarrow A] \Leftrightarrow A, \\ &\mathbf{True}_{\infty, Nst}(g(A)) \Leftrightarrow \exists i Bew_{i, Nst}(\#A) \end{aligned} \quad (3.5.20)$$

holds. Thus in contrast with naive definition of the sets  $\mathfrak{T}_{\infty, Nst}$  and  $\mathfrak{R}_{\infty, Nst}$  there is no any problem which arises from Tarski's undefinability theorem.

**Remark 3.5.14.** In order to prove that set theory  $ZFC_2^{Hs} + \exists M^{ZFC_2^{Hs}}$  is inconsistent without

any reference to the set  $\mathfrak{T}_{\infty}$ , notice that by the properties of the extension  $\mathbf{Th}_{\infty}^{\#}$  follows that

definition given by formula (1.5.18) is correct, i.e., for every  $ZFC_2^{Hs}$ -formula  $\Phi$  such that  $M^{ZFC_2^{Hs}} \models \Phi$  the following equivalence  $\Phi \Leftrightarrow \mathbf{True}_{\infty}(g(\Phi), \Phi)$  holds.

**Theorem 3.5.2. (Generalized Tarski's undefinability theorem)** (see Part II, section 2,

Proposition 2.30). Let  $\mathbf{Th}_{\mathcal{L}}$  be a first order theory or the second order theory with Henkin

semantics and with formal language  $\mathcal{L}$ , which includes negation and has a Gödel encoding

$g(\cdot)$  such that for every  $\mathcal{L}$ -formula  $A(x)$  there is a formula  $B$  such that the equivalence  $B \Leftrightarrow A(g(B)) \wedge [A(g(B)) \Rightarrow B]$  holds. Assume that  $\mathbf{Th}_{\mathcal{L}}$  has an standard Model  $M_{st}^{\mathbf{Th}}$ .

Then there is no  $\mathcal{L}$ -formula  $\mathbf{True}(n), n \in \mathbb{N}$ , such that for every  $\mathcal{L}$ -formula  $A$  such that  $M \models A$ , the following equivalence

$$A \Leftrightarrow \mathbf{True}(g(A)) \wedge [\mathbf{True}(g(A)) \Rightarrow A] \quad (3.5.21)$$

holds.

**Theorem 3.5.3.** (i) Set theory  $\mathbf{Th}_1^{\#} = ZFC_2^{Hs} + \exists M^{ZFC_2^{Hs}}$  is inconsistent;

(ii) Set theory  $\mathbf{Th}_{1, st}^{\#} = ZFC + \exists M_{st}^{ZFC}$  is inconsistent; (iii) Set theory  $\mathbf{Th}_{1, Nst}^{\#} = ZFC + \exists M_{Nst}^{ZFC}$  is

inconsistent; (see Part II, section 2, Proposition 2.31.(i)-(iii)).

**Proof.** (i) Notice that by the properties of the extension  $\mathbf{Th}_{\infty}^{\#}$  of the theory  $ZFC_2^{Hs} + \exists M^{ZFC_2^{Hs}} = \mathbf{Th}_1^{\#}$  follows that

$$M^{ZFC_2^{Hs}} \models \Phi \Rightarrow \mathbf{Th}_{\infty}^{\#} \vdash \Phi. \quad (3.5.22)$$

Therefore formula (3.5.18) gives generalized "truth predicate" for the set theory  $\mathbf{Th}_1^{\#}$ . By

Theorem 3.5.2 one obtains a contradiction.

(ii) Notice that by the properties of the extension  $\mathbf{Th}_{\infty, Nst}^{\#}$  of the theory  $ZFC + \exists M_{st}^{ZFC} = \mathbf{Th}_{1, st}^{\#}$  follows that

$$M_{st}^{ZFC} \models \Phi \Rightarrow \mathbf{Th}_{\infty, st}^{\#} \vdash \Phi. \quad (3.5.23)$$

Therefore formula (3.5.19) gives generalized "truth predicate" for the set theory  $\mathbf{Th}_{1, st}^{\#}$ . By

Theorem 3.5.2 one obtains a contradiction.

(iii) Notice that by the properties of the extension  $\mathbf{Th}_{\infty, Nst}^{\#}$  of the theory  $ZFC + \exists M_{Nst}^{ZFC} = \mathbf{Th}_{1, st}^{\#}$  follows that

$$M_{Nst}^{ZFC} \models \Phi \Rightarrow \mathbf{Th}_{\infty, Nst}^{\#} \vdash \Phi. \quad (3.5.24)$$

Therefore (3.5.20) gives generalized "truth predicate" for the set theory  $\mathbf{Th}_{1, Nst}^{\#}$ . By Theorem 3.5.2 one obtains a contradiction.

### 3.6. Avoiding the contradictions from set theory $\overline{ZFC}_2^{Hs}$ , $ZFC_{st}$ and set theory $ZFC_{Nst}$ using Quinean approach.

In order to avoid difficulties mentioned above we use well known Quinean approach.

#### 3.6.1. Quinean set theory $NF$ .

Remind that the primitive predicates of Russellian unramified typed set theory (TST), a streamlined version of the theory of types, are equality = and membership  $\in$ . TST has a linear hierarchy of types: type 0 consists of individuals otherwise undescribed. For each (meta-) natural number  $n$ , type  $n + 1$  objects are sets of type  $n$  objects; sets of type  $n$  have members of type  $n - 1$ . Objects connected by identity must have the same type. The following two atomic formulas succinctly describe the typing rules:  $x^n = y^n$  and  $x^n \in y^{n+1}$ .

The axioms of TST are:

**Extensionality:** sets of the same (positive) type with the same members are equal;

**Axiom schema of comprehension:**

If  $\Phi(x^n)$  is a formula, then the set  $\{x^n \mid \Phi(x^n)\}^{n+1}$  exists i.e., given any formula  $\Phi(x^n)$ , the formula

$$\exists A^{n+1} \forall x^n [x^n \in A^{n+1} \leftrightarrow \Phi(x^n)] \quad (3.6.1)$$

is an axiom where  $A^{n+1}$  represents the set  $\{x^n \mid \Phi(x^n)\}^{n+1}$  and is not free in  $\Phi(x^n)$ .

Quinean set theory.(New Foundations) seeks to eliminate the need for such superscripts.

New Foundations has a universal set, so it is a non-well founded set theory. That is to say, it is a logical theory that allows infinite descending chains of membership such as

...

$x_n \in x_{n-1} \in \dots x_3 \in x_2 \in x_1$ . It avoids Russell's paradox by only allowing stratifiable formulae in the axiom of comprehension. For instance  $x \in y$  is a stratifiable formula, but  $x \in x$  is not (for details of how this works see below).

**Definition 3.6.1.** In New Foundations ( $NF$ ) and related set theories, a formula  $\Phi$  in the language of first-order logic with equality and membership is said to be stratified if and only if there is a function  $\sigma$  which sends each variable appearing in  $\Phi$  [considered as an item of syntax] to a natural number (this works equally well if all integers are used) in such a way that any atomic formula  $x \in y$  appearing in  $\Phi$  satisfies  $\sigma(x) + 1 = \sigma(y)$  and any atomic formula  $x = y$  appearing in  $\Phi$  satisfies  $\sigma(x) = \sigma(y)$ .

#### Quinean set theory $NF$ .

**Axioms and stratification are:**

The well-formed formulas of New Foundations ( $NF$ ) are the same as the well-formed formulas of TST, but with the type annotations erased. The axioms of  $NF$  are:

**Extensionality:** Two objects with the same elements are the same object;

A comprehension schema: All instances of TST Comprehension but with type indices dropped (and without introducing new identifications between variables).

By convention, NF's Comprehension schema is stated using the concept of stratified formula and making no direct reference to types. Comprehension then becomes.

**Stratified Axiom schema of comprehension:**

$\{x \mid \Phi^s\}$  exists for each stratified formula  $\Phi^s$ .

Even the indirect reference to types implicit in the notion of stratification can be eliminated. Theodore Hailperin showed in 1944 that Comprehension is equivalent to a finite conjunction of its instances, so that  $NF$  can be finitely axiomatized without any reference to the notion of type. Comprehension may seem to run afoul of problems similar to those in naive set theory, but this is not the case. For example, the existence of the impossible Russell class  $\{x \mid x \notin x\}$  is not an axiom of  $NF$ , because  $x \notin x$  cannot be stratified.

### 3.6.2. Set theory $\overline{ZFC}_2^{Hs}$ , $ZFC_{st}$ and set theory $ZFC_{Nst}$ with stratified axiom schema of replacement.

The stratified axiom schema of replacement asserts that the image of a set under any function definable by stratified formula of the theory  $ZFC_{st}$  will also fall inside a set.

**Stratified Axiom schema of replacement:**

Let  $\Phi^s(x, y, w_1, w_2, \dots, w_n)$  be any stratified formula in the language of  $ZFC_{st}$  whose free variables are among  $x, y, A, w_1, w_2, \dots, w_n$ , so that in particular  $B$  is not free in  $\Phi^s$ . Then

$$\begin{aligned} \forall A \forall w_1 \forall w_2 \dots \forall w_n [\forall x (x \in A \Rightarrow \exists! y \Phi^s(x, y, w_1, w_2, \dots, w_n))] &\Rightarrow \\ \Rightarrow \exists B \forall x (x \in A \Rightarrow \exists y (y \in B \wedge \Phi^s(x, y, w_1, w_2, \dots, w_n))) &], \end{aligned} \quad (3.6.2)$$

i.e., if the relation  $\Phi^s(x, y, \dots)$  represents a definable function  $f$ ,  $A$  represents its domain, and  $f(x)$  is a set for every  $x \in A$ , then the range of  $f$  is a subset of some set  $B$ .

**Stratified Axiom schema of separation:**

Let  $\Phi^s(x, w_1, w_2, \dots, w_n)$  be any stratified formula in the language of  $ZFC_{st}$  whose free variables are among  $x, A, w_1, w_2, \dots, w_n$ , so that in particular  $B$  is not free in  $\Phi^s$ . Then

$$\forall w_1 \forall w_2 \dots \forall w_n \forall A \exists B \forall x [x \in B \Leftrightarrow (x \in A \wedge \Phi^s(x, w_1, w_2, \dots, w_n))], \quad (3.6.3)$$

**Remark 3.6.1.** Notice that the stratified axiom schema of separation follows from the stratified axiom schema of replacement together with the axiom of empty set.

**Remark 3.6.2.** Notice that the stratified axiom schema of replacement (separation) obviously violated any contradictions (2.1.20), (2.2.18) and (2.3.18) mentioned above. The existence of the countable Russell sets  $\mathfrak{R}_2^{*Hs}$ ,  $\mathfrak{R}_{st}^*$  and  $\mathfrak{R}_{Nst}^*$  impossible, because  $x \notin x$  cannot be stratified.

**Designation 3.6.1.**

## Part II. Generalized Löbs Theorem.

1.

## 2. Generalized Löbs Theorem

**Remark 2.1.** In this section we use second-order arithmetic  $Z_2^{Hs}$  with Henkin semantics.

Notice that any standard model  $M_{st}^{Z_2^{Hs}}$  of second-order arithmetic  $Z_2^{Hs}$  consists of a set  $\mathbb{N}$  of usual natural numbers (which forms the range of individual variables) together with a constant 0 (an element of  $\mathbb{N}$ ), a function  $S$  from  $\mathbb{N}$  to  $\mathbb{N}$ , two binary operations  $+$  and  $\cdot$  on  $\mathbb{N}$ , a binary relation  $<$  on  $\mathbb{N}$ , and a collection  $D \subseteq 2^{\mathbb{N}}$  of subsets of  $\mathbb{N}$ , which is the range of the set variables. Omitting  $D$  produces a model of the first order Peano arithmetic.

When  $D = 2^{\mathbb{N}}$  is the full powerset of  $\mathbb{N}$ , the model  $M_{st}^{Z_2}$  is called a full model. The use of full second-order semantics is equivalent to limiting the models of second-order arithmetic to the full models. In fact, the axioms of second-order arithmetic  $Z_2^{fss}$  have only one full model. This follows from the fact that the axioms of Peano arithmetic with the second-order induction axiom have only one model under second-order semantics, see section 3.

Let **Th** be some fixed, but unspecified, consistent formal theory. For later convenience, we assume that the encoding is done in some fixed formal second order theory **S** and that **Th** contains **S**. We assume throughout this paper that formal second order theory **S** has an  $\omega$ -model  $M_\omega^S$ . The sense in which **S** is contained in **Th** is better exemplified than explained: if **S** is a formal system of a second order arithmetic  $Z_2^{Hs}$  and **Th** is, say,  $ZFC_2^{Hs}$ , then **Th** contains **S** in the sense that there is a well-known embedding, or interpretation, of **S** in **Th**. Since encoding is to take place in  $M_\omega^S$ , it will have to have a large supply of constants and closed terms to be used as codes. (e.g. in formal arithmetic, one has  $\bar{0}, \bar{1}, \dots$ .) **S** will also have certain function symbols to be described shortly. To each formula,  $\Phi$ , of the language of **Th** is assigned a closed term,  $[\Phi]^c$ , called the code of  $\Phi$ . We note that if  $\Phi(x)$  is a formula with free variable  $x$ , then  $[\Phi(x)]^c$  is a closed term encoding the formula  $\Phi(x)$  with  $x$  viewed as a syntactic object and not as a parameter. Corresponding to the logical connectives and quantifiers are the function symbols,  $neg(\cdot)$ ,  $imp(\cdot)$ , etc., such that for all formulae

$\Phi, \Psi : \mathbf{S} \vdash neg([\Phi]^c) = [\neg\Phi]^c$ ,  $\mathbf{S} \vdash imp([\Phi]^c, [\Psi]^c) = [\Phi \rightarrow \Psi]^c$  etc. Of particular importance is the substitution operator, represented by the function symbol  $sub(\cdot, \cdot)$ . For formulae  $\Phi(x)$ , terms  $t$  with codes  $[t]^c$  :

$$\mathbf{S} \vdash sub([\Phi(x)]^c, [t]^c) = [\Phi(t)]^c. \quad (2.1)$$

It is well known [8] that one can also encode derivations and have a binary relation  $\mathbf{Prov}_{\mathbf{Th}}(x, y)$  (read " $x$  proves  $y$ " or " $x$  is a proof of  $y$ ") such that for closed  $t_1, t_2 : \mathbf{S} \vdash \mathbf{Prov}_{\mathbf{Th}}(t_1, t_2)$  iff  $t_1$  is the code of a derivation in **Th** of the formula with code  $t_2$ . It follows that

$$\mathbf{Th} \vdash \Phi \text{ iff } \mathbf{S} \vdash \mathbf{Prov}_{\mathbf{Th}}(t, [\Phi]^c) \quad (2.2)$$

for some closed term  $t$ . Thus one can define

$$\mathbf{Pr}_{\mathbf{Th}}(y) \leftrightarrow \exists x \mathbf{Prov}_{\mathbf{Th}}(x, y), \quad (2.3)$$

and therefore one obtain a predicate asserting provability.

**Remark 2.2. (I)** We note that it is not always the case that [8]:

$$\mathbf{Th} \vdash \Phi \text{ iff } \mathbf{S} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^c), \quad (2.4)$$

unless **S** is fairly sound, e.g. this is a case when **S** and **Th** replaced by  $\mathbf{S}_\omega = \mathbf{S} \upharpoonright M_\omega^{\mathbf{Th}}$  and  $\mathbf{Th}_\omega = \mathbf{Th} \upharpoonright M_\omega^{\mathbf{Th}}$  correspondingly (see Designation 2.1 below).

**(II)** Notice that it is always the case that:



$$\mathbf{Th}_\omega \vdash \Phi_\omega \text{ iff } \mathbf{S}_\omega \vdash \mathbf{Pr}_{\mathbf{Th}_\omega}([\Phi_\omega]^c), \quad (2.5)$$

i.e. that is the case when predicate  $\mathbf{Pr}_{\mathbf{Th}_\omega}(y), y \in M_\omega^{\mathbf{Th}}$  :

$$\mathbf{Pr}_{\mathbf{Th}_\omega}(y) \leftrightarrow \exists x(x \in M_\omega^{\mathbf{Th}}) \mathbf{Prov}_{\mathbf{Th}_\omega}(x, y) \quad (2.6)$$

really asserts provability.

It is well known [8] that the above encoding can be carried out in such a way that the following important conditions **D1**, **D2** and **D3** are meet for all sentences [8]:

$$\begin{aligned} \mathbf{D1. Th} \vdash \Phi \text{ implies } \mathbf{S} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^c), \\ \mathbf{D2. S} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^c) \rightarrow \mathbf{Pr}_{\mathbf{Th}}([\mathbf{Pr}_{\mathbf{Th}}([\Phi]^c)]^c), \\ \mathbf{D3. S} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^c) \wedge \mathbf{Pr}_{\mathbf{Th}}([\Phi \rightarrow \Psi]^c) \rightarrow \mathbf{Pr}_{\mathbf{Th}}([\Psi]^c). \end{aligned} \quad (2.7)$$

Conditions **D1**, **D2** and **D3** are called the Derivability Conditions.

**Remark 2.3.** From (2.5)-(2.6) follows that

$$\begin{aligned} \mathbf{D4. Th}_\omega \vdash \Phi \text{ iff } \mathbf{S}_\omega \vdash \mathbf{Pr}_{\mathbf{Th}_\omega}([\Phi_\omega]^c), \\ \mathbf{D5. S}_\omega \vdash \mathbf{Pr}_{\mathbf{Th}_\omega}([\Phi_\omega]^c) \leftrightarrow \mathbf{Pr}_{\mathbf{Th}_\omega}([\mathbf{Pr}_{\mathbf{Th}_\omega}([\Phi_\omega]^c)]^c), \\ \mathbf{D6. S}_\omega \vdash \mathbf{Pr}_{\mathbf{Th}_\omega}([\Phi_\omega]^c) \wedge \mathbf{Pr}_{\mathbf{Th}_\omega}([\Phi_\omega \rightarrow \Psi_\omega]^c) \rightarrow \mathbf{Pr}_{\mathbf{Th}_\omega}([\Psi_\omega]^c). \end{aligned} \quad (2.8)$$

Conditions **D4**, **D5** and **D6** are called the Strong Derivability Conditions.

**Definition 2.1.** Let  $\Phi$  be well formed formula (wff) of **Th**. Then wff  $\Phi$  is called **Th**-sentence iff it has no free variables.

**Designation 2.1.** (i) Assume that a theory **Th** has an  $\omega$ -model  $M_\omega^{\mathbf{Th}}$  and  $\Phi$  is a **Th**-sentence, then:

$\Phi_{M_\omega^{\mathbf{Th}}} \triangleq \Phi \upharpoonright M_\omega^{\mathbf{Th}}$  (we will write  $\Phi_\omega$  instead  $\Phi_{M_\omega^{\mathbf{Th}}}$ ) is a **Th**-sentence  $\Phi$  with all quantifiers relativized to  $\omega$ -model  $M_\omega^{\mathbf{Th}}$  [11] and

$\mathbf{Th}_\omega \triangleq \mathbf{Th} \upharpoonright M_\omega^{\mathbf{Th}}$  is a theory **Th** relativized to model  $M_\omega^{\mathbf{Th}}$ , i.e., any **Th** $_\omega$ -sentence has the form  $\Phi_\omega$  for some **Th**-sentence  $\Phi$ .

(ii) Assume that a theory **Th** has a standard model  $M_{St}^{\mathbf{Th}}$  and  $\Phi$  is a **Th**-sentence, then:

(iii) Assume that a theory **Th** has a non-standard model  $M_{Nst}^{\mathbf{Th}}$  and  $\Phi$  is a **Th**-sentence, then:

$\Phi_{M_{Nst}^{\mathbf{Th}}} \triangleq \Phi \upharpoonright M_{Nst}^{\mathbf{Th}}$  (we will write  $\Phi_{Nst}$  instead  $\Phi_{M_{Nst}^{\mathbf{Th}}}$ ) is a **Th**-sentence with all quantifiers relativized to non-standard model  $M_{Nst}^{\mathbf{Th}}$ , and

$\mathbf{Th}_{Nst} \triangleq \mathbf{Th} \upharpoonright M_{Nst}^{\mathbf{Th}}$  is a theory **Th** relativized to model  $M_{Nst}^{\mathbf{Th}}$ , i.e., any **Th** $_{Nst}$ -sentence has a form  $\Phi_{Nst}$  for some **Th**-sentence  $\Phi$ .

(iv) Assume that a theory **Th** has a model  $M = M^{\mathbf{Th}}$  and  $\Phi$  is a **Th**-sentence, then:

$\Phi_M \triangleq \Phi \upharpoonright M$  is a **Th**-sentence with all quantifiers relativized to model  $M^{\mathbf{Th}}$ , and  $\mathbf{Th}_M$  is a theory **Th** relativized to model  $M^{\mathbf{Th}}$ , i.e. any **Th** $_M$ -sentence has a form  $\Phi_M$  for some **Th**-sentence  $\Phi$ .

**Designation 2.2.** (i) Assume that a theory **Th** with a language  $\mathcal{L}$  has an  $\omega$ -model  $M_\omega^{\mathbf{Th}}$  and

there exists **Th**-sentence  $S_\mathcal{L}$  such that: (a)  $S_\mathcal{L}$  expressible by language  $\mathcal{L}$  and (b)  $S_\mathcal{L}$  asserts

that **Th** has a model  $M_\omega^{\mathbf{Th}}$ ; we denote such **Th**-sentence  $S_\mathcal{L}$  by  $Con(\mathbf{Th}; M_\omega^{\mathbf{Th}})$ .

(ii) Assume that a theory **Th** with a language  $\mathcal{L}$  has a non-standard model  $M_{Nst}^{\mathbf{Th}}$  and there

exists  $\mathbf{Th}$ -sentence  $S_{\mathcal{L}}$  such that: (a)  $S_{\mathcal{L}}$  expressible by language  $\mathcal{L}$  and (b)  $S_{\mathcal{L}}$  asserts that  $\mathbf{Th}$  has a non-standard model  $M_{Nst}^{\mathbf{Th}}$ ; we denote such  $\mathbf{Th}$ -sentence  $S_{\mathcal{L}}$  by  $Con(\mathbf{Th}; M_{Nst}^{\mathbf{Th}})$ .

(iii) Assume that a theory  $\mathbf{Th}$  with a language  $\mathcal{L}$  has an model  $M^{\mathbf{Th}}$  and there exists  $\mathbf{Th}$ -sentence  $S_{\mathcal{L}}$  such that: (a)  $S_{\mathcal{L}}$  expressible by language  $\mathcal{L}$  and (b)  $S_{\mathcal{L}}$  asserts that  $\mathbf{Th}$

has a model  $M^{\mathbf{Th}}$ ; we denote such  $\mathbf{Th}$ -sentence  $S_{\mathcal{L}}$  by  $Con(\mathbf{Th}; M^{\mathbf{Th}})$

**Remark 2.4.** We emphasize that: (i) it is well known that there exist a  $ZFC$ -sentence  $Con(ZFC; M^{ZFC})$  [10],[11],(ii) obviously there exists a  $ZFC_2^{Hs}$ -sentence  $Con(ZFC_2^{Hs}; M^{ZFC_2^{Hs}})$

and there exists a  $Z_2^{Hs}$ -sentence  $Con(Z_2^{Hs}; M^{Z_2^{Hs}})$ .

