Natural Squarefree Numbers: Statistical Properties.

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December 13, 2017
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Abstract

In this paper we calculate for various sets X (some subsets of the natural numbers) the probability of an element a of X is also squarefree. Furthermore we calculate the probability of c is squarefree, where c=a+b, a is an element of the set X and b is an element of the set Y.

1 Notation and Results

Notation 1. ***[Subsets of*** $\mathbb{N}$***]***

N ... *natural (i.e.* $\mathbb{N}$*),*
NE ... *natural and even,*
NO ... *natural and odd*

S ... *squarefree (i.e.* $a \in \mathbb{N}$ *and a is squarefree),*
SE ... *squarefree and even,*
SO ... *squarefree and odd*

F ... *none squarefree (i.e.* $a \in \mathbb{N}$ *and a is none squarefree),*
FE ... *none squarefree and even,*
FO ... *none squarefree and odd*

Notation 2. ***[Probabilities]***
P_X*: Probability:* $a \in X$ *and is squarefree*
$P_{X,Y}$*: Let* $a \in X$, $b \in Y$ *and* $c = a + b$*. Probability: c is squarefree.*

1.1 Results

Note, for our numerical calculations we use the programming language PureBasic (see http://www.purebasic.com).

1.1.1 Probability of $a \in X$ and a is squarefree

choosen set X	N	NE	NO
ProbabilityP_X	P_N	$\frac{2P_N}{3}$	$\frac{4P_N}{3}$

where $P_N := \frac{6}{\pi^2}$

1.1.2 Probability of $a \in X$, $b \in Y$ and $a + b$ is squarefree

$+$	N	NE	NO	S	SE	SO	F	FE	FO
N	P_N	P_N	P_N	P_N	P_N	P_N	P_N	P_N	P_N
NE		$\frac{2P_N}{3}$	$\frac{4P_N}{3}$	$\frac{10P_N}{9}$	$\frac{2P_N}{3}$	$\frac{4P_N}{3}$	$\frac{(9-10P_N)P_N}{9(1-P_N)}$	$\frac{2P_N}{3}$	$\frac{4P_N}{3}$
NO			$\frac{2P_N}{3}$	$\frac{8P_N}{9}$	$\frac{4P_N}{3}$	$\frac{2P_N}{3}$	$\frac{(9-8P_N)P_N}{9(1-P_N)}$	$\frac{4P_N}{3}$	$\frac{2P_N}{3}$
S				$P_{S,S}$	$P_{S,S}$	$P_{S,S}$	$\frac{P_N(1-P_{S,S})}{1-P_N}$	$\frac{2P_N(5-3P_{S,S})}{3(1-2P_N)}$	$\frac{4P_N(2-3P_{S,S})}{3(3-4P_N)}$
SE					0	$\frac{3P_{S,S}}{2}$	$\frac{P_N(1-P_{S,S})}{1-P_N}$	$\frac{2P_N}{3-2P_N}$	$\frac{2P_N(2-3P_{S,S})}{3-4P_N}$
SO						$\frac{3P_{S,S}}{4}$	$\frac{P_N(1-P_{S,S})}{1-P_N}$	$\frac{P_N(4-3P_{S,S})}{3-2P_N}$	$\frac{P_N(2-3P_{S,S})}{3-4P_N}$
F							$\frac{P_N(1-P_N(2-PP_{S,S}))}{(1-P_N)^2}$	$P_{F,FE}$	$P_{F,FO}$
FE								$\frac{2P_N(3-4P_N)}{(3-2P_N)^2}$	$P_{FE,FO}$
FO									$P_{FO,FO}$

where

$P_{F,FE} = \frac{P_N(9-16P_N+6P_NP_{S,S})}{3(1-P_N)(3-2P_N)}$

$P_{F,FO} = \frac{P_N(9-20P_N+12P_NP_{S,S})}{3(3-4P_N)(1-P_N)}$

$P_{FE,FO} = \frac{12P_N(1-2P_N+P_NP_{S,S})}{(3-2P_N)(3-4P_N)}$

$P_{FO,FO} = \frac{2P_N(3-8P_N+6P_NP_{S,S})}{(3-4P_N)^2}$

2 Some Notes About Summation

Definition 3. ***[Alternate Summation:*** $\sigma^n(\gamma)$***]***

Let $\gamma = \gamma_1, \gamma_2, \ldots$ *a sequence of real numbers,* $n \in \mathbb{N}$*. We also assume* $\sum_{i=1}^{\infty} |\gamma_i| < \infty$

Let $\sigma^n(\gamma) = \sigma_1^n(\gamma) - \sigma_2^n(\gamma) + \sigma_3^n(\gamma) - \cdots$*, where:*

$$\begin{aligned}
\sigma_1^n(\gamma) &= \sum_{i=1}^{n} \gamma_i \\
\sigma_2^n(\gamma) &= \sum_{i=1, i<j}^{n} \gamma_i \gamma_j \\
&\vdots \\
\sigma_k^n(\gamma) &= \sum_{i_1=1, i_{k-1}<i_k}^{n} \gamma_{1_{i_1}} \cdots \gamma_{1_{i_k}} \\
&\vdots
\end{aligned}$$

We set the alternate summation $\sigma^n(\gamma)$ *as:*

$$\sigma^n(\gamma) = \sum_{i=1}^{n} (-1)^{i-1} \sigma_i^n(\gamma)$$

If γ *is an infinite sequence, we write*

$$\sigma(\gamma) = \sum_{i=1}^{\infty} (-1)^{i-1} \sigma_i(\gamma)$$

Notation 4. ***[Some Sequences]***
Let $A = \{a_1, a_2, \dots\}$ *an sequence of real numbers.*
Let B_- *a sequence of* n*, not necessary consecutive, elements of* A*.*
Let $B = A - B_-$ *. Note* $b_i \in B$*,* $b_i \in A$ *and* b_i *occur* $n-$*times in* A *then* b_i *occur* $(n-1)-$*times in* B*.*
Let C_- *a sequence of* n*, not necessary consecutive and not necessary elements of* A*, real numbers.*
Let $C = A + C_-$*. Note, if* $c_i \in C_-$ *and* c_i *occur in* A $n-$*times then* c_i *occur in* C $(n+1)-$*times.*
Let $D = (A \backslash B_-) + C_-$*.*
For convenience we set $\sigma_0^\infty(Y) = 1$ *(*$Y = A, B, C, D$*)*