**Designation 2.3.** Let  $\widetilde{Con}(\mathbf{Th})$  be the formula:

$$\left\{ \begin{array}{l} \widetilde{Con}(\mathbf{Th}) \triangleq \\ \forall t_1(t_1 \in M_{\omega}^{\mathbf{Th}}) \forall t'_1(t'_1 \in M_{\omega}^{\mathbf{Th}}) \forall t_2(t_2 \in M_{\omega}^{\mathbf{Th}}) \forall t'_2(t'_2 \in M_{\omega}^{\mathbf{Th}}) \\ \neg[\mathbf{Prov}_{\mathbf{Th}}(t_1, [\Phi]^c) \wedge \mathbf{Prov}_{\mathbf{Th}}(t_2, neg([\Phi]^c))], \\ \\ t'_1 = [\Phi]^c, t'_2 = neg([\Phi]^c) \\ \text{or} \\ \widetilde{Con}(\mathbf{Th}) \triangleq \\ \forall \Phi \forall t_1(t_1 \in M_{\omega}^{\mathbf{Th}}) \forall t_2(t_2 \in M_{\omega}^{\mathbf{Th}}) \neg[\mathbf{Prov}_{\mathbf{Th}}(t_1, [\Phi]^c) \wedge \mathbf{Prov}_{\mathbf{Th}}(t_2, neg([\Phi]^c))] \end{array} \right. \quad (2.9)$$

and where  $t_1, t'_1, t_2, t'_2$  is a closed term.

**Lemma 2.1.** (I) Assume that: (i)  $Con(\mathbf{Th}; M^{\mathbf{Th}})$ , (ii)  $M^{\mathbf{Th}} \models \widetilde{Con}(\mathbf{Th})$  and (iii)  $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^c)$ , where  $\Phi$  is a closed formula. Then  $\mathbf{Th} \not\vdash \mathbf{Pr}_{\mathbf{Th}}([\neg\Phi]^c)$ ,

(II) Assume that: (i)  $Con(\mathbf{Th}; M_{\omega}^{\mathbf{Th}})$  (ii)  $M_{\omega}^{\mathbf{Th}} \models \widetilde{Con}(\mathbf{Th})$  and (iii)  $\mathbf{Th}_{\omega} \vdash \mathbf{Pr}_{\mathbf{Th}_{\omega}}([\Phi_{\omega}]^c)$ ,

where

$\Phi_{\omega}$  is a closed formula. Then  $\mathbf{Th}_{\omega} \not\vdash \mathbf{Pr}_{\mathbf{Th}_{\omega}}([\neg\Phi_{\omega}]^c)$ .

**Proof.** (I) Let  $\widetilde{Con}_{\mathbf{Th}}(\Phi)$  be the formula :

$$\left\{ \begin{array}{l} \widetilde{Con}_{\mathbf{Th}}(\Phi) \triangleq \\ \forall t_1(t_1 \in M_{\omega}^{\mathbf{Th}}) \forall t_2(t_2 \in M_{\omega}^{\mathbf{Th}}) \neg[\mathbf{Prov}_{\mathbf{Th}}(t_1, [\Phi]^c) \wedge \mathbf{Prov}_{\mathbf{Th}}(t_2, neg([\Phi]^c))], \\ \forall t_1(t_1 \in M_{\omega}^{\mathbf{Th}}) \forall t_2(t_2 \in M_{\omega}^{\mathbf{Th}}) \neg[\mathbf{Prov}_{\mathbf{Th}}(t_1, [\Phi]^c) \wedge \mathbf{Prov}_{\mathbf{Th}}(t_2, neg([\Phi]^c))] \leftrightarrow \\ \leftrightarrow \{ \neg \exists t_1(t_1 \in M_{\omega}^{\mathbf{Th}}) \neg \exists t_2(t_2 \in M_{\omega}^{\mathbf{Th}}) [\mathbf{Prov}_{\mathbf{Th}}(t_1, [\Phi]^c) \wedge \mathbf{Prov}_{\mathbf{Th}}(t_2, neg([\Phi]^c))] \}. \end{array} \right. \quad (2.10)$$

where  $t_1, t_2$  is a closed term. From (i)-(ii) follows that theory  $\mathbf{Th} + \widetilde{Con}(\mathbf{Th})$  is consistent.

We note that  $\mathbf{Th} + \widetilde{Con}(\mathbf{Th}) \vdash \widetilde{Con}_{\mathbf{Th}}(\Phi)$  for any closed  $\Phi$ . Suppose that  $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\neg\Phi]^c)$ , then (iii) gives

$$\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^c) \wedge \mathbf{Pr}_{\mathbf{Th}}([\neg\Phi]^c). \quad (2.11)$$

From (2.3) and (2.11) we obtain

$$\exists t_1 \exists t_2 [\mathbf{Prov}_{\mathbf{Th}}(t_1, [\Phi]^c) \wedge \mathbf{Prov}_{\mathbf{Th}}(t_2, neg([\Phi]^c))]. \quad (2.12)$$

But the formula (2.10) contradicts the formula (2.12). Therefore  $\mathbf{Th} \not\vdash \mathbf{Pr}_{\mathbf{Th}}([\neg\Phi]^c)$ .

(II) This case is trivial because formula  $\mathbf{Pr}_{\mathbf{Th}_{\omega}}([\neg\Phi_{\omega}]^c)$  by the Strong Derivability

Condition **D4**, see formulae (2.8), really asserts provability of the  $\mathbf{Th}_\omega$ -sentence  $\neg\Phi_\omega$ . But this is a contradiction.

**Lemma 2.2.** (I) Assume that: (i)  $\text{Con}(\mathbf{Th}; M^{\mathbf{Th}})$ , (ii)  $M^{\mathbf{Th}} \models \widetilde{\text{Con}}(\mathbf{Th})$  and (iii)  $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\neg\Phi]^c)$ , where  $\Phi$  is a closed formula. Then  $\mathbf{Th} \not\vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^c)$ ,

(II) Assume that: (i)  $\text{Con}(\mathbf{Th}; M_\omega^{\mathbf{Th}})$  (ii)  $M_\omega^{\mathbf{Th}} \models \widetilde{\text{Con}}(\mathbf{Th})$  and (iii)  $\mathbf{Th}_\omega \vdash \mathbf{Pr}_{\mathbf{Th}_\omega}([\neg\Phi_\omega]^c)$ , where  $\Phi_\omega$  is a closed formula. Then  $\mathbf{Th}_\omega \not\vdash \mathbf{Pr}_{\mathbf{Th}_\omega}([\Phi_\omega]^c)$ .

**Proof.** Similarly as Lemma 2.1 above.

**Example 2.1.** (i) Let  $\mathbf{Th} = \mathbf{PA}$  be Peano arithmetic and  $\Phi \Leftrightarrow 0 = 1$ . Then obviously by Löb's theorem  $\mathbf{PA} \vdash \mathbf{Pr}_{\mathbf{PA}}(0 \neq 1)$ , and therefore by Lemma 2.1  $\mathbf{PA} \not\vdash \mathbf{Pr}_{\mathbf{PA}}(0 = 1)$ .

(ii) Let  $\mathbf{PA}^* = \mathbf{PA} + \neg\widetilde{\text{Con}}(\mathbf{PA})$  and  $\Phi \Leftrightarrow 0 = 1$ . Then obviously by Löb's theorem

$$\mathbf{PA}^* \vdash \mathbf{Pr}_{\mathbf{PA}^*}(0 \neq 1),$$

and therefore

$$\mathbf{PA}^* \not\vdash \mathbf{Pr}_{\mathbf{PA}^*}(0 = 1).$$

However obviously

$$\mathbf{PA}^* \vdash [\mathbf{Pr}_{\mathbf{PA}}(0 \neq 1)] \wedge [\mathbf{Pr}_{\mathbf{PA}}(0 = 1)].$$

**Remark 2.5.** Notice that there is no standard model of  $\mathbf{PA}^*$ .

**Assumption 2.1.** Let  $\mathbf{Th}$  be a first order or a second order theory with the Henkin semantics. We assume now that:

(i) the language of  $\mathbf{Th}$  consists of:

numerals  $\bar{0}, \bar{1}, \dots$

countable set of the numerical variables:  $\{v_0, v_1, \dots\}$

countable set  $\mathcal{F}$  of the set variables:  $\mathcal{F} = \{x, y, z, X, Y, Z, \mathfrak{X}, \mathfrak{Y}, \dots\}$

countable set of the  $n$ -ary function symbols:  $f_0^n, f_1^n, \dots$

countable set of the  $n$ -ary relation symbols:  $R_0^n, R_1^n, \dots$

connectives:  $\neg, \rightarrow$

quantifier:  $\forall$ .

(ii)  $\mathbf{Th}$  contains  $ZFC_2^{Hs}$  or  $ZFC$

(iii)  $\mathbf{Th}$  has an  $\omega$ -model  $M_\omega^{\mathbf{Th}}$  or

(iv)  $\mathbf{Th}$  has a nonstandard model  $M_{Nst}^{\mathbf{Th}}[PA]$ .

**Definition 2.1.** A  $\mathbf{Th}$ -wff  $\Phi$  (well-formed formula  $\Phi$ ) is closed - i.e.  $\Phi$  is a sentence - if it has no free variables; a wff is open if it has free variables. We'll use the slang ' $k$ -place open wff' to mean a wff with  $k$  distinct free variables.

**Definition 2.2.** We will say that,  $\mathbf{Th}_\omega^\#$  is a nice theory or a nice extension of the  $\mathbf{Th}$  iff the

following

(i)  $\mathbf{Th}_\omega^\#$  contains  $\mathbf{Th}$ ;

(ii) Let  $\Phi$  be any closed formula of  $\mathbf{Th}$ , then  $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^c)$  implies  $\mathbf{Th}_\omega^\# \vdash \Phi$ ;

(iii) Let  $\Phi_\omega$  be any closed formula of  $\mathbf{Th}_\omega^\#$ , then  $M_\omega^{\mathbf{Th}} \models \Phi_\omega$  implies  $\mathbf{Th}_\omega^\# \vdash \Phi_\omega$ , i.e.

$\text{Con}(\mathbf{Th} + \Phi_\omega; M_\omega^{\mathbf{Th}})$  implies  $\mathbf{Th}_\omega^\# \vdash \Phi_\omega$ .

**Remark 2.6.** Notice that formulae  $\text{Con}(\mathbf{Th} + \Phi_\omega; M_\omega^{\mathbf{Th}})$  and  $\text{Con}(\mathbf{Th}_\omega^\# + \Phi_\omega; M_\omega^{\mathbf{Th}})$  are expressible in  $\mathbf{Th}_\omega^\#$ .

**Definition 2.3.** Let  $L$  be a classical propositional logic  $L$ . Recall that a set  $\Delta$  of  $L$ -wff's is said to be  $L$ -consistent, or consistent for short, if  $\Delta \not\vdash \perp$  and there are other equivalent formulations of consistency: (1)  $\Delta$  is consistent, (2)  $\text{Ded}(\Delta) := \{A \mid \Delta \vdash A\}$  is not the

set

of all wff's,(3) there is a formula such that  $\Delta \nVdash A$ . (4) there are no formula  $A$  such that  $\Delta \vdash A$  and  $\Delta \vdash \neg A$ .

We will say that,  $\mathbf{Th}_\infty^\#$  is a maximally nice theory or a maximally nice extension of the  $\mathbf{Th}$  iff

$\mathbf{Th}_\infty^\#$  is consistent and for any consistent nice extension  $\mathbf{Th}_\infty^{\#i}$  of the  $\mathbf{Th}$  :

$\mathbf{Ded}(\mathbf{Th}_\infty^\#) \subseteq \mathbf{Ded}(\mathbf{Th}_\infty^{\#i})$  implies  $\mathbf{Ded}(\mathbf{Th}_\infty^\#) = \mathbf{Ded}(\mathbf{Th}_\infty^{\#i})$ .

**Remark 2.7.** We note that a theory  $\mathbf{Th}_\infty^\#$  depend on model  $M_\omega^{\mathbf{Th}}$  or  $M_{Nst}^{\mathbf{Th}}$ , i.e.

$\mathbf{Th}_\infty^\# = \mathbf{Th}_\infty^\#[M_\omega^{\mathbf{Th}}]$  or  $\mathbf{Th}_\infty^\# = \mathbf{Th}_\infty^\#[M_{Nst}^{\mathbf{Th}}]$  correspondingly. We will consider now the case

$\mathbf{Th}_\infty^\# \triangleq \mathbf{Th}_\infty^\#[M_\omega^{\mathbf{Th}}]$  without loss of generality.

**Remark 2.8.a.** Notice that in order to prove the statements: (i)  $\neg \text{Con}(ZFC_2^{Hs}; M_\omega^{\mathbf{Th}})$ ,

(ii)  $\neg \text{Con}(ZFC; M_\omega^{\mathbf{Th}})$  the following Proposition 2.1 is not necessary, see Proposition

2.18.

**Proposition 2.1.(Generalized Löbs Theorem).**

(I) Assume that:

(i)  $\widetilde{\text{Con}}(\mathbf{Th})$ , where predicate  $\widetilde{\text{Con}}(\mathbf{Th})$  defined by formula 2.9

(ii)  $\mathbf{Th}$  has an  $\omega$ -model  $M_\omega^{\mathbf{Th}}$ , and

(iii) the statement  $\exists M_\omega^{\mathbf{Th}}$  is expressible by language of  $\mathbf{Th}$  as a single sentence of  $\mathbf{Th}$ .

Then theory  $\mathbf{Th}$  can be extended to a maximally consistent nice theory

$\mathbf{Th}_{\infty, st}^\# = \mathbf{Th}_{\infty, st}^\#[M_\omega^{\mathbf{Th}}]$ . Below we write for short  $\mathbf{Th}_{\infty, st}^\# \triangleq \mathbf{Th}_\infty^\# = \mathbf{Th}_\infty^\#[M_\omega^{\mathbf{Th}}]$ .

**Remark 2.8.b.** We emphasize that (iii) valid for  $ZFC$  despite the fact that the axioms of  $ZFC$  are infinite, see [10] Chapter II, section 7, p.78.

(II) Assume that:

(i)  $\widetilde{\text{Con}}(\mathbf{Th})$ , where predicate  $\widetilde{\text{Con}}(\mathbf{Th})$  defined by formula 2.9,

(ii)  $\mathbf{Th}$  has an  $\omega$ -model  $M_\omega^{\mathbf{Th}}$  and

(iii) the statement  $\exists M_\omega^{\mathbf{Th}}$  is expressible by language of  $\mathbf{Th}$  as a single sentence of  $\mathbf{Th}$ .

Then theory  $\mathbf{Th}_\omega \triangleq \mathbf{Th} \upharpoonright M_\omega^{\mathbf{Th}}$  can be extended to a maximally consistent nice theory

$\mathbf{Th}_\omega^\#$ .

(III) Assume that:

(i)  $\widetilde{\text{Con}}(\mathbf{Th})$ , where predicate  $\widetilde{\text{Con}}(\mathbf{Th})$  defined by formula 2.9,

(ii)  $\mathbf{Th}$  has a nonstandard model  $M_{Nst}^{\mathbf{Th}}[PA]$  and

(iii) the statement  $\exists M_{Nst}^{\mathbf{Th}}[PA]$  is expressible by language of  $\mathbf{Th}$  as a single sentence of

$\mathbf{Th}$ .

Then theory  $\mathbf{Th}$  can be extended to a maximally consistent nice theory

$\mathbf{Th}_{\infty, Nst}^\# = \mathbf{Th}_{\infty, Nst}^\#[M_{Nst}^{\mathbf{Th}}]$ .

**Remark 2.8.c.** We emphasize that (iii) valid for  $ZFC$  despite the fact that the axioms of  $ZFC$  are infinite, see [10] Ch.II, section 7, p.78.

**Proof.(I)** Let  $\Phi_1 \dots \Phi_i \dots$  be an enumeration of all closed wff's of the theory  $\mathbf{Th}$  (this can

be achieved if the set of propositional variables can be enumerated). Define a chain

$\emptyset = \{\mathbf{Th}_{i, st}^\# | i \in \mathbb{N}\}$ ,  $\mathbf{Th}_{1, st}^\# = \mathbf{Th}$  of consistent theories inductively as follows: assume

that

theory  $\mathbf{Th}_{i, st}^\#$  is defined. Notice that below we write for short  $\mathbf{Th}_{i, st}^\# \triangleq \mathbf{Th}_i^\#$ .

(i) Suppose that the statement (2.13) is satisfied

$$[\mathbf{Th}_i^\# \vDash \mathbf{Pr}_{\mathbf{Th}_i^\#}([\Phi_i]^c)] \wedge [\mathbf{Th}_i^\# \vDash \Phi_i] \text{ and } M_\omega^{\mathbf{Th}} \vDash \Phi_i. \quad (2.13)$$

Then we define a theory  $\mathbf{Th}_{i+1}^\#$  as follows  $\mathbf{Th}_{i+1}^\# \triangleq \mathbf{Th}_i^\# \cup \{\Phi_i\}$ . We will rewrite the condition

(2.13) using predicate  $\mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}(\cdot)$  symbolically as follows:

$$\left\{ \begin{array}{l} \mathbf{Th}_{i+1}^\# \vdash \mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}([\Phi_i]^c), \\ \mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}([\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i^\#}([\Phi_i]^c) \wedge [M_\omega^{\mathbf{Th}} \vDash \Phi_i], \\ M_\omega^{\mathbf{Th}} \vDash \Phi_i \Leftrightarrow \text{Con}(\mathbf{Th}_i^\# + \Phi_i; M_\omega^{\mathbf{Th}}), \\ \text{i.e.} \\ \mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}([\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i^\#}([\Phi_i]^c) \wedge \text{Con}(\mathbf{Th}_i^\# + \Phi_i; M_\omega^{\mathbf{Th}}), \\ \mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}([\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}([\Phi_i]^c), \\ \mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}([\Phi_i]^c) \Rightarrow \Phi_i, \\ \mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}([\Phi_i]^c) \Rightarrow \Phi_i. \end{array} \right. \quad (2.14)$$

(ii) Suppose that the statement (2.15) is satisfied

$$[\mathbf{Th}_i^\# \vDash \mathbf{Pr}_{\mathbf{Th}_i^\#}([\neg\Phi_i]^c)] \wedge [\mathbf{Th}_i^\# \vDash \neg\Phi_i] \text{ and } M_\omega^{\mathbf{Th}} \vDash \neg\Phi_i. \quad (2.15)$$

Then we define a theory  $\mathbf{Th}_{i+1}^\#$  as follows  $\mathbf{Th}_{i+1}^\# \triangleq \mathbf{Th}_i^\# \cup \{\Phi_i\}$ . We will rewrite the condition

(2.15) using predicate  $\mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}(\cdot)$ , symbolically as follows:

$$\left\{ \begin{array}{l} \mathbf{Th}_{i+1}^\# \vdash \mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}([\neg\Phi_i]^c), \\ \mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}([\neg\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i^\#}([\neg\Phi_i]^c) \wedge [M_\omega^{\mathbf{Th}} \vDash \neg\Phi_i], \\ M_\omega^{\mathbf{Th}} \vDash \neg\Phi_i \Leftrightarrow \text{Con}(\mathbf{Th}_i^\# + \neg\Phi_i; M_\omega^{\mathbf{Th}}), \\ \text{i.e.} \\ \mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}([\neg\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i^\#}([\neg\Phi_i]^c) \wedge \text{Con}(\mathbf{Th}_i^\# + \neg\Phi_i; M_\omega^{\mathbf{Th}}), \\ \mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}([\neg\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}([\neg\Phi_i]^c), \\ \mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}([\neg\Phi_i]^c) \Rightarrow \neg\Phi_i, \\ \mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}([\Phi_i]^c) \Rightarrow \neg\Phi_i. \end{array} \right. \quad (2.16)$$

(iii) Suppose that the statement (2.17) is satisfied

$$\mathbf{Th}_i^\# \vdash \mathbf{Pr}_{\mathbf{Th}_i^\#}([\Phi_i]^c) \text{ and } [\mathbf{Th}_i^\# \vDash \Phi_i] \wedge [M_\omega^{\mathbf{Th}} \vDash \Phi_i]. \quad (2.17)$$

Then we define a theory  $\mathbf{Th}_{i+1}^\#$  as follows  $\mathbf{Th}_{i+1}^\# \triangleq \mathbf{Th}_i^\# \cup \{\Phi_i\}$ . Using Lemma 2.1 and predicate  $\mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}(\cdot)$ , we will rewrite the condition (2.17) symbolically as follows:

$$\left\{ \begin{array}{l}
\mathbf{Th}_{i+1}^\# \vdash \mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}^\#([\Phi_i]^c), \\
\mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}^\#([\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i^\#}^\#([\Phi_i]^c) \wedge [M_\omega^{\mathbf{Th}} \models \Phi_i], \\
M_\omega^{\mathbf{Th}} \models \Phi_i \Leftrightarrow \text{Con}(\mathbf{Th}_i^\# + \Phi_i; M_\omega^{\mathbf{Th}}), \\
\text{i.e.} \\
\mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}^\#([\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i^\#}^\#([\Phi_i]^c) \wedge \text{Con}(\mathbf{Th}_i^\# + \Phi_i; M_\omega^{\mathbf{Th}}), \\
\mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}^\#([\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}^\#([\Phi_i]^c), \\
\mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}^\#([\Phi_i]^c) \Rightarrow \Phi_i, \\
\mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}^\#([\Phi_i]^c) \Rightarrow \Phi_i.
\end{array} \right. \quad (2.18)$$

**Remark 2.9.** Notice that predicate  $\mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}^\#([\Phi_i]^c)$  is expressible in  $\mathbf{Th}_i^\#$  because  $\mathbf{Th}_i^\#$  is a finite extension of the recursive theory  $\mathbf{Th}$  and  $\text{Con}(\mathbf{Th}_i^\# + \Phi_i; M_\omega^{\mathbf{Th}}) \in \mathbf{Th}_i^\#$ .

(iv) Suppose that a statement (2.19) is satisfied

$$\mathbf{Th}_i^\# \vdash \mathbf{Pr}_{\mathbf{Th}_i^\#}^\#([\neg\Phi_i]^c) \text{ and } [\mathbf{Th}_i^\# \vdash \neg\Phi_i] \wedge [M_\omega^{\mathbf{Th}} \models \neg\Phi_i]. \quad (2.19)$$

Then we define theory  $\mathbf{Th}_{i+1}^\#$  as follows:  $\mathbf{Th}_{i+1}^\# \triangleq \mathbf{Th}_i^\# \cup \{\neg\Phi_i\}$ . Using Lemma 2.2 and predicate  $\mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}^\#(\cdot)$ , we will rewrite the condition (2.15) symbolically as follows

$$\left\{ \begin{array}{l}
\mathbf{Th}_i^\# \vdash \mathbf{Pr}_{\mathbf{Th}_i^\#}^\#([\neg\Phi_i]^c), \\
\mathbf{Pr}_{\mathbf{Th}_i^\#}^\#([\neg\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i^\#}^\#([\neg\Phi_i]^c) \wedge [M_\omega^{\mathbf{Th}} \models \neg\Phi_i], \\
M_\omega^{\mathbf{Th}} \models \neg\Phi_i \Leftrightarrow \text{Con}(\mathbf{Th}_i^\# + \neg\Phi_i; M_\omega^{\mathbf{Th}}), \\
\text{i.e.} \\
\mathbf{Pr}_{\mathbf{Th}_i^\#}^\#([\neg\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i^\#}^\#([\neg\Phi_i]^c) \wedge \text{Con}(\mathbf{Th}_i^\# + \neg\Phi_i; M_\omega^{\mathbf{Th}}), \\
\mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}^\#([\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}^\#([\Phi_i]^c), \\
\mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}^\#([\Phi_i]^c) \Rightarrow \Phi_i, \\
\mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}^\#([\Phi_i]^c) \Rightarrow \Phi_i.
\end{array} \right. \quad (2.20)$$

**Remark 2.10.** Notice that predicate  $\mathbf{Pr}_{\mathbf{Th}_i^\#}^\#([\neg\Phi_i]^c)$  is expressible in  $\mathbf{Th}_i^\#$  because  $\mathbf{Th}_i^\#$  is a finite extension of the recursive theory  $\mathbf{Th}$  and  $\text{Con}(\mathbf{Th}_i^\# + \neg\Phi_i; M_\omega^{\mathbf{Th}}) \in \mathbf{Th}_i^\#$ .

(v) Suppose that the statement (2.21) is satisfied

$$\mathbf{Th}_i^\# \vdash \mathbf{Pr}_{\mathbf{Th}_i^\#}^\#([\Phi_i]^c) \text{ and } \mathbf{Th}_i^\# \vdash \mathbf{Pr}_{\mathbf{Th}_i^\#}^\#([\Phi_i]^c) \Rightarrow \Phi_i. \quad (2.21)$$

We will rewrite now the conditions (2.21) symbolically as follows

$$\left\{ \begin{array}{l}
\mathbf{Th}_i^\# \vdash \mathbf{Pr}_{\mathbf{Th}_i^\#}^*([\Phi_i]^c) \\
\mathbf{Pr}_{\mathbf{Th}_i^\#}^*([\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i^\#}^*([\Phi_i]^c) \wedge [\mathbf{Pr}_{\mathbf{Th}_i^\#}^*([\Phi_i]^c) \Rightarrow \Phi_i]
\end{array} \right. \quad (2.22)$$

Then we define a theory  $\mathbf{Th}_{i+1}^\#$  as follows:  $\mathbf{Th}_{i+1}^\# \triangleq \mathbf{Th}_i^\#$ .

(iv) Suppose that the statement (2.23) is satisfied

$$\mathbf{Th}_i^\# \vdash \mathbf{Pr}_{\mathbf{Th}_i^\#}^\#([\neg\Phi_i]^c) \text{ and } \mathbf{Th}_i^\# \vdash \mathbf{Pr}_{\mathbf{Th}_i^\#}^\#([\neg\Phi_i]^c) \Rightarrow \neg\Phi_i. \quad (2.23)$$

We will rewrite now the condition (2.23) symbolically as follows

$$\left\{ \begin{array}{l} \mathbf{Th}_i^\# \vdash \mathbf{Pr}_{\mathbf{Th}_i^\#}^*([\neg\Phi_i]^c) \\ \mathbf{Pr}_{\mathbf{Th}_i^\#}^*([\neg\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i^\#}([\neg\Phi_i]^c) \wedge [\mathbf{Pr}_{\mathbf{Th}_i^\#}([\neg\Phi_i]^c) \Rightarrow \neg\Phi_i] \end{array} \right. \quad (2.24)$$

Then we define a theory  $\mathbf{Th}_{i+1}^\#$  as follows:  $\mathbf{Th}_{i+1}^\# \triangleq \mathbf{Th}_i^\#$ . We define now a theory  $\mathbf{Th}_\infty^\#$  as follows:

$$\mathbf{Th}_\infty^\# \triangleq \bigcup_{i \in \mathbb{N}} \mathbf{Th}_i^\#. \quad (2.25)$$

First, notice that each  $\mathbf{Th}_i^\#$  is consistent. This is done by induction on  $i$  and by Lemmas 2.1-2.2. By assumption, the case is true when  $i = 1$ . Now, suppose  $\mathbf{Th}_i^\#$  is consistent. Then its deductive closure  $\mathbf{Ded}(\mathbf{Th}_i^\#)$  is also consistent. If the statement (2.14) is satisfied, i.e.  $\mathbf{Th}_{i+1}^\# \vdash \mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}^\#([\Phi_i]^c)$  and  $\mathbf{Th}_{i+1}^\# \vdash \Phi_i$ , then clearly  $\mathbf{Th}_{i+1}^\# \triangleq \mathbf{Th}_i^\# \cup \{\Phi_i\}$  is consistent since it is a subset of closure  $\mathbf{Ded}(\mathbf{Th}_{i+1}^\#)$ . If a statement (2.16) is satisfied, i.e.  $\mathbf{Th}_{i+1}^\# \vdash \mathbf{Pr}_{\mathbf{Th}_{i+1}^\#}^\#([\neg\Phi_i]^c)$  and  $\mathbf{Th}_{i+1}^\# \vdash \neg\Phi_i$ , then clearly  $\mathbf{Th}_{i+1}^\# \triangleq \mathbf{Th}_i^\# \cup \{\neg\Phi_i\}$  is consistent since it is a subset of closure  $\mathbf{Ded}(\mathbf{Th}_{i+1}^\#)$ . If the statement (2.18) is satisfied, i.e.  $\mathbf{Th}_i^\# \vdash \mathbf{Pr}_{\mathbf{Th}_i^\#}([\Phi_i]^c)$  and  $[\mathbf{Th}_i^\# \not\vdash \Phi_i] \wedge [M_\omega^{\mathbf{Th}} \models \Phi_i]$  then clearly  $\mathbf{Th}_{i+1}^\# \triangleq \mathbf{Th}_i^\# \cup \{\Phi_i\}$  is consistent by Lemma 2.1 and by one of the standard properties of consistency:  $\Delta \cup \{A\}$  is consistent iff  $\Delta \not\vdash \neg A$ . If the statement (2.20) is satisfied, i.e.  $\mathbf{Th}_i^\# \vdash \mathbf{Pr}_{\mathbf{Th}_i^\#}([\neg\Phi_i]^c)$  and  $[\mathbf{Th}_i^\# \not\vdash \neg\Phi_i] \wedge [M_\omega^{\mathbf{Th}} \models \neg\Phi_i]$  then clearly  $\mathbf{Th}_{i+1}^\# \triangleq \mathbf{Th}_i^\# \cup \{\neg\Phi_i\}$  is consistent by Lemma 2.2 and by one of the standard properties of consistency:  $\Delta \cup \{\neg A\}$  is consistent iff  $\Delta \not\vdash A$ . Next, notice  $\mathbf{Ded}(\mathbf{Th}_\infty^\#)$  is maximally consistent nice extension of the  $\mathbf{Ded}(\mathbf{Th})$ .  $\mathbf{Ded}(\mathbf{Th}_\infty^\#)$  is consistent because, by the standard Lemma 2.3 below, it is the union of a chain of consistent sets. To see that  $\mathbf{Ded}(\mathbf{Th}_\infty^\#)$  is maximal, pick any wff  $\Phi$ . Then  $\Phi$  is some  $\Phi_i$  in the enumerated list of all wff's. Therefore for any  $\Phi$  such that  $\mathbf{Th}_i \vdash \mathbf{Pr}_{\mathbf{Th}_i}([\Phi]^c)$  or  $\mathbf{Th}_i^\# \vdash \mathbf{Pr}_{\mathbf{Th}_i^\#}([\neg\Phi]^c)$ , either  $\Phi \in \mathbf{Th}_\infty^\#$  or  $\neg\Phi \in \mathbf{Th}_\infty^\#$ . Since  $\mathbf{Ded}(\mathbf{Th}_{i+1}^\#) \subseteq \mathbf{Ded}(\mathbf{Th}_\infty^\#)$ , we have  $\Phi \in \mathbf{Ded}(\mathbf{Th}_\infty^\#)$  or  $\neg\Phi \in \mathbf{Ded}(\mathbf{Th}_\infty^\#)$ , which implies that  $\mathbf{Ded}(\mathbf{Th}_\infty^\#)$  is maximally consistent nice extension of the  $\mathbf{Ded}(\mathbf{Th})$ .

**Proof.(II)** Let  $\Phi_{\omega,1} \dots \Phi_{\omega,i} \dots$  be an enumeration of all closed wff's of the theory  $\mathbf{Th}_\omega$  (this can be achieved if the set of propositional variables can be enumerated). Define a chain  $\wp = \{\mathbf{Th}_{\omega,i}^\# | i \in \mathbb{N}\}$ ,  $\mathbf{Th}_{\omega,1}^\# = \mathbf{Th}_\omega$  of consistent theories inductively as follows: assume that theory  $\mathbf{Th}_{\omega,i}^\#$  is defined.

(i) Suppose that a statement (2.26) is satisfied

$$\mathbf{Th}_{\omega,i}^\# \not\vdash \mathbf{Pr}_{\mathbf{Th}_{\omega,i}^\#}([\Phi_{\omega,i}]^c) \text{ and } M_\omega^{\mathbf{Th}} \models \Phi_i. \quad (2.26)$$

Then we define a theory  $\mathbf{Th}_{\omega,i+1}^\#$  as follows

$$\mathbf{Th}_{\omega,i+1}^\# \triangleq \mathbf{Th}_{\omega,i}^\# \cup \{\Phi_{\omega,i}\}. \quad (2.27)$$

We will rewrite now the conditions (2.26) and (2.27) symbolically as follows

$$\left\{ \begin{array}{l} \mathbf{Th}_{\omega,i+1}^\# \vdash \mathbf{Pr}_{\mathbf{Th}_{\omega,i+1}^\#}([\Phi_{\omega,i}]^c) \Leftrightarrow \mathbf{Th}_{\omega,i+1}^\# \vdash \Phi_{\omega,i}, \\ \mathbf{Pr}_{\mathbf{Th}_{\omega,i+1}^\#}([\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{\omega,i+1}^\#}([\Phi_i]^c) \wedge \Phi_{\omega,i}. \end{array} \right. \quad (2.28)$$

(ii) Suppose that a statement (2.29) is satisfied

$$\mathbf{Th}_{\omega,i}^{\#} \not\vdash \mathbf{Pr}_{\mathbf{Th}_{\omega,i}^{\#}}([\neg\Phi_{\omega,i}]^c) \text{ and } M_{\omega}^{\mathbf{Th}} \models \neg\Phi_i. \quad (2.29)$$

Then we define theory  $\mathbf{Th}_{\omega,i+1}^{\#}$  as follows:

$$\mathbf{Th}_{\omega,i+1}^{\#} \triangleq \mathbf{Th}_{\omega,i}^{\#} \cup \{\neg\Phi_{\omega,i}\}. \quad (2.30)$$

We will rewrite the conditions (2.25) and (2.26) symbolically as follows

$$\left\{ \begin{array}{l} \mathbf{Th}_{\omega,i+1} \vdash \mathbf{Pr}_{\mathbf{Th}_{\omega,i+1}}([\neg\Phi_{\omega,i}]^c) \Leftrightarrow \mathbf{Th}_{\omega,i+1} \vdash \neg\Phi_{\omega,i}, \\ \mathbf{Pr}_{\mathbf{Th}_{\omega,i+1}}^{\#}([\neg\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{\omega,i+1}}([\neg\Phi_i]^c). \end{array} \right. \quad (2.31)$$

(iii) Suppose that the following statement (2.32) is satisfied

$$\mathbf{Th}_{\omega,i} \vdash \mathbf{Pr}_{\mathbf{Th}_{\omega,i}}([\Phi_{\omega,i}]^c), \quad (2.32)$$

and therefore by Derivability Conditions (2.8)

$$\mathbf{Th}_{\omega,i} \vdash \Phi_{\omega,i}. \quad (2.33)$$

We will rewrite now the conditions (2.28) and (2.29) symbolically as follows

$$\mathbf{Pr}_{\mathbf{Th}_{\omega,i}}^*([\Phi_{\omega,i}]^c) \Leftrightarrow \mathbf{Th}_{\omega,i} \vdash \mathbf{Pr}_{\mathbf{Th}_{\omega,i}}([\Phi_{\omega,i}]^c) \quad (2.34)$$

Then we define a theory  $\mathbf{Th}_{\omega,i+1}$  as follows:  $\mathbf{Th}_{\omega,i+1} \triangleq \mathbf{Th}_{\omega,i}$ .

(iv) Suppose that the following statement (2.35) is satisfied

$$\mathbf{Th}_{\omega,i} \vdash \mathbf{Pr}_{\mathbf{Th}_{\omega,i}}([\neg\Phi_{\omega,i}]^c), \quad (2.35)$$

and therefore by Derivability Conditions (2.8)

$$\mathbf{Th}_{\omega,i} \vdash \neg\Phi_{\omega,i}. \quad (2.36)$$

We will rewrite now the conditions (2.35) and (2.36) symbolically as follows

$$\mathbf{Pr}_{\mathbf{Th}_{\omega,i}}^*([\neg\Phi_{\omega,i}]^c) \Leftrightarrow \mathbf{Th}_{\omega,i} \vdash \mathbf{Pr}_{\mathbf{Th}_{\omega,i}}([\neg\Phi_{\omega,i}]^c) \quad (2.37)$$

Then we define a theory  $\mathbf{Th}_{\omega,i+1}$  as follows:  $\mathbf{Th}_{\omega,i+1} \triangleq \mathbf{Th}_{\omega,i}$ . We define now a theory  $\mathbf{Th}_{\infty;\omega}^{\#}$  as follows:

$$\mathbf{Th}_{\infty;\omega}^{\#} \triangleq \bigcup_{i \in \mathbb{N}} \mathbf{Th}_{\omega,i}. \quad (2.38)$$

First, notice that each  $\mathbf{Th}_{\omega,i}$  is consistent. This is done by induction on  $i$ . Now, suppose  $\mathbf{Th}_{\omega,i}$  is consistent. Then its deductive closure  $\mathbf{Ded}(\mathbf{Th}_{\omega,i})$  is also consistent. If statement (2.22) is satisfied, i.e.  $\mathbf{Th}_{\omega,i} \not\vdash \mathbf{Pr}_{\mathbf{Th}_{\omega,i}}([\Phi_{\omega,i}]^c)$  and  $M_{\omega}^{\mathbf{Th}} \models \Phi_i$  then clearly  $\mathbf{Th}_{\omega,i+1} \triangleq \mathbf{Th}_{\omega,i} \cup \{\Phi_{\omega,i}\}$  is consistent. If statement (2.25) is satisfied, i.e.  $\mathbf{Th}_{\omega,i} \not\vdash \mathbf{Pr}_{\mathbf{Th}_{\omega,i}}([\neg\Phi_{\omega,i}]^c)$  and  $M_{\omega}^{\mathbf{Th}} \models \neg\Phi_i$ , then clearly  $\mathbf{Th}_{\omega,i+1} \triangleq \mathbf{Th}_{\omega,i} \cup \{\neg\Phi_{\omega,i}\}$  is consistent. If the statement (2.28) is satisfied, i.e.  $\mathbf{Th}_{\omega,i} \vdash \mathbf{Pr}_{\mathbf{Th}_{\omega,i}}([\Phi_{\omega,i}]^c)$ , then clearly  $\mathbf{Th}_{\omega,i+1} \triangleq \mathbf{Th}_{\omega,i}$  is also consistent. If the statement (2.35) is satisfied, i.e.  $\mathbf{Th}_{\omega,i} \vdash \mathbf{Pr}_{\mathbf{Th}_{\omega,i}}([\neg\Phi_{\omega,i}]^c)$ , then clearly  $\mathbf{Th}_{\omega,i+1} \triangleq \mathbf{Th}_{\omega,i}$  is also consistent. Next, notice  $\mathbf{Ded}(\mathbf{Th}_{\infty;\omega}^{\#})$  is a maximally consistent nice extension of the  $\mathbf{Ded}(\mathbf{Th}_{\infty;\omega})$ . The set  $\mathbf{Ded}(\mathbf{Th}_{\infty;\omega}^{\#})$  is consistent because, by the standard Lemma 2.3 below, it is the union of a chain of consistent sets.

**Lemma 2.3.** The union of a chain  $\wp = \{\Gamma_i \mid i \in \mathbb{N}\}$  of consistent sets  $\Gamma_i$ , ordered by  $\subseteq$ , is



consistent.

**Definition 2.4. (I)** We define now predicate  $\mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\Phi]^c)$  and predicate  $\mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\neg\Phi]^c)$  asserting provability in  $\mathbf{Th}_\infty^\#$  by the following formulae

$$\left\{ \begin{array}{l} \mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\Phi]^c) \Leftrightarrow \left\{ \exists i(\Phi \in \mathbf{Th}_i^\#) \left[ \mathbf{Pr}_{\mathbf{Th}_i^\#}([\Phi]^c) \right] \vee \left[ \mathbf{Pr}_{\mathbf{Th}_i^\#}^*([\Phi]^c) \right] \right\} \vee \\ \vee [(\Phi \in \mathbf{Th}_\infty^\#) \wedge \text{Con}(\mathbf{Th}_\infty^\# + \Phi; M_\omega^{\text{Th}})], \\ \text{Con}(\mathbf{Th}_\infty^\# + \Phi; M_\omega^{\text{Th}}) \Leftrightarrow \\ \mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\neg\Phi]^c) \Leftrightarrow \left\{ \exists i(\Phi \in \mathbf{Th}_i^\#) \left[ \mathbf{Pr}_{\mathbf{Th}_i^\#}([\neg\Phi]^c) \right] \vee \left[ \mathbf{Pr}_{\mathbf{Th}_i^\#}^*([\neg\Phi]^c) \right] \right\} \vee \\ \vee [(\Phi \in \mathbf{Th}_\infty^\#) \wedge \text{Con}(\mathbf{Th}_\infty^\# + \neg\Phi; M_\omega^{\text{Th}})], \\ \text{Con}(\mathbf{Th}_\infty^\# + \neg\Phi; M_\omega^{\text{Th}}) \Leftrightarrow \end{array} \right. \quad (2.39)$$

**(II)** We define now predicate  $\mathbf{Pr}_{\mathbf{Th}_{\infty;\omega}^\#}([\Phi_\omega]^c)$  and predicate  $\mathbf{Pr}_{\mathbf{Th}_{\infty;\omega}^\#}([\neg\Phi_\omega]^c)$  asserting provability in  $\mathbf{Th}_{\infty;\omega}^\#$  by the following formulae

$$\left\{ \begin{array}{l} \mathbf{Pr}_{\mathbf{Th}_{\infty;\omega}^\#}([\Phi_\omega]^c) \Leftrightarrow \\ \left\{ \exists i(\Phi_\omega \in \mathbf{Th}_{\omega,i}^\#) \left[ \mathbf{Pr}_{\mathbf{Th}_{\omega,i}^\#}([\Phi_\omega]^c) \right] \vee \left[ \mathbf{Pr}_{\mathbf{Th}_{\omega,i}^\#}^*([\Phi_\omega]^c) \right] \right\} \vee \\ \vee [(\Phi_\omega \in \mathbf{Th}_{\infty;\omega}^\#) \wedge \text{Con}(\mathbf{Th}_{\infty;\omega}^\# + \Phi_\omega; M_\omega^{\text{Th}})], \\ \text{Con}(\mathbf{Th}_{\infty;\omega}^\# + \Phi_\omega; M_\omega^{\text{Th}}) \Leftrightarrow \\ \mathbf{Pr}_{\mathbf{Th}_{\infty;\omega}^\#}([\neg\Phi_\omega]^c) \Leftrightarrow \\ \left\{ \exists i(\Phi_\omega \in \mathbf{Th}_{\omega,i}^\#) \left[ \mathbf{Pr}_{\mathbf{Th}_{\omega,i}^\#}([\neg\Phi_\omega]^c) \right] \vee \left[ \mathbf{Pr}_{\mathbf{Th}_{\omega,i}^\#}^*([\neg\Phi_\omega]^c) \right] \right\} \vee \\ \vee [(\Phi_\omega \in \mathbf{Th}_{\infty;\omega}^\#) \wedge \text{Con}(\mathbf{Th}_{\infty;\omega}^\# + \neg\Phi_\omega; M_\omega^{\text{Th}})], \\ \text{Con}(\mathbf{Th}_{\infty;\omega}^\# + \neg\Phi_\omega; M_\omega^{\text{Th}}) \Leftrightarrow \end{array} \right. \quad (2.40)$$

**Remark 2.11.(I)** Notice that both predicate  $\mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\Phi]^c)$  and predicate  $\mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\neg\Phi]^c)$  are

expressible in  $\mathbf{Th}_\infty^\#$  because for any  $i \in \mathbb{N}$ ,  $\mathbf{Th}_i^\#$  is an finite extension of the recursive theory

$\mathbf{Th}$  and  $\text{Con}(\mathbf{Th}_i^\# + \Phi; M^{\text{Th}}) \in \mathbf{Th}_i$ ,  $\text{Con}(\mathbf{Th}_i^\# + \neg\Phi; M^{\text{Th}}) \in \mathbf{Th}_i$ .

**(II)** Notice that both predicate  $\mathbf{Pr}_{\mathbf{Th}_{\infty;\omega}^\#}([\Phi_\omega]^c)$  and predicate  $\mathbf{Pr}_{\mathbf{Th}_{\infty;\omega}^\#}([\neg\Phi_\omega]^c)$  are expressible

in  $\mathbf{Th}_{\infty;\omega}^\#$  because for any  $i \in \mathbb{N}$ ,  $\mathbf{Th}_{\omega,i}^\#$  is an finite extension of the recursive theory  $\mathbf{Th}_\omega$  and

$\text{Con}(\mathbf{Th}_{\omega,i}^\# + \Phi_\omega; M^{\text{Th}}) \in \mathbf{Th}_{\omega,i}$ ,  $\text{Con}(\mathbf{Th}_{\omega,i}^\# + \neg\Phi_\omega; M^{\text{Th}}) \in \mathbf{Th}_{\omega,i}$ .

**Definition 2.5.** Let  $\Psi = \Psi(x)$  be one-place open  $\mathbf{Th}$ -wff such that the following condition:

$$\mathbf{Th} \triangleq \mathbf{Th}_1^\# \vdash \exists! x_\Psi [\Psi(x_\Psi)] \quad (2.41)$$

is satisfied.