Notation 5. ***[Elementary Symmetric Functions]***

Let $X = \{X_1, X_2, \ldots, X_n\}$ a set of n parameters. We set:

$$\begin{aligned}
e_0(X) &= 1 \\
e_1(X) &= X_1 + X_2 + \cdots + X_n \\
e_2(X) &= X_1X_2 + \cdots + X_1X_n + X_2X_3 + \cdots + X_2X_n + \cdots + X_{n-1}X_n \\
&\vdots \\
e_k(X) &= \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} X_{i_1} \cdots X_{i_k} \\
&\vdots \\
e_n(X) &= X_1 \cdots X_n
\end{aligned}$$

For convenience we set $e_j(X) = 0$ for all $j < 0$.

Notation 6. ***[Alternate Summation of Elementary Symmetric Functions]*** *We set $S(X) = \sum_{i=1}^{n} (-1)^{i-1} e_i(X)$.*

Proposition 7. *With the above definition and notations we have:*

$$\begin{aligned}
\sigma_1(C) &= \sigma_1(A) + e_1(C_-) \\
\sigma_2(C) &= \sigma_2(A) + \sigma_1(A)e_1(C_-) + e_2(C_-) \\
&\vdots \\
\sigma_n(C) &= \sum_{i=0}^{n} A_{n-i} e_i(C_-) \\
&\vdots
\end{aligned}$$

and finally

$$\sigma(C) = \sigma(A)(1 - S(C_-)) + S(C_-)$$

Proof. Consider the sum $\sigma_n(C)$ and let $m = |C_-|$. It consist of:
1) All n tuple of $\sigma_n(A)$.
2) All $n-1$ tuple of $\sigma_{n-1}(A) \cup c_i \in C_-$.
...
m) All $n-m$ tuple of $\sigma_{n-m}(A) \cup \Pi_{c_i \in C_-} c_i$.

Rearrange the terms and summing up is the expected result. □

Corollary 8. *With the above definition and notations, $S(C_-) = A_0$ and $A_n = A_0 + \cdots + A_0$ $n-$times, we have:*

$$\sigma(A_n) = 1 - (1 - \sigma(A_0))^{n+1}$$

Proof. Let $n = 1$. We have

$$\begin{aligned}\sigma(A_1) &= \sigma(A_0)(1 - A_0) + A_0 \\ &= 2\sigma(A_0) - \sigma(A_0)^2 \\ &= 1 - (1 - \sigma(A_0))^2\end{aligned}$$

and the recursion

$$\begin{aligned}\sigma(A_{n+1}) &= (1 - (1 - \sigma(A_0))^n)(1 - A_0) + A_0 \\ &= 1 - (1 - \sigma(A_0))^{n+1}\end{aligned}$$

□

Corollary 9. *With the above definition, notations and* $1 - S(B_-) \neq 0$ *we have:*

$$\sigma(B) = \frac{\sigma(A) - S(B_-)}{1 - S(B_-)}.$$

Proof. With $A = B + B_-$ we have

$$\sigma(A) = \sigma(B)(1 - S(B_-)) + S(B_-)$$

and therefore

$$\sigma(B) = \frac{\sigma(A) - S(B_-)}{1 - S(B_-)}$$

□

Corollary 10. *With the above definition, notations,* $1 - S(B_-) \neq 0$ *and* $D = A - B_- + C_-$ *we have:*

$$\sigma(D) = \frac{(\sigma(A) - S(B_-))\,(1 - S(C_-))}{1 - S(B_-)} + S(C_-)$$

Proof. With $B = A - B_-$ we have

$$\sigma(B) = \frac{\sigma(A) - S(B_-)}{1 - S(B_-)}.$$

and with $D = B + C_- = A - B_- + C_-$ we have

$$\sigma(D) = \sigma(B)(1 - S(C_-)) + S(C_-)$$

□

Corollary 11. *With the above definition, notations and* $1 \in A$ *we have:*

$$\sigma(A) = 1$$

Proof. Let $A = \{1, a_2, a_3, \dots\}$, $A' = \{0, a_2, a_3, \dots\}$ and therefore $S(A_-) = 1$ then we get

$$\sigma(A) = \sigma(B)(1 - S(A_-)) + S(A_-) = 1$$

□

2.1 Error Term, Lower Bound, Upper Bound

Let $n \in \mathbb{N}$, $R_n \geq \sum_{i=n+1}^{\infty} \gamma_i$, and $\sigma_0^n(\gamma) = 1$. We have:

$$
\begin{aligned}
\sigma_1(\gamma) &= \sigma_1^n(\gamma) + \sum_{i=n+1}^{\infty} \gamma_i \leq \sigma_1^n(\gamma) + R_n \\
\sigma_2(\gamma) &= \sigma_2^n(\gamma) + \left(\sum_{i=1}^{n} \gamma_i\right)\left(\sum_{j=n+1}^{\infty} \gamma_j\right) + \left(\sum_{k=n+1}^{\infty} \gamma_k\right)\left(\sum_{s=k+1}^{\infty} \gamma_s\right) \\
\sigma_2(\gamma) &\leq \sigma_2^n + \sigma_1^n R_n + R_n^2 \\
&\vdots \\
\sigma_k(\gamma) &= \cdots \leq \sigma_k^n(\gamma) + \sigma_{k-1}^n(\gamma) R_n + \cdots + R_n^k
\end{aligned}
$$

where $k \leq n$.
Therefore we have for every $\sigma_k(\gamma)$ with $k \leq n$:

$$
\sigma_k^n(\gamma) \leq \sigma_k(\gamma) \leq \sum_{i=0}^{n} \sigma_{n-i}^n(\gamma) R_n^i \tag{1}
$$