**Remark 2.12.** We rewrite now the condition (2.41) using only the language of the theory

$\mathbf{Th}_1^\#$  :

$$\begin{aligned} \{\mathbf{Th}_1^\# \vdash \exists!x_\Psi[\Psi(x_\Psi)]\} &\Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_1^\#}([\exists!x_\Psi[\Psi(x_\Psi)]])^c \wedge \\ &\wedge \{\mathbf{Pr}_{\mathbf{Th}_1^\#}([\exists!x_\Psi[\Psi(x_\Psi)]])^c \Rightarrow \exists!x_\Psi[\Psi(x_\Psi)]\}. \end{aligned} \quad (2.42)$$

**Definition 2.6.** We will say that, a set  $y$  is a  $\mathbf{Th}_1^\#$ -set if there exist one-place open wff  $\Psi(x)$

such that  $y = x_\Psi$ . We write  $y[\mathbf{Th}_1^\#]$  iff  $y$  is a  $\mathbf{Th}_1^\#$ -set.

**Remark 2.13.** Note that

$$\begin{aligned} y[\mathbf{Th}_1^\#] &\Leftrightarrow \exists\Psi \left\{ (y = x_\Psi) \wedge \mathbf{Pr}_{\mathbf{Th}_1^\#}([\exists!x_\Psi[\Psi(x_\Psi)]])^c \right. \\ &\left. \wedge \{\mathbf{Pr}_{\mathbf{Th}_1^\#}([\exists!x_\Psi[\Psi(x_\Psi)]])^c \Rightarrow \exists!x_\Psi[\Psi(x_\Psi)]\} \right\}. \end{aligned} \quad (2.43)$$

**Definition 2.7.** Let  $\mathfrak{S}_1$  be a collection such that :

$$\forall x [x \in \mathfrak{S}_1 \leftrightarrow x \text{ is a } \mathbf{Th}_1^\# \text{-set}]. \quad (2.44)$$

**Proposition 2.2.** Collection  $\mathfrak{S}_1$  is a  $\mathbf{Th}_1^\#$ -set.

**Proof.** Let us consider an one-place open wff  $\Psi(x)$  such that conditions (2.41) are satisfied, i.e.  $\mathbf{Th}_1^\# \vdash \exists!x_\Psi[\Psi(x_\Psi)]$ . We note that there exists countable collection  $\mathcal{F}_\Psi$  of the

one-place open wff's  $\mathcal{F}_\Psi = \{\Psi_n(x)\}_{n \in \mathbb{N}}$  such that: (i)  $\Psi(x) \in \mathcal{F}_\Psi$  and (ii)

$$\left\{ \begin{aligned} \mathbf{Th} \triangleq \mathbf{Th}_1^\# \vdash \exists!x_\Psi[[\Psi(x_\Psi)] \wedge \{\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]]\}] \\ \text{or in the equivalent form} \\ \mathbf{Th} \triangleq \mathbf{Th}_1^\# \vdash \\ \mathbf{Pr}_{\mathbf{Th}_1^\#}([\exists!x_\Psi[\Psi(x_\Psi)]])^c \wedge \\ \{\mathbf{Pr}_{\mathbf{Th}_1^\#}([\exists!x_\Psi[\Psi(x_\Psi)]])^c \Rightarrow \exists!x_\Psi[\Psi(x_\Psi)]\} \wedge \\ [\mathbf{Pr}_{\mathbf{Th}_1^\#}([\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]]^c)] \wedge \\ \mathbf{Pr}_{\mathbf{Th}_1^\#}([\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]]^c \Rightarrow \forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]) \end{aligned} \right. \quad (2.45)$$

or in the following equivalent form

$$\left\{ \begin{aligned} \mathbf{Th}_1^\# \vdash \exists!x_1[[\Psi_1(x_1)] \wedge \{\forall n(n \in \mathbb{N})[\Psi_1(x_1) \leftrightarrow \Psi_{n,1}(x_1)]]\}] \\ \text{or} \\ \mathbf{Th}_1^\# \vdash \\ \mathbf{Pr}_{\mathbf{Th}_1^\#}([\exists!x_1\Psi(x_1)]^c) \wedge \\ \{\mathbf{Pr}_{\mathbf{Th}_1^\#}([\exists!x_1\Psi(x_1)]^c) \Rightarrow \exists!x_1\Psi(x_1)\} \wedge \\ [\mathbf{Pr}_{\mathbf{Th}_1^\#}([\forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]]^c)] \wedge \\ \mathbf{Pr}_{\mathbf{Th}_1^\#}([\forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]]^c \Rightarrow \forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]), \end{aligned} \right. \quad (2.46)$$

where we have set  $\Psi(x) = \Psi_1(x_1)$ ,  $\Psi_n(x_1) = \Psi_{n,1}(x_1)$  and  $x_\Psi = x_1$ . We note that any collection  $\mathcal{F}_{\Psi_k} = \{\Psi_{n,k}(x)\}_{n \in \mathbb{N}}$ ,  $k = 1, 2, \dots$  such as mentioned above, defines an unique set  $x_{\Psi_k}$ , i.e.  $\mathcal{F}_{\Psi_{k_1}} \cap \mathcal{F}_{\Psi_{k_2}} = \emptyset$  iff  $x_{\Psi_{k_1}} \neq x_{\Psi_{k_2}}$ . We note that collections  $\mathcal{F}_{\Psi_k}$ ,  $k = 1, 2, \dots$  are not a part of the  $ZFC_2^{Hs}$  or  $ZFC$ , i.e. collection  $\mathcal{F}_{\Psi_k}$  is not a set in sense of  $ZFC_2^{Hs}$  or  $ZFC$ . However this is no problem, because by using Gödel numbering one can to replace any collection  $\mathcal{F}_{\Psi_k}$ ,  $k = 1, 2, \dots$  by collection  $\Theta_k = g(\mathcal{F}_{\Psi_k})$  of the corresponding Gödel numbers

such that

$$\Theta_k = g(\mathcal{F}_{\Psi_k}) = \{g(\Psi_{n,k}(x_k))\}_{n \in \mathbb{N}}, k = 1, 2, \dots \quad (2.47)$$

It is easy to prove that any collection  $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1, 2, \dots$  is a  $\mathbf{Th}_1^\#$ -set. This is done by Gödel encoding [7],[10] (2.47), by the statement (2.45) and by axiom schemata of separation [10]. Let  $g_{n,k} = g(\Psi_{n,k}(x_k)), k = 1, 2, \dots$  be a Gödel number of the wff  $\Psi_{n,k}(x_k)$ . Therefore  $g(\mathcal{F}_k) = \{g_{n,k}\}_{n \in \mathbb{N}}$ , where we have set  $\mathcal{F}_k = \mathcal{F}_{\Psi_k}, k = 1, 2, \dots$  and

$$\forall k_1 \forall k_2 [\{g_{n,k_1}\}_{n \in \mathbb{N}} \cap \{g_{n,k_2}\}_{n \in \mathbb{N}} = \emptyset \leftrightarrow x_{k_1} \neq x_{k_2}]. \quad (2.48)$$

Let  $\{\{g_{n,k}\}_{n \in \mathbb{N}}\}_{k \in \mathbb{N}}$  be a family of the sets  $\{g_{n,k}\}_{n \in \mathbb{N}}, k = 1, 2, \dots$ . By the axiom of choice [10] one obtains unique set  $\mathfrak{S}'_1 = \{g_k\}_{k \in \mathbb{N}}$  such that  $\forall k [g_k \in \{g_{n,k}\}_{n \in \mathbb{N}}]$ . Finally one obtains a set  $\mathfrak{S}_1$  from the set  $\mathfrak{S}'_1$  by the axiom schema of replacement [10].

**Proposition 2.3.** Any collection  $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1, 2, \dots$  is a  $\mathbf{Th}_1^\#$ -set.

**Proof.** We define  $g_{n,k} = g(\Psi_{n,k}(x_k)) = [\Psi_{n,k}(x_k)]^c, v_k = [x_k]^c$ . Therefore  $g_{n,k} = g(\Psi_{n,k}(x_k)) \leftrightarrow \mathbf{Fr}(g_{n,k}, v_k)$  (see [7]). Let us define now predicate  $\Pi(g_{n,k}, v_k)$

$$\begin{aligned} \Pi(g_{n,k}, v_k) &\leftrightarrow \mathbf{Pr}_{\mathbf{Th}_1^\#}([\exists! x_k [\Psi_{1,k}(x_1)]]^c) \wedge \\ &\wedge \exists! x_k (v_k = [x_k]^c) [\forall n (n \in \mathbb{N}) [\mathbf{Pr}_{\mathbf{Th}_1^\#}([\Psi_{1,k}(x_k)]]^c) \leftrightarrow \mathbf{Pr}_{\mathbf{Th}_1^\#}(\mathbf{Fr}(g_{n,k}, v_k))]]. \end{aligned} \quad (2.49)$$

We define now a set  $\Theta_k$  such that

$$\left\{ \begin{array}{l} \Theta_k = \Theta'_k \cup \{g_k\}, \\ \forall n (n \in \mathbb{N}) [g_{n,k} \in \Theta'_k \leftrightarrow \Pi(g_{n,k}, v_k)] \end{array} \right. \quad (2.50)$$

Obviously definitions (2.45) and (2.50) are equivalent.

**Definition 2.7.** We define now the following  $\mathbf{Th}_1^\#$ -set  $\mathfrak{R}_1 \subseteq \mathfrak{S}_1$  :

$$\forall x [x \in \mathfrak{R}_1 \leftrightarrow (x \in \mathfrak{S}_1) \wedge \mathbf{Pr}_{\mathbf{Th}_1^\#}([x \notin x]^c) \wedge [\mathbf{Pr}_{\mathbf{Th}_1^\#}([x \notin x]^c) \Rightarrow x \notin x]]. \quad (2.51)$$

**Proposition 2.4.** (i)  $\mathbf{Th}_1^\# \vdash \exists \mathfrak{R}_1$ , (ii)  $\mathfrak{R}_1$  is a countable  $\mathbf{Th}_1^\#$ -set.

**Proof.** (i) Statement  $\mathbf{Th}_1^\# \vdash \exists \mathfrak{R}_1$  follows immediately from the statement  $\exists \mathfrak{S}_1$  and the axiom schema of separation [4], (ii) follows immediately from countability of a set  $\mathfrak{S}_1$ . Notice that  $\mathfrak{R}_1$  is nonempty countable set such that  $\mathbb{N} \subset \mathfrak{R}_1$ , because for any  $n \in \mathbb{N}$  :

$\mathbf{Th}_1^\# \vdash n \notin n$ .

**Proposition 2.5.** A set  $\mathfrak{R}_1$  is inconsistent.

**Proof.** From formula (2.51) we obtain

$$\mathbf{Th}_1^\# \vdash \mathfrak{R}_1 \in \mathfrak{R}_1 \leftrightarrow \mathbf{Pr}_{\mathbf{Th}_1^\#}([\mathfrak{R}_1 \notin \mathfrak{R}_1]^c) \wedge [\mathbf{Pr}_{\mathbf{Th}_1^\#}([\mathfrak{R}_1 \notin \mathfrak{R}_1]^c) \Rightarrow \mathfrak{R}_1 \notin \mathfrak{R}_1]. \quad (2.52)$$

From (2.52) we obtain

$$\mathbf{Th}_1^\# \vdash \mathfrak{R}_1 \in \mathfrak{R}_1 \leftrightarrow \mathfrak{R}_1 \notin \mathfrak{R}_1 \quad (2.53)$$

and therefore

$$\mathbf{Th}_1^\# \vdash (\mathfrak{R}_1 \in \mathfrak{R}_1) \wedge (\mathfrak{R}_1 \notin \mathfrak{R}_1). \quad (2.54)$$

But this is a contradiction.

**Definition 2.8.** Let  $\Psi = \Psi(x)$  be one-place open  $\mathbf{Th}$ -wff such that the following

condition:

$$\mathbf{Th}_i^\# \vdash \exists!x_\Psi[\Psi(x_\Psi)] \quad (2.55)$$

is satisfied.

**Remark 2.14.** We rewrite now the condition (2.55) using only the language of the theory

$\mathbf{Th}_i^\#$  :

$$\begin{aligned} \{\mathbf{Th}_i^\# \vdash \exists!x_\Psi[\Psi(x_\Psi)]\} &\Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i^\#}([\exists!x_\Psi[\Psi(x_\Psi)]])^c \wedge \\ &\wedge \{\mathbf{Pr}_{\mathbf{Th}_i^\#}([\exists!x_\Psi[\Psi(x_\Psi)]])^c \Rightarrow \exists!x_\Psi[\Psi(x_\Psi)]\}. \end{aligned} \quad (2.56)$$

**Definition 2.9.** We will say that, a set  $y$  is a  $\mathbf{Th}_i^\#$ -set if there exist one-place open wff  $\Psi(x)$

such that  $y = x_\Psi$ . We write  $y[\mathbf{Th}_i^\#]$  iff  $y$  is a  $\mathbf{Th}_i^\#$ -set.

**Remark 2.15.** Note that

$$\begin{aligned} y[\mathbf{Th}_i^\#] &\Leftrightarrow \exists\Psi \{ (y = x_\Psi) \wedge \mathbf{Pr}_{\mathbf{Th}_i^\#}([\exists!x_\Psi[\Psi(x_\Psi)]])^c \\ &\quad \{ \mathbf{Pr}_{\mathbf{Th}_i^\#}([\exists!x_\Psi[\Psi(x_\Psi)]])^c \Rightarrow \exists!x_\Psi[\Psi(x_\Psi)] \} \}. \end{aligned} \quad (2.57)$$

**Definition 2.10.** Let  $\mathfrak{S}_i$  be a collection such that :

$$\forall x [x \in \mathfrak{S}_i \leftrightarrow x \text{ is a } \mathbf{Th}_i^\# \text{-set}]. \quad (2.58)$$

**Proposition 2.6.** Collection  $\mathfrak{S}_i$  is a  $\mathbf{Th}_i^\#$ -set.

**Proof.** Let us consider an one-place open wff  $\Psi(x)$  such that conditions (2.51) are satisfied, i.e.  $\mathbf{Th}_i^\# \vdash \exists!x_\Psi[\Psi(x_\Psi)]$ . We note that there exists countable collection  $\mathcal{F}_\Psi$  of the one-place open wff's  $\mathcal{F}_\Psi = \{\Psi_n(x)\}_{n \in \mathbb{N}}$  such that: (i)  $\Psi(x) \in \mathcal{F}_\Psi$  and (ii)

$$\left\{ \begin{aligned} &\mathbf{Th}_i^\# \vdash \exists!x_\Psi[[\Psi(x_\Psi)] \wedge \{\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]\}] \\ &\quad \text{or in the equivalent form} \\ &\quad \mathbf{Th}_i^\# \vdash \mathbf{Pr}_{\mathbf{Th}_i^\#}([\exists!x_\Psi[\Psi(x_\Psi)]])^c \wedge \\ &\quad \{ \mathbf{Pr}_{\mathbf{Th}_i^\#}([\exists!x_\Psi[\Psi(x_\Psi)]])^c \Rightarrow \exists!x_\Psi[\Psi(x_\Psi)] \} \wedge \\ &\quad [ \mathbf{Pr}_{\mathbf{Th}_i^\#}([\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]])^c ] \wedge \\ &\quad \mathbf{Pr}_{\mathbf{Th}_i^\#}([\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]])^c \Rightarrow \forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)] \end{aligned} \right. \quad (2.59)$$

or in the following equivalent form

$$\left\{ \begin{aligned} &\mathbf{Th}_i^\# \vdash \exists!x_1[[\Psi_1(x_1)] \wedge \{\forall n(n \in \mathbb{N})[\Psi_1(x_1) \leftrightarrow \Psi_{n,1}(x_1)]\}] \\ &\quad \text{or} \\ &\quad \mathbf{Th}_i^\# \vdash \\ &\quad \mathbf{Pr}_{\mathbf{Th}_i^\#}([\exists!x_1\Psi(x_1)]^c) \wedge \\ &\quad \{ \mathbf{Pr}_{\mathbf{Th}_i^\#}([\exists!x_1\Psi(x_1)]^c) \Rightarrow \exists!x_1\Psi(x_1) \} \wedge \\ &\quad [ \mathbf{Pr}_{\mathbf{Th}_i^\#}([\forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]])^c ] \wedge \\ &\quad \mathbf{Pr}_{\mathbf{Th}_i^\#}([\forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]])^c \Rightarrow \forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]. \end{aligned} \right. \quad (2.60)$$

where we have set  $\Psi(x) \triangleq \Psi_1(x_1)$ ,  $\Psi_n(x_1) \triangleq \Psi_{n,1}(x_1)$  and  $x_\Psi \triangleq x_1$ . We note that any collection  $\mathcal{F}_{\Psi_k} = \{\Psi_{n,k}(x)\}_{n \in \mathbb{N}}$ ,  $k = 1, 2, \dots$  such as mentioned above, defines an unique

set  $x_{\Psi_k}$ , i.e.  $\mathcal{F}_{\Psi_{k_1}} \cap \mathcal{F}_{\Psi_{k_2}} = \emptyset$  iff  $x_{\Psi_{k_1}} \neq x_{\Psi_{k_2}}$ . We note that collections  $\mathcal{F}_{\Psi_k}, k = 1, 2, \dots$  are not a part of the  $ZFC_2^{Hs}$ , i.e. collection  $\mathcal{F}_{\Psi_k}$  there is no set in the sense of  $ZFC_2^{Hs}$ . However that is no problem, because by using Gödel numbering one can to replace any collection  $\mathcal{F}_{\Psi_k}, k = 1, 2, \dots$  by collection  $\Theta_k = g(\mathcal{F}_{\Psi_k})$  of the corresponding Gödel numbers such that

$$\Theta_k = g(\mathcal{F}_{\Psi_k}) = \{g(\Psi_{n,k}(x_k))\}_{n \in \mathbb{N}}, k = 1, 2, \dots \quad (2.61)$$

It is easy to prove that any collection  $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1, 2, \dots$  is a  $\mathbf{Th}_i^\#$ -set. This is done by Gödel encoding [7],[10] (2.61), by the statement (2.55) and by the axiom schema of separation [10]. Let  $g_{n,k} = g(\Psi_{n,k}(x_k)), k = 1, 2, \dots$  be a Gödel number of the wff  $\Psi_{n,k}(x_k)$ . Therefore  $g(\mathcal{F}_k) = \{g_{n,k}\}_{n \in \mathbb{N}}$ , where we have set  $\mathcal{F}_k = \mathcal{F}_{\Psi_k}, k = 1, 2, \dots$  and

$$\forall k_1 \forall k_2 [\{g_{n,k_1}\}_{n \in \mathbb{N}} \cap \{g_{n,k_2}\}_{n \in \mathbb{N}} = \emptyset \leftrightarrow x_{k_1} \neq x_{k_2}]. \quad (2.62)$$

Let  $\{\{g_{n,k}\}_{n \in \mathbb{N}}\}_{k \in \mathbb{N}}$  be a family of the all sets  $\{g_{n,k}\}_{n \in \mathbb{N}}$ . By axiom of choice [10] one obtains a unique set  $\mathfrak{S}'_i = \{g_k\}_{k \in \mathbb{N}}$  such that  $\forall k [g_k \in \{g_{n,k}\}_{n \in \mathbb{N}}]$ . Finally for any  $i \in \mathbb{N}$  one obtains a set  $\mathfrak{S}_i$  from the set  $\mathfrak{S}'_i$  by the axiom schema of replacement [10].

**Proposition 2.8.** Any collection  $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1, 2, \dots$  is a  $\mathbf{Th}_i^\#$ -set.

**Proof.** We define  $g_{n,k} = g(\Psi_{n,k}(x_k)) = [\Psi_{n,k}(x_k)]^c, v_k = [x_k]^c$ . Therefore  $g_{n,k} = g(\Psi_{n,k}(x_k)) \leftrightarrow \mathbf{Fr}(g_{n,k}, v_k)$  (see [7]). Let us define now predicate  $\Pi_i(g_{n,k}, v_k)$

$$\begin{aligned} \Pi_i(g_{n,k}, v_k) &\leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i^\#}([\exists! x_k [\Psi_{1,k}(x_1)]]^c) \wedge \\ &\wedge \exists! x_k (v_k = [x_k]^c) [\forall n (n \in \mathbb{N}) [\mathbf{Pr}_{\mathbf{Th}_i^\#}([\Psi_{1,k}(x_k)]]^c) \leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i^\#}(\mathbf{Fr}(g_{n,k}, v_k))]. \end{aligned} \quad (2.63)$$

We define now a set  $\Theta_k$  such that

$$\begin{cases} \Theta_k = \Theta'_k \cup \{g_k\}, \\ \forall n (n \in \mathbb{N}) [g_{n,k} \in \Theta'_k \leftrightarrow \Pi_i(g_{n,k}, v_k)]. \end{cases} \quad (2.64)$$

Obviously definitions (2.59) and (2.64) are equivalent.

**Definition 2.11.** We define now the following  $\mathbf{Th}_i^\#$ -set  $\mathfrak{R}_i \subseteq \mathfrak{S}_i$  :

$$\forall x [x \in \mathfrak{R}_i \leftrightarrow (x \in \mathfrak{S}_i) \wedge \mathbf{Pr}_{\mathbf{Th}_i^\#}([x \notin x]^c) \wedge [\mathbf{Pr}_{\mathbf{Th}_i^\#}([x \notin x]^c) \Rightarrow x \notin x]]. \quad (2.65)$$

**Proposition 2.9.** (i)  $\mathbf{Th}_i^\# \vdash \exists \mathfrak{R}_i$ , (ii)  $\mathfrak{R}_i$  is a countable  $\mathbf{Th}_i^\#$ -set,  $i \in \mathbb{N}$ .

**Proof.** (i) Statement  $\mathbf{Th}_i^\# \vdash \exists \mathfrak{R}_i$  follows immediately by using statement  $\exists \mathfrak{S}_i$  and axiom schema of separation [4]. (ii) follows immediately from countability of a set  $\mathfrak{S}_i$ .

**Proposition 2.10.** Any set  $\mathfrak{R}_i, i \in \mathbb{N}$  is inconsistent.

**Proof.** From the formula (2.65) we obtain

$$\mathbf{Th}_i^\# \vdash \mathfrak{R}_i \in \mathfrak{R}_i \leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i^\#}([\mathfrak{R}_i \notin \mathfrak{R}_i]^c) \wedge [\mathbf{Pr}_{\mathbf{Th}_i^\#}([\mathfrak{R}_i \notin \mathfrak{R}_i]^c) \Rightarrow \mathfrak{R}_i \notin \mathfrak{R}_i]. \quad (2.66)$$

From the formula (2.66) we obtain

$$\mathbf{Th}_i^\# \vdash \mathfrak{R}_i \in \mathfrak{R}_i \leftrightarrow \mathfrak{R}_i \notin \mathfrak{R}_i \quad (2.67)$$

and therefore

$$\mathbf{Th}_i^\# \vdash (\mathfrak{R}_i \in \mathfrak{R}_i) \wedge (\mathfrak{R}_i \notin \mathfrak{R}_i). \quad (2.68)$$

But this is a contradiction.

**Definition 2.12.** A  $\mathbf{Th}_\infty^\#$ -wff  $\Phi_\infty$  that is: (i)  $\mathbf{Th}$ -wff  $\Phi$  or (ii) well-formed formula  $\Phi_\infty$  which contains predicate  $\mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\Phi]^c)$  given by formula (2.39). An  $\mathbf{Th}_\infty^\#$ -wff  $\Phi_\infty$  (well-formed formula  $\Phi_\infty$ ) is closed - i.e.  $\Phi_\infty$  is a sentence - if it has no free variables; a wff is open if it

has free variables.

**Definition 2.13.** Let  $\Psi = \Psi(x)$  be one-place open  $\mathbf{Th}_\infty^\#$ -wff such that the following condition:

$$\mathbf{Th}_\infty^\# \vdash \exists!x_\Psi[\Psi(x_\Psi)] \quad (2.69)$$

is satisfied.

**Remark 2.16.** We rewrite now the condition (2.69) using only the language of the theory  $\mathbf{Th}_\infty^\#$  :

$$\begin{aligned} \{\mathbf{Th}_\infty^\# \vdash \exists!x_\Psi[\Psi(x_\Psi)]\} &\Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\exists!x_\Psi[\Psi(x_\Psi)]]^c) \wedge \\ &\wedge \{\mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\exists!x_\Psi[\Psi(x_\Psi)]]^c) \Rightarrow \exists!x_\Psi[\Psi(x_\Psi)]\}. \end{aligned} \quad (2.70)$$

**Definition 2.14.** We will say that, a set  $y$  is a  $\mathbf{Th}_\infty^\#$ -set if there exists one-place open wff  $\Psi(x)$  such that  $y = x_\Psi$ . We write  $y[\mathbf{Th}_\infty^\#]$  iff  $y$  is a  $\mathbf{Th}_\infty^\#$ -set.

**Definition 2.15.** Let  $\mathfrak{S}_\infty$  be a collection such that :  $\forall x[x \in \mathfrak{S}_\infty \leftrightarrow x \text{ is a } \mathbf{Th}_\infty^\# \text{-set}]$ .

**Proposition 2.11.** Collection  $\mathfrak{S}_\infty$  is a  $\mathbf{Th}_\infty^\#$ -set.

**Proof.** Let us consider an one-place open wff  $\Psi(x)$  such that condition (2.69) is satisfied, i.e.  $\mathbf{Th}_\infty^\# \vdash \exists!x_\Psi[\Psi(x_\Psi)]$ . We note that there exists countable collection  $\mathcal{F}_\Psi$  of the one-place open wff's  $\mathcal{F}_\Psi = \{\Psi_n(x)\}_{n \in \mathbb{N}}$  such that: (i)  $\Psi(x) \in \mathcal{F}_\Psi$  and (ii)

$$\left\{ \begin{aligned} &\mathbf{Th}_\infty^\# \vdash \exists!x_\Psi[[\Psi(x_\Psi)] \wedge \{\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]]\}] \\ &\quad \text{or in the equivalent form} \\ &\quad \mathbf{Th}_\infty^\# \vdash \mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\exists!x_\Psi[\Psi(x_\Psi)]]^c) \wedge \\ &\quad \{\mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\exists!x_\Psi[\Psi(x_\Psi)]]^c) \Rightarrow \exists!x_\Psi[\Psi(x_\Psi)]\} \wedge \\ &\quad [\mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]]^c)] \wedge \\ &\mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]]^c) \Rightarrow \forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)] \end{aligned} \right. \quad (2.71)$$

or in the following equivalent form

$$\left\{ \begin{aligned} &\mathbf{Th}_\infty^\# \vdash \exists!x_1[[\Psi_1(x_1)] \wedge \{\forall n(n \in \mathbb{N})[\Psi_1(x_1) \leftrightarrow \Psi_{n,1}(x_1)]]\}] \\ &\quad \text{or} \\ &\quad \mathbf{Th}_\infty^\# \vdash \mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\exists!x_1\Psi(x_1)]^c) \wedge \\ &\quad \{\mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\exists!x_1\Psi(x_1)]^c) \Rightarrow \exists!x_1\Psi(x_1)\} \wedge \\ &\quad [\mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]]^c)] \wedge \\ &\mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]]^c) \Rightarrow \forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]. \end{aligned} \right. \quad (2.72)$$

where we set  $\Psi(x) = \Psi_1(x_1)$ ,  $\Psi_n(x_1) = \Psi_{n,1}(x_1)$  and  $x_\Psi = x_1$ . We note that any collection  $\mathcal{F}_{\Psi_k} = \{\Psi_{n,k}(x)\}_{n \in \mathbb{N}}$ ,  $k = 1, 2, \dots$  such as mentioned above defines a unique set  $x_{\Psi_k}$ , i.e.  $\mathcal{F}_{\Psi_{k_1}} \cap \mathcal{F}_{\Psi_{k_2}} = \emptyset$  iff  $x_{\Psi_{k_1}} \neq x_{\Psi_{k_2}}$ . We note that collections  $\mathcal{F}_{\Psi_k}$ ,  $k = 1, 2, \dots$  are not a part of the  $ZFC_2^{Hs}$ , i.e. collection  $\mathcal{F}_{\Psi_k}$  there is no set in sense of  $ZFC_2^{Hs}$ . However that is not a problem, because by using Gödel numbering one can to replace any collection  $\mathcal{F}_{\Psi_k}$ ,  $k = 1, 2, \dots$  by collection  $\Theta_k = g(\mathcal{F}_{\Psi_k})$  of the corresponding Gödel numbers such that

$$\Theta_k = g(\mathcal{F}_{\Psi_k}) = \{g(\Psi_{n,k}(x_k))\}_{n \in \mathbb{N}}, k = 1, 2, \dots \quad (2.73)$$

It is easy to prove that any collection  $\Theta_k = g(\mathcal{F}_{\Psi_k})$ ,  $k = 1, 2, \dots$  is a  $\mathbf{Th}_\infty^\#$ -set. This is done by Gödel encoding [8],[10] by the statement (2.66) and by axiom schema of separation [9].

Let  $g_{n,k} = g(\Psi_{n,k}(x_k)), k = 1, 2, \dots$  be a Gödel number of the wff  $\Psi_{n,k}(x_k)$ . Therefore  $g(\mathcal{F}_k) = \{g_{n,k}\}_{n \in \mathbb{N}}$ , where we have set  $\mathcal{F}_k \triangleq \mathcal{F}_{\Psi_k}, k = 1, 2, \dots$  and

$$\forall k_1 \forall k_2 [\{g_{n,k_1}\}_{n \in \mathbb{N}} \cap \{g_{n,k_2}\}_{n \in \mathbb{N}} = \emptyset \leftrightarrow x_{k_1} \neq x_{k_2}]. \quad (2.74)$$

Let  $\{\{g_{n,k}\}_{n \in \mathbb{N}}\}_{k \in \mathbb{N}}$  be a family of the sets  $\{g_{n,k}\}_{n \in \mathbb{N}}, k = 1, 2, \dots$ . By axiom of choice [9] one obtains an unique set  $\mathfrak{T}' = \{g_k\}_{k \in \mathbb{N}}$  such that  $\forall k [g_k \in \{g_{n,k}\}_{n \in \mathbb{N}}]$ . Finally one obtains a set  $\mathfrak{T}_\infty$  from the set  $\mathfrak{T}'_\infty$  by axiom schema of replacement [9]. Thus one can define  $\mathbf{Th}_\infty^\#$ -set

$\mathfrak{R}_\infty \subseteq \mathfrak{T}_\infty$  :

$$\forall x [x \in \mathfrak{R}_\infty \leftrightarrow (x \in \mathfrak{T}_\infty) \wedge [\mathbf{Pr}_{\mathbf{Th}_\infty^\#}([x \notin x]^c) \wedge \{\mathbf{Pr}_{\mathbf{Th}_\infty^\#}([x \notin x]^c) \Rightarrow x \notin x\}]]. \quad (2.75)$$

**Proposition 2.12.** Any collection  $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1, 2, \dots$  is a  $\mathbf{Th}_\infty^\#$ -set.

**Proof.** We define  $g_{n,k} = g(\Psi_{n,k}(x_k)) = [\Psi_{n,k}(x_k)]^c, v_k = [x_k]^c$ . Therefore  $g_{n,k} = g(\Psi_{n,k}(x_k)) \leftrightarrow \mathbf{Fr}(g_{n,k}, v_k)$  (see [10]). Let us define now predicate  $\Pi_\infty(g_{n,k}, v_k)$

$$\left\{ \begin{array}{l} \Pi_\infty(g_{n,k}, v_k) \Leftrightarrow \\ \mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\exists! x_k [\Psi_{1,k}(x_1)]]^c) \wedge [\mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\exists! x_k [\Psi_{1,k}(x_1)]]^c) \Rightarrow \exists! x_1 \Psi(x_1)] \\ \wedge \exists! x_k (v_k = [x_k]^c) [\forall n (n \in \mathbb{N}) [\mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\Psi_{1,k}(x_k)]]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_\infty^\#}(\mathbf{Fr}(g_{n,k}, v_k))]]. \end{array} \right. \quad (2.76)$$

We define now a set  $\Theta_k$  such that

$$\left\{ \begin{array}{l} \Theta_k = \Theta'_k \cup \{g_k\}, \\ \forall n (n \in \mathbb{N}) [g_{n,k} \in \Theta'_k \Leftrightarrow \Pi(g_{n,k}, v_k)] \end{array} \right. \quad (2.77)$$

Obviously definitions (2.70) and (2.77) are equivalent by Proposition 2.1.

**Proposition 2.13.** (i)  $\mathbf{Th}_\infty^\# \vdash \exists \mathfrak{R}_\infty$ , (ii)  $\mathfrak{R}_\infty$  is a countable  $\mathbf{Th}_\infty^\#$ -set.

**Proof.** (i) Statement  $\mathbf{Th}_\infty^\# \vdash \exists \mathfrak{R}_\infty$  follows immediately from the statement  $\exists \mathfrak{T}_\infty$  and axiom

schema of separation [9], (ii) follows immediately from countability of the set  $\mathfrak{T}_\infty$ .

**Proposition 2.14.** Set  $\mathfrak{R}_\infty$  is inconsistent.

**Proof.** From the formula (2.75) we obtain

$$\mathbf{Th}_\infty^\# \vdash \mathfrak{R}_\infty \in \mathfrak{R}_\infty \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\mathfrak{R}_\infty \notin \mathfrak{R}_\infty]^c) \wedge \{\mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\mathfrak{R}_\infty \notin \mathfrak{R}_\infty]^c) \Rightarrow \mathfrak{R}_\infty \notin \mathfrak{R}_\infty\}. \quad (2.78)$$

From (2.74) one obtains

$$\mathbf{Th}_\infty^\# \vdash \mathfrak{R}_\infty \in \mathfrak{R}_\infty \Leftrightarrow \mathfrak{R}_\infty \notin \mathfrak{R}_\infty \quad (2.79)$$

and therefore

$$\mathbf{Th}_\infty^\# \vdash (\mathfrak{R}_\infty \in \mathfrak{R}_\infty) \wedge (\mathfrak{R}_\infty \notin \mathfrak{R}_\infty). \quad (2.80)$$

But this is a contradiction.

**Definition 2.16.** An  $\mathbf{Th}_{\infty;\omega}^\#$ -wff  $\Phi_{\infty;\omega}$  that is: (i)  $\mathbf{Th}_\omega$ -wff  $\Phi_\omega$  or (ii) well-formed formula  $\Phi_{\infty;\omega}$

which contains predicate  $\mathbf{Pr}_{\mathbf{Th}_{\infty;\omega}^\#}([\Phi]^c)$  given by formula (2.36). An  $\mathbf{Th}_{\infty;\omega}^\#$ -wff  $\Phi_{\infty;\omega}$  (well-formed formula  $\Phi_{\infty;\omega}$ ) is closed - i.e.  $\Phi_{\infty;\omega}$  is a sentence - if it has no free variables; a

wff is open if it has free variables.

**Definition 2.17.** Let  $\Psi = \Psi(x)$  be one-place open  $\mathbf{Th}$ -wff such that the following condition:

$$\mathbf{Th}_\omega \triangleq \mathbf{Th}_{\omega,1}^\# \vdash \exists!x_\Psi[\Psi(x_\Psi)] \quad (2.81)$$

is satisfied.

**Remark 2.17.** We rewrite now the condition (2.81) using only the language of the theory

$\mathbf{Th}_{\omega,1}^\#$  :

$$\{\mathbf{Th}_{\omega,1}^\# \vdash \exists!x_\Psi[\Psi(x_\Psi)]\} \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{\omega,1}^\#}([\exists!x_\Psi[\Psi(x_\Psi)]]^c). \quad (2.82)$$

**Definition 2.18.** We will say that, a set  $y$  is a  $\mathbf{Th}_{\omega,1}^\#$ -set if there exist one-place open wff

$\Psi(x)$  such that  $y = x_\Psi$ . We write  $y[\mathbf{Th}_{\omega,1}^\#]$  iff  $y$  is a  $\mathbf{Th}_{\omega,1}^\#$ -set.

**Remark 2.18.** Note that

$$y[\mathbf{Th}_{\omega,1}^\#] \Leftrightarrow \exists\Psi \left\{ (y = x_\Psi) \wedge \mathbf{Pr}_{\mathbf{Th}_{\omega,1}^\#}([\exists!x_\Psi[\Psi(x_\Psi)]]^c) \right. \\ \left. \left\{ \mathbf{Pr}_{\mathbf{Th}_{\omega,1}^\#}([\exists!x_\Psi[\Psi(x_\Psi)]]^c) \Rightarrow \exists!x_\Psi[\Psi(x_\Psi)] \right\} \right\}. \quad (2.83)$$

**Definition 2.19.** Let  $\mathfrak{T}_{\omega,1}$  be a collection such that :

$$\forall x [x \in \mathfrak{T}_{\omega,1} \leftrightarrow x \text{ is a } \mathbf{Th}_{\omega,1}^\# \text{-set}]. \quad (2.84)$$

**Proposition 2.15.** Collection  $\mathfrak{T}_{\omega,1}$  is a  $\mathbf{Th}_{\omega,1}^\#$ -set.

**Proof.** Let us consider an one-place open wff  $\Psi(x)$  such that conditions (2.37) are satisfied, i.e.  $\mathbf{Th}_{\omega,1}^\# \vdash \exists!x_\Psi[\Psi(x_\Psi)]$ . We note that there exists countable collection  $\mathcal{F}_\Psi$  of the one-place open wff's  $\mathcal{F}_\Psi = \{\Psi_n(x)\}_{n \in \mathbb{N}}$  such that: (i)  $\Psi(x) \in \mathcal{F}_\Psi$  and (ii)

$$\left\{ \begin{array}{l} \mathbf{Th}_\omega \triangleq \mathbf{Th}_{\omega,1}^\# \vdash \exists!x_\Psi[[\Psi(x_\Psi)] \wedge \{\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]]\}] \\ \text{or in the equivalent form} \\ \mathbf{Th}_\omega \triangleq \mathbf{Th}_{\omega,1}^\# \vdash \mathbf{Pr}_{\mathbf{Th}_{\omega,1}^\#}([\exists!x_\Psi[\Psi(x_\Psi)]]^c) \wedge \\ [\mathbf{Pr}_{\mathbf{Th}_{\omega,1}^\#}([\forall n(n \in \mathbb{N})[\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]]^c)], \end{array} \right. \quad (2.85)$$

or in the following equivalent form

$$\left\{ \begin{array}{l} \mathbf{Th}_{\omega,1}^\# \vdash \exists!x_1[[\Psi_1(x_1)] \wedge \{\forall n(n \in \mathbb{N})[\Psi_1(x_1) \leftrightarrow \Psi_{n,1}(x_1)]]\}] \\ \text{or} \\ \mathbf{Th}_{\omega,1}^\# \vdash \mathbf{Pr}_{\mathbf{Th}_{\omega,1}^\#}([\exists!x_1\Psi(x_1)]^c) \wedge \\ [\mathbf{Pr}_{\mathbf{Th}_{\omega,1}^\#}([\forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]]^c)], \end{array} \right. \quad (2.86)$$

where we have set  $\Psi(x) \triangleq \Psi_1(x_1)$ ,  $\Psi_n(x_1) \triangleq \Psi_{n,1}(x_1)$  and  $x_\Psi \triangleq x_1$ . We note that any collection  $\mathcal{F}_{\Psi_k} = \{\Psi_{n,k}(x)\}_{n \in \mathbb{N}}$ ,  $k = 1, 2, \dots$  such as mentioned above, defines an unique set  $x_{\Psi_k}$ , i.e.  $\mathcal{F}_{\Psi_{k_1}} \cap \mathcal{F}_{\Psi_{k_2}} = \emptyset$  iff  $x_{\Psi_{k_1}} \neq x_{\Psi_{k_2}}$ . We note that collections  $\mathcal{F}_{\Psi_k}$ ,  $k = 1, 2, \dots$  are not a part of the  $ZFC_2^{Hs}$ , i.e. collection  $\mathcal{F}_{\Psi_k}$  is not a set in the sense of  $ZFC_2$ . However that is not a problem, because by using Gödel numbering one can to replace any collection  $\mathcal{F}_{\Psi_k}$ ,  $k = 1, 2, \dots$  by collection  $\Theta_k = g(\mathcal{F}_{\Psi_k})$  of the corresponding Gödel numbers such that

$$\Theta_k = g(\mathcal{F}_{\Psi_k}) = \{g(\Psi_{n,k}(x_k))\}_{n \in \mathbb{N}}, k = 1, 2, \dots \quad (2.87)$$

It is easy to prove that any collection  $\Theta_k = g(\mathcal{F}_{\Psi_k})$ ,  $k = 1, 2, \dots$  is a  $\mathbf{Th}_{\omega,1}^\#$ -set. This is done by Gödel encoding [7],[10] (2.87), by the statement (2.85) and by the axiom schema of separation [7]. Let  $g_{n,k} = g(\Psi_{n,k}(x_k))$ ,  $k = 1, 2, \dots$  be a Gödel number of the wff



$\Psi_{n,k}(x_k)$ . Therefore  $g(\mathcal{F}_k) = \{g_{n,k}\}_{n \in \mathbb{N}}$ , where we have set  $\mathcal{F}_k = \mathcal{F}_{\Psi_k}$ ,  $k = 1, 2, \dots$  and

$$\forall k_1 \forall k_2 [\{g_{n,k_1}\}_{n \in \mathbb{N}} \cap \{g_{n,k_2}\}_{n \in \mathbb{N}} = \emptyset \leftrightarrow x_{k_1} \neq x_{k_2}]. \quad (2.88)$$

Let  $\{\{g_{n,k}\}_{n \in \mathbb{N}}\}_{k \in \mathbb{N}}$  be a family of the sets  $\{g_{n,k}\}_{n \in \mathbb{N}}$ ,  $k = 1, 2, \dots$ . By the axiom of choice [7] one obtains an unique set  $\mathfrak{S}'_1 = \{g_k\}_{k \in \mathbb{N}}$  such that  $\forall k [g_k \in \{g_{n,k}\}_{n \in \mathbb{N}}]$ . Finally one obtains a set  $\mathfrak{S}_{\omega,1}$  from the set  $\mathfrak{S}'_{\omega,1}$  by the axiom schema of replacement [7].

**Proposition 2.16.** Any collection  $\Theta_k = g(\mathcal{F}_{\Psi_k})$ ,  $k = 1, 2, \dots$  is a  $\mathbf{Th}_{\omega,1}^\#$ -set.

**Proof.** We define  $g_{n,k} = g(\Psi_{n,k}(x_k)) = [\Psi_{n,k}(x_k)]^c$ ,  $v_k = [x_k]^c$ . Therefore  $g_{n,k} = g(\Psi_{n,k}(x_k)) \leftrightarrow \mathbf{Fr}(g_{n,k}, v_k)$  (see [10]). Let us define now predicate  $\Pi(g_{n,k}, v_k)$

$$\begin{aligned} \Pi(g_{n,k}, v_k) &\leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{\omega,1}^\#}([\exists! x_k [\Psi_{1,k}(x_k)]]^c) \wedge \\ &\wedge \exists! x_k (v_k = [x_k]^c) [\forall n (n \in \mathbb{N}) [\mathbf{Pr}_{\mathbf{Th}_{\omega,1}^\#}([\Psi_{1,k}(x_k)]]^c) \leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{\omega,1}^\#}(\mathbf{Fr}(g_{n,k}, v_k))]]. \end{aligned} \quad (2.89)$$

We define now a set  $\Theta_k$  such that

$$\begin{cases} \Theta_k = \Theta'_k \cup \{g_k\}, \\ \forall n (n \in \mathbb{N}) [g_{n,k} \in \Theta'_k \leftrightarrow \Pi(g_{n,k}, v_k)] \end{cases} \quad (2.90)$$

Obviously definitions (2.85) and (2.90) are equivalent.

**Definition 2.20.** We define now the following  $\mathbf{Th}_{\omega,1}^\#$ -set  $\mathfrak{R}_{\omega,1} \subseteq \mathfrak{S}_{\omega,1}$  :

$$\forall x [x \in \mathfrak{R}_{\omega,1} \leftrightarrow (x \in \mathfrak{S}_{\omega,1}) \wedge \mathbf{Pr}_{\mathbf{Th}_{\omega,1}^\#}([x \notin x]^c)]. \quad (2.91)$$

**Proposition 2.17.** (i)  $\mathbf{Th}_{\omega,1}^\# \vdash \exists \mathfrak{R}_{\omega,1}$ , (ii)  $\mathfrak{R}_{\omega,1}$  is a countable  $\mathbf{Th}_{\omega,1}^\#$ -set.

**Proof.** (i) Statement  $\mathbf{Th}_{\omega,1}^\# \vdash \exists \mathfrak{R}_{\omega,1}$  follows immediately from the statement  $\exists \mathfrak{S}_{\omega,1}$  and axiom schema of separation [7], (ii) follows immediately from countability of the set  $\mathfrak{S}_{\omega,1}$ .

**Proposition 2.18.** A set  $\mathfrak{R}_{\omega,1}$  is inconsistent.

**Proof.** From formula (2.87) we obtain

$$\mathbf{Th}_{\omega,1}^\# \vdash \mathfrak{R}_{\omega,1} \in \mathfrak{R}_{\omega,1} \leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{\omega,1}^\#}([\mathfrak{R}_{\omega,1} \notin \mathfrak{R}_{\omega,1}]^c). \quad (2.92)$$

From (2.92) we obtain

$$\mathbf{Th}_{\omega,1}^\# \vdash \mathfrak{R}_{\omega,1} \in \mathfrak{R}_{\omega,1} \leftrightarrow \mathfrak{R}_{\omega,1} \notin \mathfrak{R}_{\omega,1} \quad (2.93)$$

and therefore

$$\mathbf{Th}_{\omega,1}^\# \vdash (\mathfrak{R}_{\omega,1} \in \mathfrak{R}_{\omega,1}) \wedge (\mathfrak{R}_{\omega,1} \notin \mathfrak{R}_{\omega,1}). \quad (2.94)$$

But this is a contradiction.