Now we consider the sum $\sigma(\gamma)$ and with the above results we get an **Lower Bound**

$$
\sigma(\gamma) \geq \sum_{i=0,\, 2i+1 \leq n}^{n} \sigma_{2i+1}^n - \sum_{i=0,\, 2i \leq n}^{n} \sum_{j=0}^{2i} \sigma_{2i-j}^n R_n^j \tag{2}
$$

and a **Upper Bound**

$$
\sigma(\gamma) \leq \sum_{i=0,\, 2i+1 \leq n}^{n} \sum_{j=0}^{2i+1} \sigma_{2i+1-j}^n(\gamma) R_n^j - \sum_{i=0,\, 2i \leq n}^{n} \sigma_{2i}^n(\gamma) \tag{3}
$$

3 How "many" Squarefree Numbers are in the Set?

Let P the probability of a is squarefree for various sets.

Experimental Data: *Note, this data gives only a hint, we do not invest much work.* We choose an interval of $w/50$ consecutive numbers with random (between $1..N$) starting value. Then we count the squarefree numbers in this interval. We repeat this procedure 50 times.

Numerical Calculation: *Note, this data gives only a hint, we use only the 8 byte Double datatype (see: IEEE 754 standard)* . We estimate a lower and upper bound for the probability (we use $\sigma_1^n(\gamma)$, $\sigma_2^n(\gamma)$, $\sigma_3^n(\gamma)$, $\sigma_4^n(\gamma)$ with the summation over the first n primes). Let p

prime, to estimate γ_i we consider the probability of $a \equiv m \mod p^2$ where $0 \leq m < p^2$. We get

$$\gamma_i = \frac{\text{number of } m = 0}{\text{number of all possible remainders}}$$

and estimate an appropriate error term $R_n \geq (\sigma_1(\gamma) - \sigma_1^n(\gamma))$.

3.1 a is natural

Experimental Data:

Table: a is natural

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.608000	0.000070	0.008391	+/- 0.001187	+/- 0.003560
40000	2e+6	0.608275	0.000012	0.003507	+/- 0.000496	+/- 0.001488
10000	1.6e+7	0.608400	0.000076	0.008715	+/- 0.001233	+/- 0.003698
90000	1.6e+7	0.608022	0.000003	0.001627	+/- 0.000230	+/- 0.000690

Numerical Calculation: Since $f(a) = \{1, 1, \ldots, 1\}$ for $0 \leq m < p^2$ we get $\gamma_i = \frac{1}{p^2}$ and

$$P_N = 1 - \sigma\left(\frac{1}{p^2}\right).$$

With $(p_n > 2)$

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6} = \sum_{i=1}^{\infty} \frac{1}{(2i)^2} + \sum_{i=0}^{\infty} \frac{1}{(2i+1)^2} = \frac{1}{4} \cdot \frac{\pi^2}{6} + \sum_{i=0}^{\frac{p_n-1}{2}} \frac{1}{(2i+1)^2} + \sum_{i=\frac{p_n+1}{2}}^{\infty} \frac{1}{(2i+1)^2}$$

we get

$$R_n = \frac{\pi^2}{8} - \sum_{i=0}^{\frac{p_n-1}{2}} \frac{1}{(2i+1)^2}$$

Summation over the first 360 primes gives

$$0.6076327119 \leq P_N \leq 0.6080511787$$

Analytical Calculation: Well known is

$$P_N = \frac{6}{\pi^2}$$

3.2 a is natural and even

Experimental Data:

Table: a is natural and even

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.404300	0.000047	0.006852	+/- 0.000969	+/- 0.002907
40000	2e+6	0.405225	0.000007	0.002645	+/- 0.000374	+/- 0.001122
10000	1.6e+7	0.405900	0.000051	0.007121	+/- 0.001007	+/- 0.003021
90000	1.6e+7	0.405289	0.000002	0.001576	+/- 0.000223	+/- 0.000669

Numerical Calculation: We have

$$f(a) = \begin{cases} \{1, 0, 1, 0\}, & p = 2 \\ \{1, 1, \ldots, 1\}, & \text{otherwise} \end{cases}$$

and therefore $\gamma = \{1/2, 1/p_2^2, 1/p_3^2, \ldots\}$ and $R_n = \frac{\pi^2}{8} - \sum_{i=0}^{\frac{p_n-1}{2}} \frac{1}{(2i+1)^2}$ (see section 3.1) Summation over the first 360 primes give:

$$0.4048734678 \leq P_{NE} = 1 - \sigma(\gamma) \leq 0.4054531897$$

Analytical Calculation: We have $\gamma_i = 1/p_i^2$, $B_- = \{1/4\}$ and $C_- = \{1/2\}$ (i.e. replace 1/4 by 1/2).

$$P_{NE} = 1 - \sigma(\gamma) = 1 - \left(\frac{((1 - P_N) - 1/4)(1 - 1/2)}{3/4} \right) = \frac{2P_N}{3} = \frac{4}{\pi^2}$$

3.3 a is natural and odd

Experimental Data:

Table: a is natural and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.811500	0.000052	0.007232	+/- 0.001023	+/- 0.003068
40000	2e+6	0.810775	0.000012	0.003452	+/- 0.000488	+/- 0.001465
10000	1.6e+7	0.811900	0.000087	0.009307	+/- 0.001316	+/- 0.003949
90000	1.6e+7	0.810711	0.000004	0.001882	+/- 0.000266	+/- 0.000798