**Definition 2.21.** Let  $\Psi = \Psi(x)$  be one-place open  $\mathbf{Th}$ -wff such that the following condition:

$$\mathbf{Th}_{\omega,i}^\# \vdash \exists! x_\Psi [\Psi(x_\Psi)] \quad (2.95)$$

is satisfied.

**Remark 2.19.** We rewrite now the condition (2.95) using only the language of the theory

$\mathbf{Th}_{\omega,i}^\#$  :

$$\{\mathbf{Th}_{\omega,i}^\# \vdash \exists! x_\Psi [\Psi(x_\Psi)]\} \leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{\omega,i}^\#}([\exists! x_\Psi [\Psi(x_\Psi)]]^c). \quad (2.96)$$

**Definition 2.22.** We will say that, a set  $y$  is a  $\mathbf{Th}_{\omega,i}^\#$ -set if there exist one-place open wff  $\Psi(x)$  such that  $y = x\Psi$ . We write  $y[\mathbf{Th}_{\omega,i}^\#]$  iff  $y$  is a  $\mathbf{Th}_{\omega,i}^\#$ -set.

**Remark 2.20.** Note that

$$y[\mathbf{Th}_{\omega,i}^\#] \Leftrightarrow \exists \Psi \left[ (y = x\Psi) \wedge \mathbf{Pr}_{\mathbf{Th}_{\omega,i}^\#}([\exists!x\Psi[\Psi(x\Psi)]]^c) \right]. \quad (2.97)$$

**Definition 2.23.** Let  $\mathfrak{S}_{\omega,i}$  be a collection such that :

$$\forall x [x \in \mathfrak{S}_{\omega,i} \leftrightarrow x \text{ is a } \mathbf{Th}_{\omega,i}^\# \text{-set}]. \quad (2.98)$$

**Proposition 2.19.** Collection  $\mathfrak{S}_{\omega,i}$  is a  $\mathbf{Th}_{\omega,i}^\#$ -set.

**Proof.** Let us consider an one-place open wff  $\Psi(x)$  such that conditions (2.95) is satisfied, i.e.  $\mathbf{Th}_{\omega,i}^\# \vdash \exists!x\Psi[\Psi(x\Psi)]$ . We note that there exists countable collection  $\mathcal{F}_\Psi$  of the one-place open wff's  $\mathcal{F}_\Psi = \{\Psi_n(x)\}_{n \in \mathbb{N}}$  such that: (i)  $\Psi(x) \in \mathcal{F}_\Psi$  and (ii)

$$\begin{aligned} \mathbf{Th}_{\omega,i}^\# \vdash \exists!x\Psi[\Psi(x\Psi)] \wedge \{ \forall n(n \in \mathbb{N})[\Psi(x\Psi) \leftrightarrow \Psi_n(x\Psi)] \} \\ \text{or in the equivalent form} \\ \mathbf{Th}_{\omega,i}^\# \vdash \mathbf{Pr}_{\mathbf{Th}_{\omega,i}^\#}([\exists!x\Psi[\Psi(x\Psi)]]^c) \wedge \\ \left[ \mathbf{Pr}_{\mathbf{Th}_{\omega,i}^\#}([\forall n(n \in \mathbb{N})[\Psi(x\Psi) \leftrightarrow \Psi_n(x\Psi)]]^c) \right], \end{aligned} \quad (2.99)$$

or in the following equivalent form

$$\begin{aligned} \mathbf{Th}_{\omega,i}^\# \vdash \exists!x_1[\Psi_1(x_1)] \wedge \{ \forall n(n \in \mathbb{N})[\Psi_1(x_1) \leftrightarrow \Psi_{n,1}(x_1)] \} \\ \text{or} \\ \mathbf{Th}_{\omega,i}^\# \vdash \\ \mathbf{Pr}_{\mathbf{Th}_{\omega,i}^\#}([\exists!x_1\Psi(x_1)]^c) \wedge \\ \left[ \mathbf{Pr}_{\mathbf{Th}_{\omega,i}^\#}([\forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]]^c) \right]. \end{aligned} \quad (2.100)$$

where we have set  $\Psi(x) \triangleq \Psi_1(x_1)$ ,  $\Psi_n(x_1) \triangleq \Psi_{n,1}(x_1)$  and  $x_\Psi \triangleq x_1$ . We note that any collection  $\mathcal{F}_{\Psi_k} = \{\Psi_{n,k}(x)\}_{n \in \mathbb{N}}$ ,  $k = 1, 2, \dots$  such as mentioned above, defines an unique set  $x_{\Psi_k}$ , i.e.  $\mathcal{F}_{\Psi_{k_1}} \cap \mathcal{F}_{\Psi_{k_2}} = \emptyset$  iff  $x_{\Psi_{k_1}} \neq x_{\Psi_{k_2}}$ . We note that collections  $\mathcal{F}_{\Psi_k}$ ,  $k = 1, 2, \dots$  is not a part of the  $ZFC_{st}$ , i.e. collection  $\mathcal{F}_{\Psi_k}$  is not a set in the sense of  $ZFC_{st}$ . However that is not a problem, because by using Gödel numbering one can to replace any collection  $\mathcal{F}_{\Psi_k}$ ,  $k = 1, 2, \dots$  by collection  $\Theta_k = g(\mathcal{F}_{\Psi_k})$  of the corresponding Gödel numbers such that

$$\Theta_k = g(\mathcal{F}_{\Psi_k}) = \{g(\Psi_{n,k}(x_k))\}_{n \in \mathbb{N}}, k = 1, 2, \dots \quad (2.101)$$

It is easy to prove that any collection  $\Theta_k = g(\mathcal{F}_{\Psi_k})$ ,  $k = 1, 2, \dots$  is a  $\mathbf{Th}_{\omega,i}^\#$ -set. This is done by Gödel encoding [8],[10] (2.101), by the statement (2.95) and by axiom schema of separation [9]. Let  $g_{n,k} = g(\Psi_{n,k}(x_k))$ ,  $k = 1, 2, \dots$  be a Gödel number of the wff  $\Psi_{n,k}(x_k)$ . Therefore  $g(\mathcal{F}_k) = \{g_{n,k}\}_{n \in \mathbb{N}}$ , where we have set  $\mathcal{F}_k \triangleq \mathcal{F}_{\Psi_k}$ ,  $k = 1, 2, \dots$  and

$$\forall k_1 \forall k_2 [\{g_{n,k_1}\}_{n \in \mathbb{N}} \cap \{g_{n,k_2}\}_{n \in \mathbb{N}} = \emptyset \leftrightarrow x_{k_1} \neq x_{k_2}]. \quad (2.102)$$

Let  $\{\{g_{n,k}\}_{n \in \mathbb{N}}\}_{k \in \mathbb{N}}$  be the family of the sets  $\{g_{n,k}\}_{n \in \mathbb{N}}$ . By axiom of choice [9] one obtains an unique set  $\mathfrak{S}'_i = \{g_k\}_{k \in \mathbb{N}}$  such that  $\forall k [g_k \in \{g_{n,k}\}_{n \in \mathbb{N}}]$ . Finally one obtains a set  $\mathfrak{S}_{\omega,i}$  from the set  $\mathfrak{S}'_i$  by axiom schema of replacement [9].

**Proposition 2.20.** Any collection  $\Theta_k = g(\mathcal{F}_{\Psi_k})$ ,  $k = 1, 2, \dots$  is a  $\mathbf{Th}_{\omega,i}^\#$ -set.

**Proof.** We define  $g_{n,k} = g(\Psi_{n,k}(x_k)) = [\Psi_{n,k}(x_k)]^c$ ,  $v_k = [x_k]^c$ . Therefore  $g_{n,k} = g(\Psi_{n,k}(x_k)) \leftrightarrow \mathbf{Fr}(g_{n,k}, v_k)$  (see [10]). Let us define now predicate  $\Pi_{\omega,i}(g_{n,k}, v_k)$

$$\begin{aligned} & \Pi_{\omega,i}(g_{n,k}, v_k) \leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{\omega,i}^{\#}}([\exists!x_k[\Psi_{1,k}(x_1)]]^c) \wedge \\ & \wedge \exists!x_k(v_k = [x_k]^c) \left[ \forall n(n \in \mathbb{N}) \left[ \mathbf{Pr}_{\mathbf{Th}_{\omega,i}^{\#}}([\Psi_{1,k}(x_k)]]^c) \leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{\omega,i}^{\#}}(\mathbf{Fr}(g_{n,k}, v_k)) \right] \right]. \end{aligned} \quad (2.103)$$

We define now a set  $\Theta_k$  such that

$$\begin{aligned} \Theta_k &= \Theta'_k \cup \{g_k\}, \\ \forall n(n \in \mathbb{N})[g_{n,k} \in \Theta'_k &\leftrightarrow \Pi_{\omega,i}(g_{n,k}, v_k)]. \end{aligned} \quad (2.104)$$

Obviously definitions (2.95) and (2.104) are equivalent.

**Definition 2.24.** We define now the following  $\mathbf{Th}_{\omega,i}^{\#}$ -set  $\mathfrak{R}_{\omega,i} \subseteq \mathfrak{S}_{\omega,i}$  :

$$\forall x[x \in \mathfrak{R}_{\omega,i} \Leftrightarrow (x \in \mathfrak{S}_{\omega,i}) \wedge \mathbf{Pr}_{\mathbf{Th}_{\omega,i}^{\#}}([x \notin x]^c)]. \quad (2.105)$$

**Proposition 2.21.** (i)  $\mathbf{Th}_{\omega,i}^{\#} \vdash \exists \mathfrak{R}_{\omega,i}$ , (ii)  $\mathfrak{R}_{\omega,i}$  is a countable  $\mathbf{Th}_{\omega,i}^{\#}$ -set,  $i \in \mathbb{N}$ .

**Proof.** (i) Statement  $\mathbf{Th}_{\omega,i}^{\#} \vdash \exists \mathfrak{R}_{\omega,i}$  follows immediately by using statement  $\exists \mathfrak{S}_{\omega,i}$  and axiom

schema of separation [9]. (ii) follows immediately from countability of a set  $\mathfrak{S}_{\omega,i}$ .

**Proposition 2.22.** Any set  $\mathfrak{R}_{\omega,i}$ ,  $i \in \mathbb{N}$  is inconsistent.

**Proof.** From formula (2.105) we obtain

$$\mathbf{Th}_{\omega,i}^{\#} \vdash \mathfrak{R}_{\omega,i} \in \mathfrak{R}_{\omega,i} \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{\omega,i}^{\#}}([\mathfrak{R}_{\omega,i} \notin \mathfrak{R}_{\omega,i}]^c). \quad (2.106)$$

From (2.106) we obtain

$$\mathbf{Th}_{\omega,i}^{\#} \vdash \mathfrak{R}_{\omega,i} \in \mathfrak{R}_{\omega,i} \Leftrightarrow \mathfrak{R}_{\omega,i} \notin \mathfrak{R}_{\omega,i} \quad (2.107)$$

and therefore

$$\mathbf{Th}_{\omega,i}^{\#} \vdash (\mathfrak{R}_{\omega,i} \in \mathfrak{R}_{\omega,i}) \wedge (\mathfrak{R}_{\omega,i} \notin \mathfrak{R}_{\omega,i}). \quad (2.108)$$

But this is a contradiction.

**Definition 2.25.** Let  $\Psi = \Psi(x)$  be one-place open  $\mathbf{Th}_{\infty;\omega}^{\#}$ -wff such that the following condition:

$$\mathbf{Th}_{\infty;\omega}^{\#} \vdash \exists!x_{\Psi}[\Psi(x_{\Psi})] \quad (2.109)$$

is satisfied.

**Remark 2.20.** We rewrite now the condition (2.109) using only the language of the theory

$\mathbf{Th}_{\infty}^{\#}$  in the following equivalent form

$$\begin{aligned} 1. \mathbf{Th}_{\infty;\omega}^{\#} \vdash \exists!x_{\Psi}[\Psi(x_{\Psi})] &\Leftrightarrow \mathbf{Th}_{\infty;\omega}^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_{\infty;\omega}^{\#}}([\exists!x_{\Psi}[\Psi(x_{\Psi})]]^c) \\ &\text{or} \\ 2. \mathbf{Th}_{\infty;\omega}^{\#} \vdash \exists!x_{\Psi}[\Psi(x_{\Psi})] &\Leftrightarrow \mathbf{Th}_{\infty;\omega}^{\#} \vdash \left( \mathbf{Pr}_{\mathbf{Th}_{\infty;\omega}^{\#}}([\exists!x_{\Psi}[\Psi(x_{\Psi})]]^c) \right) \wedge \\ &\wedge \left( \mathbf{Pr}_{\mathbf{Th}_{\infty;\omega}^{\#}}([\exists!x_{\Psi}[\Psi(x_{\Psi})]]^c) \Rightarrow \exists!x_{\Psi}[\Psi(x_{\Psi})] \right) \end{aligned} \quad (2.110)$$

**Definition 2.26.** We will say that: (i) a set  $y$  is a  $\mathbf{Th}_{\infty;\omega}^{\#}$ -set if there exist one-place open wff

$\Psi(x)$  such that  $y = x_{\Psi}$ , i.e.  $\mathbf{Th}_{\infty;\omega}^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_{\infty;\omega}^{\#}}([\exists!x_{\Psi}[\Psi(x_{\Psi})]]^c) \wedge (y = x_{\Psi})$ ;

(ii) a set  $y$  is a  $\mathbf{Th}_{\infty;\omega}^{\#}$ -set if there exist one-place open wff

We write  $y[\mathbf{Th}_{\infty;\omega}^{\#}]$  iff  $y$  is a  $\mathbf{Th}_{\infty;\omega}^{\#}$ -set.

**Definition 2.27.** Let  $\mathfrak{S}_{\infty;\omega}$  be a collection such that :  $\forall x[x \in \mathfrak{S}_{\infty;\omega} \Leftrightarrow x [\mathbf{Th}_{\infty;\omega}^{\#}]]$ .

**Proposition 2.23.** Collection  $\mathfrak{S}_{\infty;\omega}$  is a  $\mathbf{Th}_{\infty;\omega}^{\#}$ -set.

**Proof.** Let us consider an one-place open wff  $\Psi(x)$  such that condition (2.109) is satisfied, i.e.  $\mathbf{Th}_{\infty;\omega}^{\#} \vdash \exists!x_{\Psi}[\Psi(x_{\Psi})]$ . We note that there exists countable collection  $\mathcal{F}_{\Psi}$  of the one-place open wff's  $\mathcal{F}_{\Psi} = \{\Psi_n(x)\}_{n \in \mathbb{N}}$  such that: (i)  $\Psi(x) \in \mathcal{F}_{\Psi}$  and (ii)

$$\begin{aligned} \mathbf{Th}_{\infty;\omega}^{\#} \vdash \exists!x_{\Psi}[\Psi(x_{\Psi})] \wedge \{\forall n(n \in \mathbb{N})[\Psi(x_{\Psi}) \leftrightarrow \Psi_n(x_{\Psi})]\} \\ \text{or in the equivalent form} \\ \mathbf{Th}_{\infty;\omega}^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_{\infty;\omega}^{\#}}([\exists!x_{\Psi}[\Psi(x_{\Psi})]]^c) \wedge \\ [\mathbf{Pr}_{\mathbf{Th}_{\infty;\omega}^{\#}}([\forall n(n \in \mathbb{N})[\Psi(x_{\Psi}) \leftrightarrow \Psi_n(x_{\Psi})]]^c)], \end{aligned} \quad (2.111)$$

or in the following equivalent form

$$\begin{aligned} \mathbf{Th}_{\infty;\omega}^{\#} \vdash \exists!x_1[\Psi_1(x_1)] \wedge \{\forall n(n \in \mathbb{N})[\Psi_1(x_1) \leftrightarrow \Psi_{n,1}(x_1)]\} \\ \text{or} \\ \mathbf{Th}_{\infty;\omega}^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_{\infty;\omega}^{\#}}([\exists!x_1\Psi(x_1)]^c) \wedge \\ [\mathbf{Pr}_{\mathbf{Th}_{\infty;\omega}^{\#}}([\forall n(n \in \mathbb{N})[\Psi(x_1) \leftrightarrow \Psi_n(x_1)]]^c)], \end{aligned} \quad (2.112)$$

where we set  $\Psi(x) = \Psi_1(x_1)$ ,  $\Psi_n(x_1) = \Psi_{n,1}(x_1)$  and  $x_{\Psi} = x_1$ . We note that any collection  $\mathcal{F}_{\Psi_k} = \{\Psi_{n,k}(x)\}_{n \in \mathbb{N}}$ ,  $k = 1, 2, \dots$  such as mentioned above defines unique set  $x_{\Psi_k}$ , i.e.  $\mathcal{F}_{\Psi_{k_1}} \cap \mathcal{F}_{\Psi_{k_2}} = \emptyset$  iff  $x_{\Psi_{k_1}} \neq x_{\Psi_{k_2}}$ . We note that the collections  $\mathcal{F}_{\Psi_k}$ ,  $k = 1, 2, \dots$  is not a part of the ZFC, i.e. collection  $\mathcal{F}_{\Psi_k}$  is not a set in the sense of ZFC. However that is not a problem, because by using Gödel numbering one can to replace any collection  $\mathcal{F}_{\Psi_k}$ ,  $k = 1, 2, \dots$  by collection  $\Theta_k = g(\mathcal{F}_{\Psi_k})$  of the corresponding Gödel numbers such that

$$\Theta_k = g(\mathcal{F}_{\Psi_k}) = \{g(\Psi_{n,k}(x_k))\}_{n \in \mathbb{N}}, k = 1, 2, \dots \quad (2.113)$$

It is easy to prove that any collection  $\Theta_k = g(\mathcal{F}_{\Psi_k})$ ,  $k = 1, 2, \dots$  is a  $\mathbf{Th}_{\infty;\omega}^{\#}$ -set. This is done by Gödel encoding [8],[10] by the statement (2.109) and by axiom schema of separation [9]. Let  $g_{n,k} = g(\Psi_{n,k}(x_k))$ ,  $k = 1, 2, \dots$  be a Gödel number of the wff  $\Psi_{n,k}(x_k)$ . Therefore  $g(\mathcal{F}_k) = \{g_{n,k}\}_{n \in \mathbb{N}}$ , where we have set  $\mathcal{F}_k = \mathcal{F}_{\Psi_k}$ ,  $k = 1, 2, \dots$  and

$$\forall k_1 \forall k_2[\{g_{n,k_1}\}_{n \in \mathbb{N}} \cap \{g_{n,k_2}\}_{n \in \mathbb{N}} = \emptyset \Leftrightarrow x_{k_1} \neq x_{k_2}]. \quad (2.114)$$

Let  $\{\{g_{n,k}\}_{n \in \mathbb{N}}\}_{k \in \mathbb{N}}$  be the family of the sets  $\{g_{n,k}\}_{n \in \mathbb{N}}$ . By axiom of choice [9] one obtains unique set  $\mathfrak{S}' = \{g_k\}_{k \in \mathbb{N}}$  such that  $\forall k[g_k \in \{g_{n,k}\}_{n \in \mathbb{N}}]$ . Finally one obtains a set  $\mathfrak{S}_{\infty;\omega}$  from the set  $\mathfrak{S}'_{\infty;\omega}$  by axiom schema of replacement [9]. Thus one can define  $\mathbf{Th}_{\infty;\omega}^{\#}$ -set

$\mathfrak{R}_{\infty;\omega} \subseteq \mathfrak{S}_{\infty;\omega}$  :

$$\forall x[x \in \mathfrak{R}_{\infty;\omega} \Leftrightarrow (x \in \mathfrak{S}_{\infty;\omega}) \wedge \mathbf{Pr}_{\mathbf{Th}_{\infty;\omega}^{\#}}([x \notin x]^c)]. \quad (2.115)$$

**Proposition 2.24.** Any collection  $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1, 2, \dots$  is a  $\mathbf{Th}_{\infty; \omega}^{\#}$ -set.

**Proof.** We define  $g_{n,k} = g(\Psi_{n,k}(x_k)) = [\Psi_{n,k}(x_k)]^c, v_k = [x_k]^c$ . Therefore  $g_{n,k} = g(\Psi_{n,k}(x_k)) \Leftrightarrow \mathbf{Fr}(g_{n,k}, v_k)$  (see [7]). Let us define now predicate  $\Pi_{\infty; \omega}(g_{n,k}, v_k)$

$$\begin{aligned} \Pi_{\infty; \omega}(g_{n,k}, v_k) &\Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{\infty; \omega}^{\#}}([\exists! x_k [\Psi_{1,k}(x_1)]]^c) \wedge \\ &\exists! x_k (v_k = [x_k]^c) [\forall n (n \in \mathbb{N}) [\mathbf{Pr}_{\mathbf{Th}_{\infty; \omega}^{\#}}([\Psi_{1,k}(x_k)]]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{\infty; \omega}^{\#}}(\mathbf{Fr}(g_{n,k}, v_k))]]. \end{aligned} \quad (2.116)$$

We define now a set  $\Theta_k$  such that

$$\begin{cases} \Theta_k = \Theta'_k \cup \{g_k\}, \\ \forall n (n \in \mathbb{N}) [g_{n,k} \in \Theta'_k \leftrightarrow \Pi_{\infty; \omega}(g_{n,k}, v_k)] \end{cases} \quad (2.117)$$

Obviously definitions (2.114) and (2.117) is equivalent by Proposition 2.1.

**Proposition 2.25.** (i)  $\mathbf{Th}_{\infty; \omega}^{\#} \vdash \exists \mathfrak{R}_{\infty; \omega}$ , (ii)  $\mathfrak{R}_{\infty; \omega}$  is a countable  $\mathbf{Th}_{\infty; \omega}^{\#}$ -set.

**Proof.**(i) Statement  $\mathbf{Th}_{\infty; \omega}^{\#} \vdash \exists \mathfrak{R}_{\infty; \omega}$  follows immediately from the statement  $\exists \mathfrak{S}$  and axiom

schema of separation [9], (ii) follows immediately from countability of the set  $\mathfrak{S}_{\infty}$ .