Numerical Calculation: We have

$$f(a) = \begin{cases} \{0,1,0,1\}, & p=2 \\ \{1,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

and therefore $\gamma = \{0, 1/p_2^2, 1/p_3^2, \ldots\}$ and $R_n = \frac{\pi^2}{8} - \sum_{i=0}^{\frac{p_n-1}{2}} \frac{1}{(2i+1)^2}$ (see section 3.1) Summation over the first 360 primes give:

$$0.810391956 \leq P_{NO} 1 - \sigma(\gamma) \leq 0.8106491677$$

Analytical Calculation: We have $\gamma_i = 1/p_i^2$ and $B_- = \{1/4\}$ (i.e. delete 1/4).

$$P_{NO} = 1 - \sigma(\gamma) = 1 - \left(\frac{((1-P_N) - 1/4)}{1 - 1/4} \right) = \frac{4P_N}{3} = \frac{8}{\pi^2}$$

Remark 12. .
a) $P_N = (1/2)P_{NE} + (1/2)P_{NO}$
b) $P_{NE} : P_{NO} = 1 : 2$

4 Addition a+b: Experimental Data, Numerical and Analytical Calculation

Let P the probability of $a+b$ is squarefree, for various sets of a and b.

Experimental Data: *Note, this data gives only a hint, we do not invest much work.* We choose two intervals of $\sqrt{w}$ consecutive number of the appropriate form, both with random (between $1..N$) starting value. Then we test w pairs of numbers, given by the intervals. We repeat this procedure 36 times.

Numerical Calculation: *Note, this data gives only a hint, we use only the 8 byte Double datatype (see: IEEE 754 standard)* . We estimate a lower and upper bound for the probability (we use $\sigma_1^n(\gamma)$, $\sigma_2^n(\gamma)$, $\sigma_3^n(\gamma)$, $\sigma_4^n(\gamma)$ with the summation over the first n primes). Let p prime and $f(a)$ the probability of $a \equiv m \mod p^2$ where $0 \leq m < p^2$. To estimate γ we count all possible pairs m_a, m_b of $f(a)$ and $f(b)$ and all sufficient pairs m_a, m_b of $f(a)$ and $f(b)$ with $(m_a + m_b) \equiv 0 \mod p^2$. We get

$$\gamma_i = \frac{\text{number of all sufficient pairs}}{\text{number of all possible pairs}}$$

4.1 a is natural, b is natural

Experimental Data:

Table: a is natural, b is natural

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.609222	0.000122	0.011062	+/- 0.001844	+/- 0.005531
40000	2e+6	0.607781	0.000043	0.006577	+/- 0.001096	+/- 0.003288
10000	1.6e+7	0.607633	0.000087	0.009314	+/- 0.001552	+/- 0.004657
90000	1.6e+7	0.608278	0.000018	0.004250	+/- 0.000708	+/- 0.002125

Analytical Calculation: Since $f(a) = f(b) = \{1, \ldots, 1\}$ for $0 \leq m < p$ we get $\gamma_i = \frac{1}{p_i^2}$ and have (see section 3.1)

$$P_{N,N} = P_N$$

4.2 a is natural, b is natural and even

Experimental Data:

Table: a is natural, b is natural and even

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.607964	0.000002	0.001372	+/- 0.000229	+/- 0.000686
40000	2e+6	0.607930	0.000000	0.000394	+/- 0.000066	+/- 0.000197
10000	1.6e+7	0.608142	0.000002	0.001387	+/- 0.000231	+/- 0.000694
90000	1.6e+7	0.607870	0.000000	0.000233	+/- 0.000039	+/- 0.000116

Analytical Calculation: We have $f(a) = \{1, 1, \ldots, 1\}$ and

$$f(b) = \begin{cases} \{1, 0, 1, 0\}, & p = 2 \\ \{1, 1, \ldots, 1\}, & \text{otherwise} \end{cases}$$

Case $p = 2$: We have 8 possible pairs and 2 sufficient pair and therefore $1/4 = 1/p^2$.
Case $p > 2$: We have $p^2 p^2$ possible pairs and p^2 sufficient pairs.

We get $\gamma = \{1/p_1^2, 1/p_2^2, 1/p_3^3, \ldots\}$ and (see section 3.1)

$$P_{N,NE} = P_N$$

4.3 a is natural, b is natural and odd

Experimental Data:

Table: a is natural, b is natural and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.608017	0.000002	0.001274	+/- 0.000212	+/- 0.000637
40000	2e+6	0.607953	0.000000	0.000355	+/- 0.000059	+/- 0.000177
10000	1.6e+7	0.608164	0.000001	0.001211	+/- 0.000202	+/- 0.000606
90000	1.6e+7	0.607931	0.000000	0.000221	+/- 0.000037	+/- 0.000111

Analytical Calculation: We have $f(a) = \{1, 1, \dots, 1\}$ and

$$f(b) = \begin{cases} \{0,1,0,1\}, & p = 2 \\ \{1,1,\dots,1\}, & \text{otherwise} \end{cases}$$

Case $p = 2$: We have 8 possible pairs and 2 sufficient pair and therefore $1/4 = 1/p^2$.
Case $p > 2$: We have $p^2 p^2$ possible pairs and p^2 sufficient pairs.