**Proposition 2.26.** Set  $\mathfrak{R}_{\infty; \omega}$  is inconsistent.

**Proof.** From the formula (2.119) we obtain

$$\mathbf{Th}_{\infty; \omega}^{\#} \vdash \mathfrak{R}_{\infty; \omega} \in \mathfrak{R}_{\infty; \omega} \leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{\infty; \omega}^{\#}}([\mathfrak{R}_{\infty; \omega} \notin \mathfrak{R}_{\infty; \omega}]^c). \quad (2.118)$$

From the formula (2.118) and Proposition 2.1 we obtain

$$\mathbf{Th}_{\infty; \omega}^{\#} \vdash \mathfrak{R}_{\infty; \omega} \in \mathfrak{R}_{\infty; \omega} \leftrightarrow \mathfrak{R}_{\infty; \omega} \notin \mathfrak{R}_{\infty; \omega} \quad (2.115)$$

and therefore

$$\mathbf{Th}_{\infty; \omega}^{\#} \vdash (\mathfrak{R}_{\infty; \omega} \in \mathfrak{R}_{\infty; \omega}) \wedge (\mathfrak{R}_{\infty; \omega} \notin \mathfrak{R}_{\infty; \omega}). \quad (2.116)$$

But this is a contradiction.

**Proposition 2.26.** Assume that (i)  $\text{Con}(\mathbf{Th})$  and (ii)  $\mathbf{Th}$  has an nonstandard model  $M_{Nst}^{\mathbf{Th}}$  and  $M_{\omega}^{Z_2} \subset M_{Nst}^{\mathbf{Th}}$ . Then theory  $\mathbf{Th}$  can be extended to a maximally consistent nice theory  $\mathbf{Th}_{\infty}^{\#} \triangleq \mathbf{Th}_{\infty}^{\#}[M_{Nst}^{\mathbf{Th}}]$ .

**Proof.** Let  $\Phi_1 \dots \Phi_i \dots$  be an enumeration of all wff's of the theory  $\mathbf{Th}$  (this can be achieved if the set of propositional variables can be enumerated). Define a chain  $\wp = \{\mathbf{Th}_{Nst,i}^{\#} | i \in \mathbb{N}\}$ ,  $\mathbf{Th}_{Nst,1}^{\#} = \mathbf{Th}$  of consistent theories inductively as follows: assume that theory  $\mathbf{Th}_i$  is defined. (i) Suppose that a statement (2.117) is satisfied

$$\mathbf{Th}_{Nst,i}^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_{Nst,i}^{\#}}([\Phi_i]^c) \text{ and } [\mathbf{Th}_{Nst,i}^{\#} \not\vdash \Phi_i] \wedge [M_{Nst}^{\mathbf{Th}} \models \Phi_i]. \quad (2.117)$$

Then we define a theory  $\mathbf{Th}_{Nst,i+1}^{\#}$  as follows  $\mathbf{Th}_{Nst,i+1}^{\#} \triangleq \mathbf{Th}_{Nst,i}^{\#} \cup \{\Phi_i\}$ . Using Lemma 2.1 we will rewrite the condition (2.117) symbolically as follows

$$\begin{cases} \mathbf{Th}_{Nst,i}^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_{Nst,i}^{\#}}([\Phi_i]^c), \\ \mathbf{Pr}_{\mathbf{Th}_i}^{\#}([\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{Nst,i}^{\#}}([\Phi_i]^c) \wedge [M_{Nst}^{\mathbf{Th}} \models \Phi_i]. \end{cases} \quad (2.118)$$

(ii) Suppose that the statement (2.119) is satisfied

$$\mathbf{Th}_{Nst,i}^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_{Nst,i}^{\#}}([\neg \Phi_i]^c) \text{ and } [\mathbf{Th}_{Nst,i}^{\#} \not\vdash \neg \Phi_i] \wedge [M_{Nst}^{\mathbf{Th}} \models \neg \Phi_i]. \quad (2.119)$$

Then we define theory  $\mathbf{Th}_{i+1}^{\#}$  as follows:  $\mathbf{Th}_{i+1}^{\#} \triangleq \mathbf{Th}_i \cup \{\neg \Phi_i\}$ . Using Lemma 2.2 we will rewrite the condition (2.119) symbolically as follows

$$\left\{ \begin{array}{l} \mathbf{Th}_{Nst,i}^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_{Nst,i}^{\#}}^{\#}([\neg\Phi_i]^c), \\ \mathbf{Pr}_{\mathbf{Th}_{Nst,i}^{\#}}^{\#}([\neg\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{Nst,i}^{\#}}^{\#}([\neg\Phi_i]^c) \wedge [M_{\omega}^{\mathbf{Th}} \models \neg\Phi_i]. \end{array} \right. \quad (2.120)$$

(iii) Suppose that a statement (2.121) is satisfied

$$\mathbf{Th}_{Nst,i}^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_{Nst,i}^{\#}}^{\#}([\Phi_i]^c) \text{ and } \mathbf{Th}_{Nst,i}^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_{Nst,i}^{\#}}^{\#}([\Phi_i]^c) \Rightarrow \Phi_i. \quad (2.121)$$

We will rewrite the condition (2.121) symbolically as follows

$$\left\{ \begin{array}{l} \mathbf{Th}_{Nst,i}^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_{Nst,i}^{\#}}^*([\Phi_i]^c), \\ \mathbf{Pr}_{\mathbf{Th}_{Nst,i}^{\#}}^*([\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_i}([\Phi_i]^c) \wedge [\mathbf{Pr}_{\mathbf{Th}_i}([\Phi_i]^c) \Rightarrow \Phi_i] \end{array} \right. \quad (2.122)$$

Then we define a theory  $\mathbf{Th}_{Nst,i+1}^{\#}$  as follows:  $\mathbf{Th}_{Nst,i+1}^{\#} \triangleq \mathbf{Th}_{Nst,i}^{\#}$ .

(iv) Suppose that the statement (2.123) is satisfied

$$\mathbf{Th}_{Nst,i+1}^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_{Nst,i}^{\#}}^{\#}([\neg\Phi_i]^c) \text{ and } \mathbf{Th}_{Nst,i}^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_{Nst,i}^{\#}}^{\#}([\neg\Phi_i]^c) \Rightarrow \neg\Phi_i. \quad (2.123)$$

We will rewrite the condition (2.123) symbolically as follows

$$\left\{ \begin{array}{l} \mathbf{Th}_{Nst,i}^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_{Nst,i}^{\#}}^*([\neg\Phi_i]^c), \\ \mathbf{Pr}_{\mathbf{Th}_{Nst,i}^{\#}}^*([\neg\Phi_i]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{Nst,i}^{\#}}([\neg\Phi_i]^c) \wedge [\mathbf{Pr}_{\mathbf{Th}_{Nst,i}^{\#}}([\neg\Phi_i]^c) \Rightarrow \neg\Phi_i] \end{array} \right. \quad (2.124)$$

Then we define a theory  $\mathbf{Th}_{Nst,i+1}^{\#}$  as follows:  $\mathbf{Th}_{Nst,i+1}^{\#} \triangleq \mathbf{Th}_{Nst,i}^{\#}$ . We define now a theory  $\mathbf{Th}_{\infty;Nst}^{\#}$  as follows:

$$\mathbf{Th}_{\infty;Nst}^{\#} \triangleq \bigcup_{i \in \mathbb{N}} \mathbf{Th}_{Nst,i}^{\#}. \quad (2.125)$$

First, notice that each  $\mathbf{Th}_{Nst,i}^{\#}$  is consistent. This is done by induction on  $i$  and by Lemmas 2.1-2.2. By assumption, the case is true when  $i = 1$ . Now, suppose  $\mathbf{Th}_{Nst,i}^{\#}$  is consistent. Then its deductive closure  $\mathbf{Ded}(\mathbf{Th}_{Nst,i}^{\#}) \triangleq \{A \mid \mathbf{Th}_{Nst,i}^{\#} \vdash A\}$  is also consistent. If a statement (2.121) is satisfied, i.e.  $\mathbf{Th}_{Nst,i}^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_{Nst,i}^{\#}}^{\#}([\Phi_i]^c)$  and  $\mathbf{Th}_{Nst,i}^{\#} \vdash \Phi_i$ , then clearly  $\mathbf{Th}_{Nst,i+1}^{\#} \triangleq \mathbf{Th}_{Nst,i}^{\#} \cup \{\Phi_i\}$  is consistent since it is a subset of closure  $\mathbf{Ded}(\mathbf{Th}_{Nst,i}^{\#})$ . If a statement (2.123) is satisfied, i.e.  $\mathbf{Th}_{Nst,i}^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_{Nst,i}^{\#}}^{\#}([\neg\Phi_i]^c)$  and  $\mathbf{Th}_{Nst,i}^{\#} \vdash \neg\Phi_i$ , then clearly  $\mathbf{Th}_{Nst,i+1}^{\#} \triangleq \mathbf{Th}_{Nst,i}^{\#} \cup \{\neg\Phi_i\}$  is consistent since it is a subset of closure  $\mathbf{Ded}(\mathbf{Th}_{Nst,i}^{\#})$ . If a statement (2.117) is satisfied, i.e.  $\mathbf{Th}_{Nst,i}^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_{Nst,i}^{\#}}^{\#}([\Phi_i]^c)$  and  $[\mathbf{Th}_{Nst,i}^{\#} \not\vdash \Phi_i] \wedge [M_{Nst}^{\mathbf{Th}} \models \Phi_i]$  then clearly  $\mathbf{Th}_{Nst,i+1}^{\#} \triangleq \mathbf{Th}_{Nst,i}^{\#} \cup \{\Phi_i\}$  is consistent by Lemma 2.1 and by one of the standard properties of consistency:  $\Delta \cup \{A\}$  is consistent iff  $\Delta \not\vdash \neg A$ . If a statement (2.119) is satisfied, i.e.  $\mathbf{Th}_{Nst,i}^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_{Nst,i}^{\#}}^{\#}([\neg\Phi_i]^c)$  and  $[\mathbf{Th}_{Nst,i}^{\#} \not\vdash \neg\Phi_i] \wedge [M_{Nst}^{\mathbf{Th}} \models \neg\Phi_i]$  then clearly  $\mathbf{Th}_{Nst,i+1}^{\#} \triangleq \mathbf{Th}_{Nst,i}^{\#} \cup \{\neg\Phi_i\}$  is consistent by Lemma 2.2 and by one of the standard properties of consistency:  $\Delta \cup \{\neg A\}$  is consistent iff  $\Delta \not\vdash A$ . Next, notice  $\mathbf{Ded}(\mathbf{Th}_{\infty;Nst}^{\#})$  is maximally consistent nice extension of the  $\mathbf{Ded}(\mathbf{Th})$ .  $\mathbf{Ded}(\mathbf{Th}_{\infty;Nst}^{\#})$  is consistent because, by the standard Lemma 2.3 above, it is the union of a chain of consistent sets. To see that  $\mathbf{Ded}(\mathbf{Th}_{\infty;Nst}^{\#})$  is maximal, pick any wff  $\Phi$ . Then  $\Phi$  is some  $\Phi_i$  in the enumerated list of all wff's. Therefore for any  $\Phi$  such that  $\mathbf{Th}_{Nst,i}^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_{Nst,i}^{\#}}^{\#}([\Phi]^c)$  or  $\mathbf{Th}_{Nst,i}^{\#} \vdash \mathbf{Pr}_{\mathbf{Th}_{Nst,i}^{\#}}^{\#}([\neg\Phi]^c)$ , either  $\Phi \in \mathbf{Th}_{\infty;Nst}^{\#}$  or  $\neg\Phi \in \mathbf{Th}_{\infty;Nst}^{\#}$ . Since  $\mathbf{Ded}(\mathbf{Th}_{Nst,i+1}^{\#}) \subseteq \mathbf{Ded}(\mathbf{Th}_{\infty;Nst}^{\#})$ , we have  $\Phi \in \mathbf{Ded}(\mathbf{Th}_{\infty;Nst}^{\#})$  or  $\neg\Phi \in \mathbf{Ded}(\mathbf{Th}_{\infty;Nst}^{\#})$ , which implies that  $\mathbf{Ded}(\mathbf{Th}_{\infty;Nst}^{\#})$  is maximally

consistent nice extension of the **Ded(Th)**.

**Definition 2.28.** We define now predicate  $\mathbf{Pr}_{\mathbf{Th}^\#}([\Phi_i]^c)$  asserting provability in  $\mathbf{Th}_{\infty;Nst}^\#$  :

$$\left\{ \begin{array}{l} \mathbf{Pr}_{\mathbf{Th}_{\infty;Nst}^\#}([\Phi_i]^c) \Leftrightarrow [\mathbf{Pr}_{\mathbf{Th}_{\infty;Nst}^\#}([\Phi_i]^c)] \vee [\mathbf{Pr}_{\mathbf{Th}_{\infty;Nst}^\#}^*([\Phi_i]^c)], \\ \mathbf{Pr}_{\mathbf{Th}_{\infty;Nst}^\#}([\neg\Phi_i]^c) \Leftrightarrow [\mathbf{Pr}_{\mathbf{Th}_{\infty;Nst}^\#}([\neg\Phi_i]^c)] \vee [\mathbf{Pr}_{\mathbf{Th}_{\infty;Nst}^\#}^*([\neg\Phi_i]^c)]. \end{array} \right. \quad (2.126)$$

**Definition 2.29.** Let  $\Psi = \Psi(x)$  be one-place open wff such that the conditions:

(\*)  $\mathbf{Th}_{\infty;Nst}^\# \vdash \exists!x_\Psi[\Psi(x_\Psi)]$  or

(\*\*\*)  $\mathbf{Th}_{\infty;Nst}^\# \vdash \mathbf{Pr}_{\mathbf{Th}_{\infty;Nst}^\#}([\exists!x_\Psi[\Psi(x_\Psi)]]^c)$  and  $M_{Nst}^{\mathbf{Th}} \models \exists!x_\Psi[\Psi(x_\Psi)]$  is satisfied.

Then we said that, a set  $y$  is a  $\mathbf{Th}^\#$ -set iff there is exist one-place open wff  $\Psi(x)$  such that

$y = x_\Psi$ . We write  $y[\mathbf{Th}_{\infty;Nst}^\#]$  iff  $y$  is a  $\mathbf{Th}_{\infty;Nst}^\#$ -set.

**Remark 2.21.** Note that  $[(*) \vee (***)] \Rightarrow \mathbf{Th}_{\infty;Nst}^\# \vdash \exists!x_\Psi[\Psi(x_\Psi)]$ .

**Remark 2.22.** Note that  $y[\mathbf{Th}_{\infty;Nst}^\#] \Leftrightarrow \exists\Psi[(y = x_\Psi) \wedge \mathbf{Pr}_{\mathbf{Th}_{\infty;Nst}^\#}([\exists!x_\Psi[\Psi(x_\Psi)]]^c)]$

**Definition 2.30.** Let  $\mathfrak{S}_{\infty;Nst}^\#$  be a collection such that :  $\forall x[x \in \mathfrak{S}_{\infty;Nst}^\# \leftrightarrow x \text{ is a } \mathbf{Th}^\# \text{-set}]$ .

**Proposition 2.27.** Collection  $\mathfrak{S}_{\infty;Nst}^\#$  is a  $\mathbf{Th}_{\infty;Nst}^\#$ -set.

**Proof.** Let us consider an one-place open wff  $\Psi(x)$  such that conditions (\*) or (\*\*\*) is satisfied, i.e.  $\mathbf{Th}^\# \vdash \exists!x_\Psi[\Psi(x_\Psi)]$ . We note that there exists countable collection  $\mathcal{F}_\Psi$  of the one-place open wff's  $\mathcal{F}_\Psi = \{\Psi_n(x)\}_{n \in \mathbb{N}}$  such that: (i)  $\Psi(x) \in \mathcal{F}_\Psi$  and (ii)

$$\mathbf{Th}_{\infty;Nst}^\# \vdash \exists!x_\Psi \left[ [\Psi(x_\Psi)] \wedge \left\{ \forall n \left( n \in M_{\omega}^{Z_{\omega}^{Hs}} \right) [\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)] \right\} \right]$$

or

$$\mathbf{Th}_{\infty;Nst}^\# \vdash \exists!x_\Psi \left[ \mathbf{Pr}_{\mathbf{Th}_{\infty;Nst}^\#}([\Psi(x_\Psi)]^c) \wedge \left\{ \forall n \left( n \in M_{\omega}^{Z_{\omega}^{Hs}} \right) \mathbf{Pr}_{\mathbf{Th}_{\infty;Nst}^\#}([\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)]^c) \right\} \right] \quad (2.127)$$

and

$$M_{Nst}^{\mathbf{Th}} \models \exists!x_\Psi \left[ [\Psi(x_\Psi)] \wedge \left\{ \forall n \left( n \in M_{\omega}^{Z_{\omega}^{Hs}} \right) [\Psi(x_\Psi) \leftrightarrow \Psi_n(x_\Psi)] \right\} \right]$$

or of the equivalent form

$$\mathbf{Th}_{\infty;Nst}^\# \vdash \exists!x_1 \left[ [\Psi_1(x_1)] \wedge \left\{ \forall n \left( n \in M_{\omega}^{Z_{\omega}^{Hs}} \right) [\Psi_1(x_1) \leftrightarrow \Psi_{n,1}(x_1)] \right\} \right]$$

or

$$\mathbf{Th}_{\infty;Nst}^\# \vdash \exists!x_\Psi \left[ \mathbf{Pr}_{\mathbf{Th}_{\infty;Nst}^\#}([\Psi(x_1)]^c) \wedge \left\{ \forall n \left( n \in M_{\omega}^{Z_{\omega}^{Hs}} \right) \mathbf{Pr}_{\mathbf{Th}_{\infty;Nst}^\#}([\Psi(x_1) \leftrightarrow \Psi_n(x_1)]^c) \right\} \right] \quad (2.128)$$

and

$$M_{Nst}^{\mathbf{Th}} \models \exists!x_\Psi \left[ [\Psi(x_1)] \wedge \left\{ \forall n \left( n \in M_{\omega}^{Z_{\omega}^{Hs}} \right) [\Psi(x_1) \leftrightarrow \Psi_n(x_1)] \right\} \right]$$

where we set  $\Psi(x) = \Psi_1(x_1)$ ,  $\Psi_n(x_1) = \Psi_{n,1}(x_1)$  and  $x_\Psi = x_1$ . We note that any collection  $\mathcal{F}_{\Psi_k} = \{\Psi_{n,k}(x)\}_{n \in \mathbb{N}}$ ,  $k = 1, 2, \dots$  such above defines an unique set  $x_{\Psi_k}$ , i.e.

$\mathcal{F}_{\Psi_{k_1}} \cap \mathcal{F}_{\Psi_{k_2}} = \emptyset$  iff  $x_{\Psi_{k_1}} \neq x_{\Psi_{k_2}}$ . We note that collections  $\mathcal{F}_{\Psi_k}$ ,  $k = 1, 2, \dots$  is no part of the  $ZFC_2^{Hs}$ , i.e. collection  $\mathcal{F}_{\Psi_k}$  there is no set in sense of  $ZFC_2^{Hs}$ . However that is no problem, because by using Gödel numbering one can to replace any collection  $\mathcal{F}_{\Psi_k}$ ,  $k = 1, 2, \dots$  by collection  $\Theta_k = g(\mathcal{F}_{\Psi_k})$  of the corresponding Gödel numbers such that

$$\Theta_k = g(\mathcal{F}_{\Psi_k}) = \{g(\Psi_{n,k}(x_k))\}_{n \in \mathbb{N}}, k = 1, 2, \dots \quad (2.129)$$

It is easy to prove that any collection  $\Theta_k = g(\mathcal{F}_{\Psi_k})$ ,  $k = 1, 2, \dots$  is a  $\mathbf{Th}_{\infty;Nst}^\#$ -set. This is done

by Gödel encoding [8],[10] (2.129) and by axiom schema of separation [9]. Let  $g_{n,k} = g(\Psi_{n,k}(x_k)), k = 1, 2, \dots$  be a Gödel number of the wff  $\Psi_{n,k}(x_k)$ . Therefore  $g(\mathcal{F}_k) = \{g_{n,k}\}_{n \in \mathbb{N}}$ , where we set  $\mathcal{F}_k = \mathcal{F}_{\Psi_k}, k = 1, 2, \dots$  and

$$\forall k_1 \forall k_2 [\{g_{n,k_1}\}_{n \in \mathbb{N}} \cap \{g_{n,k_2}\}_{n \in \mathbb{N}} = \emptyset \leftrightarrow x_{k_1} \neq x_{k_2}]. \quad (2.130)$$

Let  $\{\{g_{n,k}\}_{n \in \mathbb{N}}\}_{k \in \mathbb{N}}$  be a family of the all sets  $\{g_{n,k}\}_{n \in \mathbb{N}}$ . By axiom of choice [9] one obtain unique set  $\mathfrak{S}_{\infty;Nst}^\# = \{g_k\}_{k \in \mathbb{N}}$  such that  $\forall k [g_k \in \{g_{n,k}\}_{n \in \mathbb{N}}]$ . Finally one obtain a set  $\mathfrak{S}_{\infty;Nst}^\#$  from a set  $\mathfrak{S}_{\infty;Nst}^\#$  by axiom schema of replacement [9]. Thus we can define a  $\mathbf{Th}_{\infty;Nst}^\#$ -set

$\mathfrak{R}_{\infty;Nst}^\# \subseteq \mathfrak{S}_{\infty;Nst}^\# :$

$$\forall x [x \in \mathfrak{R}_{\infty;Nst}^\# \leftrightarrow (x \in \mathfrak{S}_{\infty;Nst}^\#) \wedge \mathbf{Pr}_{\mathbf{Th}_{\infty;Nst}^\#}([x \notin x]^c) \wedge (\mathbf{Pr}_{\mathbf{Th}_{\infty;Nst}^\#}([x \notin x]^c \Rightarrow x \notin x))]. \quad (2.131)$$

**Proposition 2.28.** Any collection  $\Theta_k = g(\mathcal{F}_{\Psi_k}), k = 1, 2, \dots$  is a  $\mathbf{Th}_{\infty;Nst}^\#$ -set.

**Proof.** We define  $g_{n,k} = g(\Psi_{n,k}(x_k)) = [\Psi_{n,k}(x_k)]^c, v_k = [x_k]^c$ . Therefore  $g_{n,k} = g(\Psi_{n,k}(x_k)) \Leftrightarrow \mathbf{Fr}(g_{n,k}, v_k)$  (see [10]). Let us define now predicate  $\Pi_\infty(g_{n,k}, v_k)$

$$\begin{aligned} \Pi_\infty(g_{n,k}, v_k) &\Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{\infty;Nst}^\#}([\exists! x_k [\Psi_{1,k}(x_1)]]^c) \wedge \\ &\quad \wedge \exists! x_k (v_k = [x_k]^c) \\ &[\forall n (n \in M_{st}^{Z_2^{Hs}}) [\mathbf{Pr}_{\mathbf{Th}_{\infty;Nst}^\#}([\Psi_{1,k}(x_k)]]^c) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_{\infty;Nst}^\#}(\mathbf{Fr}(g_{n,k}, v_k))]]. \end{aligned} \quad (2.132)$$

We define now a set  $\Theta_k$  such that

$$\left\{ \begin{array}{l} \Theta_k = \Theta'_k \cup \{g_k\}, \\ \forall n (n \in \mathbb{N}) [g_{n,k} \in \Theta'_k \leftrightarrow \Pi_\infty(g_{n,k}, v_k)] \end{array} \right. \quad (2.133)$$

But obviously definitions (2.29) and (2.133) is equivalent by Proposition 2.26.