We get $\gamma = \{1/p_1^2, 1/p_2^2, 1/p_3^3, \dots\}$ and (see section 3.1)

$$P_{N,NO} = P_N$$

4.4 a is natural, b is squarefree

Experimental Data:

Table: a is natural, b is squarefree

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.608081	0.000003	0.001656	+/- 0.000276	+/- 0.000828
40000	2e+6	0.607936	0.000000	0.000628	+/- 0.000105	+/- 0.000314
10000	1.6e+7	0.608436	0.000003	0.001719	+/- 0.000286	+/- 0.000859
90000	1.6e+7	0.607861	0.000000	0.000252	+/- 0.000042	+/- 0.000126

Analytical Calculation: Since $f(a) = \{1, 1, \dots, 1\}$ and $f(b) = \{0, 1, \dots, 1\}$ we have $p^2(p^2 - 1)$ possible pairs and $p^2 - 1$ sufficient pairs. Therefore we get $\gamma = \{1/p_1^2, 1/p_2^2, \dots)\}$ and (see section 3.1)

$$P_{N,S} = P_N$$

4.5 a is natural, b is squarefree and even

Experimental Data:

Table: a is natural, b is squarefree and even

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.608061	0.000002	0.001342	+/- 0.000224	+/- 0.000671
40000	2e+6	0.607797	0.000000	0.000499	+/- 0.000083	+/- 0.000249
10000	1.6e+7	0.607567	0.000001	0.001200	+/- 0.000200	+/- 0.000600
90000	1.6e+7	0.607804	0.000000	0.000440	+/- 0.000073	+/- 0.000220

Analytical Calculation: Since $f(a) = \{1, 1, \dots, 1\}$ and

$$f(b) = \begin{cases} \{0,0,1,0\}, & p = 2 \\ \{0,1,\dots,1\}, & \text{otherwise} \end{cases}$$

Case $p = 2$: We have 4 possible pairs and 1 sufficient pair and therefore $1/4 = 1/p^2$
Case $p > 2$: We have $p^2(p^2 - 1)$ possible pairs and $p^2 - 1$ sufficient pairs.

We get $\gamma = \{1/p_1^2, 1/p_2^2, \dots\}$ and (see section 3.1)

$$P_{N,SE} = P_N$$

4.6 a is natural, b is squarefree and odd

Experimental Data:

Table: a is natural, b is squarefree and even

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.607992	0.000003	0.001610	+/- 0.000268	+/- 0.000805
40000	2e+6	0.608045	0.000000	0.000500	+/- 0.000083	+/- 0.000250
10000	1.6e+7	0.608011	0.000003	0.001778	+/- 0.000296	+/- 0.000889
90000	1.6e+7	0.607837	0.000000	0.000418	+/- 0.000070	+/- 0.000209

Analytical Calculation: Since $f(a) = \{1, 1, \dots, 1\}$ and

$$f(b) = \begin{cases} \{0,1,0,1\}, & p = 2 \\ \{0,1,\dots,1\}, & \text{otherwise} \end{cases}$$

Case $p = 2$: We have 8 possible pairs and 2 sufficient pair and therefore $1/4 = 1/p^2$
Case $p > 2$: We have $p^2(p^2 - 1)$ possible pairs and $p^2 - 1$ sufficient pairs.

We get $\gamma = \{1/p_1^2, 1/p_2^2, \dots\}$ and (see section 3.1)

$$P_{N,SO} = P_N$$

4.7 a is natural, b is none squarefree

Experimental Data:

Table: a is natural, b is none squarefree

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.608064	0.000002	0.001542	+/- 0.000257	+/- 0.000771
40000	2e+6	0.607884	0.000000	0.000702	+/- 0.000117	+/- 0.000351
10000	1.6e+7	0.607922	0.000003	0.001610	+/- 0.000268	+/- 0.000805
90000	1.6e+7	0.607997	0.000000	0.000572	+/- 0.000095	+/- 0.000286

Analytical Calculation: Since $f(a) = \{1, 1, \ldots, 1\}$ and $f(b) = \{g, 1, 1, \ldots\}$ where $g = \frac{p^2-1}{(1-P_N)p^2-1}$ (see section 4.40).
We have $p^2(p^2-1) + p^2 g$ possible pairs and $g + (p^2-1)$ sufficient pairs.

We get $\gamma = \{1/p_1^2, 1/p_2^2, \ldots\}$ and (see section 3.1)

$$P_{N,F} = P_N$$

4.8 a is natural, b is none squarefree and even

Experimental Data:

Table: a is natural, b is none squarefree and even

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.607658	0.000006	0.002360	+/- 0.000393	+/- 0.001180
40000	2e+6	0.607849	0.000001	0.000833	+/- 0.000139	+/- 0.000417
10000	1.6e+7	0.607322	0.000006	0.002448	+/- 0.000408	+/- 0.001224
90000	1.6e+7	0.607894	0.000000	0.000507	+/- 0.000085	+/- 0.000254

Analytical Calculation: Since $f(a) = \{1, 1, \ldots, 1\}$ and

$$f(b) = \begin{cases} \{g, 0, 1, 0\}, & p = 2 \\ \{d, 1, \ldots, 1\}, & \text{otherwise} \end{cases}$$

where $g = \frac{3}{3-4P_N}$ and $d = \frac{p^2-1}{(3-2P_N)p^2/3-1}$ (see section 4.43).

Case $p = 2$: We have $4 * g$ possible pairs and g sufficient pair and therefore $1/4 = 1/p^2$
Case $p > 2$: We have $p^2(p^2-1) + p^2 d$ possible pairs and $d + (p^2-1)$ sufficient pairs.

We get $\gamma = \{1/p_1^2, 1/p_2^2, \dots\}$ and (see section 3.1)

$$P_{N,FE} = P_N$$

4.9 a is natural, b is none squarefree and odd

Experimental Data:

Table: a is natural, b is none squarefree and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.607383	0.000005	0.002237	+/- 0.000373	+/- 0.001119
40000	2e+6	0.608197	0.000001	0.001054	+/- 0.000176	+/- 0.000527
10000	1.6e+7	0.606817	0.000007	0.002717	+/- 0.000453	+/- 0.001359
90000	1.6e+7	0.607923	0.000001	0.000830	+/- 0.000138	+/- 0.000415

Analytical Calculation: Since $f(a) = \{1, 1, \dots, 1\}$ and

$$f(b) = \begin{cases} \{0, 1, 0, 1\}, & p = 2 \\ \{g, 1, \dots, 1\}, & \text{otherwise} \end{cases}$$

where $g = \frac{p^2-1}{(\frac{3-4P_N}{3})p^2-1}$ (see section 4.45).

Case $p = 2$: We have 8 possible pairs and 2 sufficient pair and therefore $1/4 = 1/p^2$
Case $p > 2$: We have $p^2(p^2 - 1)$ possible pairs and $p^2 - 1$ sufficient pairs.

We get $\gamma = \{1/p_1^2, 1/p_2^2, \dots\}$ and (see section 3.1)

$$P_{N,FO} = P_N$$

4.10 a is natural and even, b is natural and even

Experimental Data:

Table: a is natural and even, b is natural and even

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.405228	0.000000	0.000406	+/- 0.000068	+/- 0.000203
40000	2e+6	0.405370	0.000000	0.000436	+/- 0.000073	+/- 0.000218
10000	1.6e+7	0.405286	0.000000	0.000496	+/- 0.000083	+/- 0.000248
90000	1.6e+7	0.405282	0.000000	0.000106	+/- 0.000018	+/- 0.000053

Numerical Calculation:

$$f(a) = f(b) = \begin{cases} \{1,0,1,0\}, & p = 2 \\ \{1,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

and therefore $\gamma = \{1/2, 1/p_2^2, 1/p_3^2, \ldots\}$ and $R_n = \frac{\pi^2}{8} - \sum_{i=0}^{\frac{p_n-1}{2}} \frac{1}{(2i+1)^2}$ (see section 3.1) Summation over the first 360 primes give:

$$0.4048734678 P_{NE,NE} \le 1 - \sigma(\gamma) \le 0.4054531897$$

Analytical Calculation: We have $\gamma_i = 1/p_i^2$, $B_- = \{1/4\}$ and $C_- = \{1/2\}$ (i.e. replace 1/4 by 1/2).

$$P_{NE,NE} = 1 - \left(\frac{((1-P_N) - 1/4)(1-1/2)}{3/4} \right) = \frac{2P_N}{3}$$

4.11 a is natural and even, b is natural and odd

Experimental Data:

Table: a is natural and even, b is natural and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.810506	0.000000	0.000450	+/- 0.000075	+/- 0.000225
40000	2e+6	0.810456	0.000000	0.000696	+/- 0.000116	+/- 0.000348
10000	1.6e+7	0.810631	0.000000	0.000388	+/- 0.000065	+/- 0.000194
90000	1.6e+7	0.810550	0.000000	0.000147	+/- 0.000025	+/- 0.000074

Analytical Calculation: We have

$$f(a) = \begin{cases} \{1,0,1,0\}, & p = 2 \\ \{1,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

and

$$f(b) = \begin{cases} \{0,1,0,1\}, & p = 2 \\ \{1,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

and therefore $\gamma = \{0, 1/p_2^2, 1/p_3^2, \ldots\}$ and (see section 3.3)

$$P_{NO,NE} = P_{NE,NO} = \frac{4P_N}{3}$$

Remark 13. .
$P_{N,N} = (1/4)P_{NO,NO} + (1/4)P_{NE,NE} + (1/4)P_{NE,NO} + (1/4)P_{NO,NE}$

4.12 a is natural and even, b is squarefree

Experimental Data:

Table: a is natural and even, b is squarefree

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.675981	0.000002	0.001436	+/- 0.000239	+/- 0.000718
40000	2e+6	0.675568	0.000000	0.000553	+/- 0.000092	+/- 0.000276
10000	1.6e+7	0.675206	0.000002	0.001402	+/- 0.000234	+/- 0.000701
90000	1.6e+7	0.675534	0.000000	0.000343	+/- 0.000057	+/- 0.000172

Analytical Calculation: We have

$$f(a) = \begin{cases} \{1,0,1,0\}, & p = 2 \\ \{1,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

and $f(b) = \{0,1,\ldots,1\}$
Case $p = 2$: We have 6 possible pairs and 1 sufficient pair and therefore 1/6.
Case $p > 2$: We have $p^2(p^2-1)$ possible pairs and p^2-1 sufficient pairs.

We get $\gamma = \{1/6, 1/p_2^2, 1/p_3^3, \ldots\}$ (start with $\gamma' = \{1/p_i^2\}$) and replace 1/4 with 1/6) (see section 3.1)

$$P_{NE,S} = 1 - \frac{(\sigma(\gamma') - (1-1/4))(1-1/6)}{1-1/4} - 1/6 = \frac{10P_N}{9}$$

4.13 a is natural and even, b is squarefree and even

Experimental Data:

Table: a is natural and even, b is squarefree and even

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.405775	0.000002	0.001347	+/- 0.000225	+/- 0.000674
40000	2e+6	0.405192	0.000000	0.000389	+/- 0.000065	+/- 0.000195
10000	1.6e+7	0.405181	0.000001	0.001027	+/- 0.000171	+/- 0.000514
90000	1.6e+7	0.405281	0.000000	0.000291	+/- 0.000049	+/- 0.000146

Analytical Calculation: We have

$$f(a) = \begin{cases} \{1,0,1,0\}, & p = 2 \\ \{1,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

and

$$f(b) = \begin{cases} \{0,0,1,0\}, & p=2 \\ \{0,1,\dots,1\}, & \text{otherwise} \end{cases}$$

Case $p = 2$: We have 2 possible pairs and 1 sufficient pair and therefore 1/2.
Case $p > 2$: We have $p^2(p^2-1)$ possible pairs and p^2-1 sufficient pairs.