**Proposition 2.28.** (i)  $\mathbf{Th}_{\infty;Nst}^\# \vdash \exists \mathfrak{R}_{\infty;Nst}^\#,$  (ii)  $\mathfrak{R}_{\infty;Nst}^\#$  is a countable  $\mathbf{Th}_{\infty;Nst}^\#$ -set.

**Proof.** (i) Statement  $\mathbf{Th}^\# \vdash \exists \mathfrak{R}_c$  follows immediately from the statement  $\exists \mathfrak{S}_{\infty;Nst}^\#$  and axiom

schema of separation [9]. (ii) follows immediately from countability of the set  $\mathfrak{S}_{\infty;Nst}^\#$ .

**Proposition 2.29.** A set  $\mathfrak{R}_{\infty;Nst}^\#$  is inconsistent.

**Proof.** From formula (2.131) we obtain

$$\mathbf{Th}_{\infty;Nst}^\# \vdash \mathfrak{R}_{\infty;Nst}^\# \in \mathfrak{R}_{\infty;Nst}^\# \Leftrightarrow \mathfrak{R}_{\infty;Nst}^\# \notin \mathfrak{R}_{\infty;Nst}^\#. \quad (2.134)$$

From formula (2.41) and Proposition 2.6 one obtains

$$\mathbf{Th}_{\infty;Nst}^\# \vdash \mathfrak{R}_{\infty;Nst}^\# \in \mathfrak{R}_{\infty;Nst}^\# \Leftrightarrow \mathfrak{R}_{\infty;Nst}^\# \notin \mathfrak{R}_{\infty;Nst}^\# \quad (2.135)$$

and therefore

$$\mathbf{Th}_{\infty;Nst}^\# \vdash (\mathfrak{R}_{\infty;Nst}^\# \in \mathfrak{R}_{\infty;Nst}^\#) \wedge (\mathfrak{R}_{\infty;Nst}^\# \notin \mathfrak{R}_{\infty;Nst}^\#). \quad (2.136)$$

But this is a contradiction.

## 2.3. Proof of the inconsistency of the set theory

$ZFC_2^{Hs} + \exists M^{ZFC_2^{Hs}}$  using Generalized Tarski's undefinability theorem.

In this section we will prove that a set theory  $ZFC_2^{Hs} + \exists M^{ZFC_2^{Hs}}$  is inconsistent, without



any reference to the set  $\mathfrak{T}_\infty$  and inconsistent set  $\mathfrak{R}_\infty$ .

**Proposition 2.30.** (Generalized Tarski's undefinability theorem). Let  $\mathbf{Th}_\mathcal{L}^{Hs}$  be second order

theory with Henkin semantics and with formal language  $\mathcal{L}$ , which includes negation and

has a Gödel encoding  $g(\cdot)$  such that for every  $\mathcal{L}$ -formula  $A(x)$  there is a formula  $B$  such

that  $B \Leftrightarrow A(g(B)) \wedge [A(g(B)) \Rightarrow B]$  holds. Assume that  $\mathbf{Th}_\mathcal{L}^{Hs}$  has an standard Model  $M$ .

Then there is no  $\mathcal{L}$ -formula  $\mathbf{True}(n)$  such that for every  $\mathcal{L}$ -formula  $A$  such that  $M \models A$ , the

following equivalence

$$A \Leftrightarrow \mathbf{True}(g(A)) \wedge [\mathbf{True}(g(A)) \Rightarrow A] \quad (2.137)$$

holds.

**Proof.** The diagonal lemma yields a counterexample to this equivalence, by giving a "Liar"

sentence  $S$  such that  $S \Leftrightarrow \neg \mathbf{True}(g(S))$  holds.

**Remark 2.23.** Above we defined the set  $\mathfrak{T}_\infty$  (see Definition 2.10) in fact using generalized

"truth predicate"  $\mathbf{True}_\infty^\#([\Phi]^c, \Phi)$  such that

$$\mathbf{True}_\infty^\#([\Phi]^c, \Phi) \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\Phi]^c) \wedge \{\mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\Phi]^c) \Rightarrow \Phi\}. \quad (2.138)$$

In order to prove that set theory  $ZFC_2^{Hs} + \exists M^{ZFC_2^{Hs}}$  is inconsistent without any reference to

the set  $\mathfrak{T}_\infty$ , notice that by the properties of the nice extension  $\mathbf{Th}_\infty^\#$  follows that definition given by (2.138) is correct, i.e., for every  $ZFC_2^{Hs}$ -formula  $\Phi$  such that  $M^{ZFC_2^{Hs}} \models \Phi$  the following equivalence

$$\Phi \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\Phi]^c) \wedge \{\mathbf{Pr}_{\mathbf{Th}_\infty^\#}([\Phi]^c) \Rightarrow \Phi\}. \quad (2.139)$$

holds.

**Proposition 2.31.** Set theory  $\mathbf{Th}_1^\# = ZFC_2^{Hs} + \exists M^{ZFC_2^{Hs}}$  is inconsistent.

**Proof.** Notice that by the properties of the nice extension  $\mathbf{Th}_\infty^\#$  of the  $\mathbf{Th}_1^\#$  follows that

$$M^{ZFC_2^{Hs}} \models \Phi \Rightarrow \mathbf{Th}_\infty^\# \vdash \Phi. \quad (2.140)$$

Therefore (2.138) gives generalized "truth predicate" for set theory  $\mathbf{Th}_1^\#$ . By Proposition 2.30 one obtains a contradiction.

**Remark 2.24.** A cardinal  $\kappa$  is inaccessible if and only if  $\kappa$  has the following reflection property: for all subsets  $U \subset V_\kappa$ , there exists  $\alpha < \kappa$  such that  $(V_\alpha, \in, U \cap V_\alpha)$  is an elementary substructure of  $(V_\kappa, \in, U)$ . (In fact, the set of such  $\alpha$  is closed unbounded in  $\kappa$ .) Equivalently,  $\kappa$  is  $\Pi_n^0$ -indescribable for all  $n \geq 0$ .

**Remark 2.25.** Under  $ZFC$  it can be shown that  $\kappa$  is inaccessible if and only if  $(V_\kappa, \in)$  is a model of second order  $ZFC$ , [5].

**Remark 2.26.** By the reflection property, there exists  $\alpha < \kappa$  such that  $(V_\alpha, \in)$  is a standard model of (first order)  $ZFC$ . Hence, the existence of an inaccessible cardinal is a stronger hypothesis than the existence of the standard model of  $ZFC_2^{Hs}$ .

### 3. Derivation inconsistent countable set in set theory $ZFC_2$ with the full semantics.

Let  $\mathbf{Th} = \mathbf{Th}^{fss}$  be an second order theory with the full second order semantics. We assume now that  $\mathbf{Th}$  contains  $ZFC_2^{fss}$ . We will write for short  $\mathbf{Th}$ , instead  $\mathbf{Th}^{fss}$ .

**Remark 3.1.** Notice that  $M$  is a model of  $ZFC_2^{fss}$  if and only if it is isomorphic to a model of

the form  $V_{\kappa} \in \cap (V_{\kappa} \times V_{\kappa})$ , for  $\kappa$  a strongly inaccessible ordinal.

**Remark 3.2.** Notice that a standard model for the language of first-order set theory is an ordered pair  $\{D, I\}$ . Its domain,  $D$ , is a nonempty set and its interpretation function,  $I$ , assigns a set of ordered pairs to the two-place predicate " $\in$ ". A sentence is true in  $\{D, I\}$  just in case it is satisfied by all assignments of first-order variables to members of  $D$  and second-order variables to subsets of  $D$ ; a sentence is satisfiable just in case it is true in some standard model; finally, a sentence is valid just in case it is true in all standard models.

**Remark 3.3.** Notice that:

(I) The assumption that  $D$  and  $I$  be sets is not without consequence. An immediate effect of this stipulation is that no standard model provides the language of set theory with its intended interpretation. In other words, there is no standard model  $\{D, I\}$  in which  $D$  consists of all sets and  $I$  assigns the standard element-set relation to " $\in$ ". For it is a theorem of  $ZFC$  that there is no set of all sets and that there is no set of ordered-pairs  $\{x, y\}$  for  $x$  an element of  $y$ .

(II) Thus, on the standard definition of model:

(1) it is not at all obvious that the validity of a sentence is a guarantee of its truth;

(2) similarly, it is far from evident that the truth of a sentence is a guarantee of its satisfiability in some standard model.

(3) If there is a connection between satisfiability, truth, and validity, it is not one that can be

"read off" standard model theory.

(III) Nevertheless this is not a problem in the first-order case since set theory provides us

with two reassuring results for the language of first-order set theory. One result is the first

order completeness theorem according to which first-order sentences are provable, if true in all models. Granted the truth of the axioms of the first-order predicate calculus and the truth preserving character of its rules of inference, we know that a sentence of the first-order language of set theory is true, if it is provable. Thus, since valid sentences are provable and provable sentences are true, we know that valid sentences

are true. The connection between truth and satisfiability immediately follows: if  $\phi$  is unsatisfiable, then  $\neg\phi$ , its negation, is true in all models and hence valid. Therefore,  $\neg\phi$  is true and  $\phi$  is false.

**Definition 3.1.** The language of second order arithmetic  $Z_2$  is a two-sorted language: there are two kinds of terms, numeric terms and set terms.

$\mathbf{0}$  is a numeric term,

1. There are in nitely many numeric variables,  $x_0, x_1, \dots, x_n, \dots$  each of which

- is a numeric term;
- 2.If  $s$  is a numeric term then  $Ss$  is a numeric term;
- 3.If  $s, t$  are numeric terms then  $+st$  and  $\cdot st$  are numeric terms (abbreviated  $s + t$  and  $s \cdot t$ );
- 3.There are infinitely many set variables,  $X_0, X_1, \dots, X_n \dots$  each of which is a set term;
- 4.If  $t$  is a numeric term and  $S$  then  $\in tS$  is an atomic formula (abbreviated  $t \in S$ );
- 5.If  $s$  and  $t$  are numeric terms then  $= st$  and  $< st$  are atomic formulas (abbreviated  $s = t$  and  $s < t$  correspondingly).

The formulas are built from the atomic formulas in the usual way.

As the examples in the definition suggest, we use upper case letters for set variables and lower case letters for numeric terms. (Note that the only set terms are the variables.) It will be more convenient to work with functions instead of sets, but within arithmetic, these are equivalent: one can use the pairing operation, and say that  $X$  represents a function if for each  $n$  there is exactly one  $m$  such that the pair  $(n, m)$  belongs to  $X$ .

We have to consider what we intend the semantics of this language to be. One possibility is the semantics of full second order logic: a model consists of a set  $M$ , representing the numeric objects, and interpretations of the various functions and relations (probably with the requirement that equality be the genuine equality relation), and a statement  $\forall X\Phi(X)$  is satisfied by the model if for every possible subset of  $M$ , the corresponding statement holds.

**Remark 3.1.** Full second order logic has no corresponding proof system. An easy way to see this is to observe that it has no compactness theorem. For example, the only

model (up to isomorphism) of Peano arithmetic together with the second order induction

axiom:  $\forall X(\mathbf{0} \in X \wedge \forall x(x \in X \Rightarrow \mathbf{S}x \in X) \Rightarrow \forall x(x \in X))$  is the standard model  $\mathbb{N}$ . This is

easily seen: any model of Peano arithmetic has an initial segment isomorphic to  $\mathbb{N}$ ; applying the induction axiom to this set, we see that it must be the whole of the model.

**Remark 3.2.** There is no completeness theorem for second-order logic. Nor do the axioms

of second-order ZFC imply a reflection principle which ensures that if a sentence of second-order set theory is true, then it is true in some standard model. Thus there may be sentences of the language of second-order set theory that are true but unsatisfiable, or sentences that are valid, but false. To make this possibility vivid, let  $Z$  be the conjunction of all the axioms of second-order ZFC.  $Z$  is surely true. But the existence of a model for  $Z$  requires the existence of strongly inaccessible cardinals. The axioms of second-order ZFC don't entail the existence of strongly inaccessible cardinals, and hence the satisfiability of  $Z$  is independent of second-order ZFC. Thus,  $Z$  is true but its unsatisfiability is consistent with second-order ZFC [5].

Thus with respect to  $ZFC_2^{fss}$ , this is a semantically defined system and thus it is not standard to speak about it being contradictory if anything, one might attempt to prove that

it has no models, which to be what is being done in section 2 for  $ZFC_2^{Hs}$ .

**Definition 3.2.** Using formula (2.3) one can define predicate  $\mathbf{Pr}_{\mathbf{Th}}^\#(y)$  really asserting provability in  $\mathbf{Th} = ZFC_2^{fss}$

$$\begin{aligned}\mathbf{Pr}_{\mathbf{Th}}^\#(y) &\Leftrightarrow \mathbf{Pr}_{\mathbf{Th}}(y) \wedge [\mathbf{Pr}_{\mathbf{Th}}(y) \Rightarrow \Phi], \\ \mathbf{Pr}_{\mathbf{Th}}(y) &\Leftrightarrow \exists x \left( x \in M_\omega^{Z_2^{fss}} \right) \mathbf{Prov}_{\mathbf{Th}}(x, y), \\ &y = [\Phi]^c.\end{aligned}\tag{3.1}$$

**Theorem 3.1.**[12].(Löb's Theorem for  $ZFC_2^{fss}$ ) Let  $\Phi$  be any closed formula with code  $y = [\Phi]^c \in M_\omega^{Z_2}$ , then  $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^c)$  implies  $\mathbf{Th} \vdash \Phi$  (see [12] Theorem 5.1).

**Proof.** Assume that

(#)  $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^c)$ .

Note that

(1)  $\mathbf{Th} \not\vdash \neg\Phi$ . Otherwise one obtains  $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\neg\Phi]^c) \wedge \mathbf{Pr}_{\mathbf{Th}}([\Phi]^c)$ , but this is a contradiction.

(2) Assume now that (2.i)  $\mathbf{Th} \vdash \mathbf{Pr}_{\mathbf{Th}}([\Phi]^c)$  and (2.ii)  $\mathbf{Th} \not\vdash \Phi$ .

From (1) and (2.ii) follows that

(3)  $\mathbf{Th} \not\vdash \neg\Phi$  and  $\mathbf{Th} \not\vdash \Phi$ .

Let  $\mathbf{Th}_{\neg\Phi}$  be a theory

(4)  $\mathbf{Th}_{\neg\Phi} \triangleq \mathbf{Th} \cup \{\neg\Phi\}$ . From (3) follows that

(5)  $\text{Con}(\mathbf{Th}_{\neg\Phi})$ .

From (4) and (5) follows that

(6)  $\mathbf{Th}_{\neg\Phi} \vdash \mathbf{Pr}_{\mathbf{Th}_{\neg\Phi}}([\neg\Phi]^c)$ .

From (4) and (#) follows that

(7)  $\mathbf{Th}_{\neg\Phi} \vdash \mathbf{Pr}_{\mathbf{Th}_{\neg\Phi}}([\Phi]^c)$ .

From (6) and (7) follows that

(8)  $\mathbf{Th}_{\neg\Phi} \vdash \mathbf{Pr}_{\mathbf{Th}_{\neg\Phi}}([\Phi]^c) \wedge \mathbf{Pr}_{\mathbf{Th}_{\neg\Phi}}([\neg\Phi]^c)$ , but this is a contradiction.

**Definition 3.3.** Let  $\Psi = \Psi(x)$  be one-place open wff such that:

$$\mathbf{Th} \vdash \exists! x_\Psi [\Psi(x_\Psi)]\tag{3.2}$$

Then we will say that, a set  $y$  is a **Th**-set iff there is exist one-place open wff  $\Psi(x)$  such

that  $y = x_\Psi$ . We write  $y[\mathbf{Th}]$  iff  $y$  is a **Th**-set.

**Remark 3.2.** Note that

$$\begin{aligned}y[\mathbf{Th}] &\Leftrightarrow \\ &\exists \Psi [(y = x_\Psi) \wedge \mathbf{Pr}_{\mathbf{Th}}([\exists! x_\Psi [\Psi(x_\Psi)]]^c) \wedge [(\mathbf{Pr}_{\mathbf{Th}}([\exists! x_\Psi [\Psi(x_\Psi)]]^c) \Rightarrow \exists! x_\Psi [\Psi(x_\Psi)])]]\end{aligned}\tag{3.3}$$

**Definition 3.4.** Let  $\mathfrak{S}$  be a collection such that :  $\forall x [x \in \mathfrak{S} \leftrightarrow x \text{ is a } \mathbf{Th}\text{-set}]$ .

**Proposition 3.1.** Collection  $\mathfrak{S}$  is a **Th**-set.

**Definition 3.4.** We define now a **Th**-set  $\mathfrak{R}_c \subseteq \mathfrak{S}$  :

$$\forall x [x \in \mathfrak{R}_c \leftrightarrow (x \in \mathfrak{S}) \wedge \mathbf{Pr}_{\mathbf{Th}}([x \notin x]^c) \wedge [\mathbf{Pr}_{\mathbf{Th}}([x \notin x]^c) \Rightarrow x \notin x]].\tag{3.4}$$

**Proposition 3.2.** (i)  $\mathbf{Th} \vdash \exists \mathfrak{R}_c$ , (ii)  $\mathfrak{R}_c$  is a countable **Th**-set.

**Proof.**(i) Statement  $\mathbf{Th} \vdash \exists \mathfrak{R}_c$  follows immediately by using statement  $\exists \mathfrak{S}$  and axiom schema of separation [4], (ii) follows immediately from countability of a set  $\mathfrak{S}$ .

**Proposition 3.3.** A set  $\mathfrak{R}_c$  is inconsistent.

**Proof.** From formula (3.2) one obtains

$$\mathbf{Th} \vdash \mathfrak{R}_c \in \mathfrak{R}_c \Leftrightarrow \mathbf{Pr}_{\mathbf{Th}}([\mathfrak{R}_c \notin \mathfrak{R}_c]^c) \wedge [\mathbf{Pr}_{\mathbf{Th}}([\mathfrak{R}_c \notin \mathfrak{R}_c]^c) \Rightarrow \mathfrak{R}_c \notin \mathfrak{R}_c]. \quad (3.5)$$

From formula (3.4) and definition 3.5 one obtains

$$\mathbf{Th} \vdash \mathfrak{R}_c \in \mathfrak{R}_c \Leftrightarrow \mathfrak{R}_c \notin \mathfrak{R}_c \quad (3.6)$$

and therefore

$$\mathbf{Th} \vdash (\mathfrak{R}_c \in \mathfrak{R}_c) \wedge (\mathfrak{R}_c \notin \mathfrak{R}_c). \quad (3.7)$$

But this is a contradiction.

Thus finally we obtain:

**Theorem 3.2.**[12].  $\neg \text{Con}(ZFC_2^{fss})$ .

It well known that under  $ZFC$  it can be shown that  $\kappa$  is inaccessible if and only if  $(V_{\kappa}, \in)$  is a

model of  $ZFC_2$  [5],[11]. Thus finally we obtain.

**Theorem 3.3.**[12].  $\neg \text{Con}(ZFC + \exists M_{st}^{ZFC}(M_{st}^{ZFC} = H_k))$ .

## 4. Consistency Results in Topology.

**Definition 4.1.**[19]. A Lindelöf space is indestructible if it remains Lindelöf after forcing with any countably closed partial order.

**Theorem 4.1.**[20]. If it is consistent with  $ZFC$  that there is an inaccessible cardinal, then it

is consistent with  $ZFC$  that every Lindelöf  $T_3$  indestructible space of weight  $\leq \aleph_1$  has size  $\leq \aleph_1$ .

**Corollary 4.1.**[20] The existence of an inaccessible cardinal and the statement:  $\mathcal{L}[T_3, \leq \aleph_1, \leq \aleph_1] \triangleq$  "every Lindelöf  $T_3$  indestructible space of weight  $\leq \aleph_1$  has size  $\leq \aleph_1$ "

are equiconsistent.

**Theorem 4.2.**[12].  $\neg \text{Con}(ZFC + \mathcal{L}[T_3, \leq \aleph_1, \leq \aleph_1])$ .

Proof. Theorem 4.2 immediately follows from Theorem 3.3 and Corollary 4.1.

**Definition 4.2.** The  $\aleph_1$ -Borel Conjecture is the statement:  $BC[\aleph_1] \triangleq$  "a Lindelöf space is

indestructible if and only if all of its continuous images in  $[0; 1]^{\omega_1}$  have cardinality  $\leq \aleph_1$ ".

**Theorem 4.3.**[12]. If it is consistent with  $ZFC$  that there is an inaccessible cardinal, then it

is consistent with  $ZFC$  that the  $\aleph_1$ -Borel Conjecture holds.

**Corollary 4.2.** The  $\aleph_1$ -Borel Conjecture and the existence of an inaccessible cardinal are

equiconsistent.

**Theorem 4.4.**[12]  $\neg \text{Con}(ZFC + BC[\aleph_1])$ .

Proof. Theorem 4.4 immediately follows from Theorem 3.3 and Corollary 4.2.

**Theorem 4.5.**[20]. If  $\omega_2$  is not weakly compact in  $\mathbf{L}$ , then there is a Lindelöf  $T_3$  indestructible space of pseudocharacter  $\leq \aleph_1$  and size  $\aleph_2$ .

**Corollary 4.3.** The existence of a weakly compact cardinal and the statement:

$\tilde{\mathcal{L}}[T_3, \leq \aleph_1, \aleph_2] \triangleq$  "there is no Lindelöf  $T_3$  indestructible space of pseudocharacter  $\leq \aleph_1$  and size  $\aleph_2$  are equiconsistent.

**Theorem 4.6.**[12]. There is a Lindelöf  $T_3$  indestructible space of pseudocharacter  $\leq \aleph_1$  and size  $\aleph_2$  in  $\mathbf{L}$ .

Proof. Theorem 4.6 immediately follows from Theorem 3.3 and Theorem 4.5.

**Theorem 4.7.**[12].  $\neg \text{Con}(ZFC + \tilde{\mathcal{L}}[T_3, \leq \aleph_1, \aleph_2])$ .

Proof. Theorem 3.7 immediately follows from Theorem 3.3 and Corollary 4.3.

## 5. Conclusion.

In this paper we have proved that the second order  $ZFC$  with the full second-order semantic is inconsistent, i.e.  $\neg \text{Con}(ZFC_2^{fss})$ . Main result is: let  $k$  be an inaccessible cardinal and  $H_k$  is a set of all sets having hereditary size less than  $k$ , then  $\neg \text{Con}(ZFC + (V = H_k))$ . This result also was obtained in [7],[12],[13] by using essentially another approach. For the first time this result has been declared to AMS in [14],[15]. An important applications in topology and homotopy theory are obtained in [16],[17],[18].

### 5. Acknowledgments

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