We get $\gamma = \{1/2, 1/p_2^2, 1/p_3^3, \dots\}$, and (see section 3.2)

$$P_{NE,SE} = \frac{2P_N}{3}$$

4.14 a is natural and even, b is squarefree and odd

Experimental Data:

Table: a is natural and even, b is squarefree and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.810536	0.000003	0.001821	+/- 0.000304	+/- 0.000911
40000	2e+6	0.810698	0.000000	0.000539	+/- 0.000090	+/- 0.000269
10000	1.6e+7	0.810511	0.000003	0.001585	+/- 0.000264	+/- 0.000793
90000	1.6e+7	0.810522	0.000000	0.000324	+/- 0.000054	+/- 0.000162

Analytical Calculation: We have

$$f(a) = \begin{cases} \{1,0,1,0\}, & p=2 \\ \{1,1,\dots,1\}, & \text{otherwise} \end{cases}$$

and

$$f(b) = \begin{cases} \{0,1,0,1\}, & p=2 \\ \{0,1,\dots,1\}, & \text{otherwise} \end{cases}$$

Case $p = 2$: We have 4 possible pairs and 0 sufficient pair.
Case $p > 2$: We have $p^2(p^2-1)$ possible pairs and p^2-1 sufficient pairs.

We get $\gamma = \{0, 1/p_2^2, 1/p_3^3, \dots\}$ and (see section 3.3)

$$P_{NE,SO} = \frac{4P_N}{3}$$

4.15 a is natural and even, b is none squarefree

Experimental Data:

Table: a is natural and even, b is none squarefree

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.503053	0.000002	0.001461	+/- 0.000244	+/- 0.000731
40000	2e+6	0.503290	0.000000	0.000604	+/- 0.000101	+/- 0.000302
10000	1.6e+7	0.503075	0.000003	0.001714	+/- 0.000286	+/- 0.000857
90000	1.6e+7	0.503282	0.000000	0.000367	+/- 0.000061	+/- 0.000183

Analytical Calculation: We have

$$f(a) = \begin{cases} \{1,0,1,0\}, & p=2 \\ \{1,1,\dots,1\}, & \text{otherwise} \end{cases}$$

and $f(b) = \{g, 1, 1, \dots\}$ where $g = \frac{p^2-1}{(1-P_N)p^2-1}$ (see section 4.40).
Case $p = 2$: We have $2(g+3)$ possible pairs and $g+1$ sufficient pair and therefore $(3-2P_N)/(12(1-P_N))$.
Case $p > 2$: We have $p_i^2(p_i^2-1) + p_i^2 g$ possible pairs and $g + p_i^2 - 1$ sufficient pairs and $1/p_i^2$.

We get $\gamma = \{(3-2P_N)/(12(1-P_N)), 1/p_2^2, 1/p_3^3, \dots\}$ and (start with $\gamma' = \{1/p_i^2\}$) and replace 1/4 with $(3-2P_N)/(12(1-P_N)))$ (see section 3.1)

$$P_{NE,F} = 1 - \frac{(\sigma(\gamma') - 1/4)(1-(3-2P_N)/(12(1-P_N)))}{1-1/4} + (3-2P_N)/(12(1-P_N))$$

$$P_{NE,F} = \frac{(9-10P_N)P_N}{9(1-P_N)}$$

4.16 a is natural and even, b is none squarefree and even

Experimental Data:

Table: a is natural and even, b is none squarefree and even

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.406450	0.000011	0.003376	+/- 0.000563	+/- 0.001688
40000	2e+6	0.405410	0.000001	0.000768	+/- 0.000128	+/- 0.000384
10000	1.6e+7	0.405006	0.000011	0.003348	+/- 0.000558	+/- 0.001674
90000	1.6e+7	0.405256	0.000000	0.000475	+/- 0.000079	+/- 0.000237

Analytical Calculation: We have

$$f(a) = \begin{cases} \{1,0,1,0\}, & p=2 \\ \{1,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

and

$$f(b) = \begin{cases} \{g,0,1,0\}, & p=2 \\ \{d,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

where $g = \frac{3}{3-4P_N}$ and $d = \frac{p^2-1}{(3-2P_N)p^2/3-1}$ (see section 4.43).

Case $p=2$: We have $2(g+1)$ possible pairs and $g+1$ sufficient pair and therefore $1/2$.
Case $p>2$: We have $p_i^2(p_i^2-1)+p_i^2 d$ possible pairs and $d+p_i^2-1$ sufficient pairs and $1/p_i^2$.

We get $\gamma = \{1/2, 1/p_2^2, 1/p_3^3, \ldots\}$ and (see section 3.2)

$$P_{NE,FE} = \frac{2P_N}{3}$$

4.17 a is natural and even, b is none squarefree and odd

Experimental Data:

Table: a is natural and even, b is none squarefree and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.811261	0.000005	0.002180	+/- 0.000363	+/- 0.001090
40000	2e+6	0.810206	0.000002	0.001567	+/- 0.000261	+/- 0.000783
10000	1.6e+7	0.811142	0.000002	0.001543	+/- 0.000257	+/- 0.000771
90000	1.6e+7	0.810373	0.000001	0.001020	+/- 0.000170	+/- 0.000510

Analytical Calculation: We have

$$f(a) = \begin{cases} \{1,0,1,0\}, & p=2 \\ \{1,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

and

$$f(b) = \begin{cases} \{0,1,0,1\}, & p=2 \\ \{g,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

where $g = \frac{p^2-1}{(\frac{3-4P_N}{3})p^2-1}$ (see section 4.45).

Case $p=2$: We have 4 possible pairs and 0 sufficient pair and therefore 0.
Case $p>2$: We have $p_i^2(p_i^2-1)$ possible pairs and p_i^2-1 sufficient pairs and $1/p_i^2$.

We get $\gamma = \{0, 1/p_2^2, 1/p_3^3, \dots\}$ and (see section 3.3)

$$P_{NE,FO} = \frac{4P_N}{3}$$

4.18 a is natural and odd, b is natural and odd

Experimental Data:

Table: a is natural and odd, b is natural and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.405289	0.000000	0.000361	+/- 0.000060	+/- 0.000180
40000	2e+6	0.405251	0.000000	0.000335	+/- 0.000056	+/- 0.000167
10000	1.6e+7	0.405303	0.000000	0.000422	+/- 0.000070	+/- 0.000211
90000	1.6e+7	0.405296	0.000000	0.000129	+/- 0.000021	+/- 0.000064

Analytical Calculation: We have

$$f(a) = f(b) = \begin{cases} \{0, 1, 0, 1\}, & p = 2 \\ \{1, 1, \dots, 1\}, & \text{otherwise} \end{cases}$$

We get $\gamma = \{1/2, 1/p_2^2, 1/p_3^2, \dots\}$ and (see section 3.2)

$$P_{NO,NO} = \frac{2P_N}{3}$$

4.19 a is natural and odd, b is squarefree

Experimental Data:

Table: a is natural and odd, b is squarefree

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.540239	0.000002	0.001476	+/- 0.000246	+/- 0.000738
40000	2e+6	0.540309	0.000000	0.000568	+/- 0.000095	+/- 0.000284
10000	1.6e+7	0.539994	0.000002	0.001480	+/- 0.000247	+/- 0.000740
90000	1.6e+7	0.540276	0.000000	0.000309	+/- 0.000052	+/- 0.000155

Analytical Calculation: We have

$$f(a) = f(b) = \begin{cases} \{0,1,0,1\}, & p = 2 \\ \{1,1,\dots,1\}, & \text{otherwise} \end{cases}$$

and $f(b) = \{0,1,1,1,\dots\}$
Case $p = 2$: We have 6 possible pairs and 2 sufficient pair and therefore 1/3.
Case $p > 2$: We have $p_i^2(p_i^2-1)$ possible pairs and p_i^2-1 sufficient pairs and $1/p_i^2$.

We get $\gamma = \{1/3, 1/p_2^2, 1/p_3^3, \dots\}$ and (start with $\gamma' = \{1/p_i^2\}$) and replace 1/4 with 1/3) (see section 3.1)

$$P_{NE,FO} = 1 - \frac{((1-P_N)-1/4)(1-1/3)}{1-1/4} + 1/3 = \frac{8P_N}{9}$$

4.20 a is natural and odd, b is squarefree and even

Experimental Data:

Table: a is natural and odd, b is squarefree and even

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.810922	0.000005	0.002188	+/- 0.000365	+/- 0.001094
40000	2e+6	0.810617	0.000000	0.000561	+/- 0.000094	+/- 0.000281
10000	1.6e+7	0.810403	0.000004	0.002064	+/- 0.000344	+/- 0.001032
90000	1.6e+7	0.810569	0.000000	0.000305	+/- 0.000051	+/- 0.000152

Analytical Calculation: We have

$$f(a) = \begin{cases} \{0,1,0,1\}, & p = 2 \\ \{1,1,\dots,1\}, & \text{otherwise} \end{cases}$$

and

$$f(b) = \begin{cases} \{0,0,1,0\}, & p = 2 \\ \{0,1,\dots,1\}, & \text{otherwise} \end{cases}$$

Case $p = 2$: We have 2 possible pairs and 0 sufficient pair.
Case $p > 2$: We have $p^2(p^2-1)$ possible pairs and p^2-1 sufficient pairs.

We get $\gamma = \{0, 1/p_2^2, 1/p_3^3, \dots\}$ and (see section 3.3)

$$P_{NO,SE} = \frac{4P_N}{3}$$

4.21 a is natural and odd, b is squarefree and odd

Experimental Data:

Table: a is natural and odd, b is squarefree and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.406036	0.000006	0.002517	+/- 0.000419	+/- 0.001258
40000	2e+6	0.405307	0.000000	0.000493	+/- 0.000082	+/- 0.000246
10000	1.6e+7	0.405247	0.000007	0.002572	+/- 0.000429	+/- 0.001286
90000	1.6e+7	0.405256	0.000000	0.000396	+/- 0.000066	+/- 0.000198

Analytical Calculation: We have

$$f(a) = \begin{cases} \{0,1,0,1\}, & p=2 \\ \{1,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

and

$$f(b) = \begin{cases} \{0,1,0,1\}, & p=2 \\ \{1,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

Case $p=2$: We have 4 possible pairs and 2 sufficient pairs and therefore 1/2.
Case $p>2$: We have p^2p^2 possible pairs and p^2 sufficient pairs.

We get $\gamma = \{1/2, 1/p_2^2, 1/p_3^3, \ldots\}$ and (see section 3.2)

$$P_{NO,SO} = \frac{2P_N}{3}$$

4.22 a is natural and odd, b is none squarefree

Experimental Data:

Table: a is natural and odd, b is none squarefree

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.712800	0.000005	0.002294	+/- 0.000382	+/- 0.001147
40000	2e+6	0.712640	0.000000	0.000641	+/- 0.000107	+/- 0.000321
10000	1.6e+7	0.712739	0.000004	0.002017	+/- 0.000336	+/- 0.001008
90000	1.6e+7	0.712760	0.000000	0.000362	+/- 0.000060	+/- 0.000181

Analytical Calculation: We have

$$f(a) = \begin{cases} \{0,1,0,1\}, & p=2 \\ \{1,1,\dots,1\}, & \text{otherwise} \end{cases}$$

and $f(b) = \{g,1,1,\dots\}$ where $g = \frac{p^2-1}{(1-P_N)p^2-1}$ (see section 4.40)
Case $p=2$: We have $2(g+3)$ possible pairs and 2 sufficient pairs and therefore $1/(g+3) = 3/(3-4P_N)$.
Case $p>2$: We have $p^2(p^2-1)+gp^2$ possible pairs and p^2-1+g sufficient pairs.

We get $\gamma = \{(3-4P_N)/(12(1-P_N)), 1/p_2^2, 1/p_3^3, \dots\}$ and (start with $\gamma' = \{1/p_i^2\}$) and replace 1/4 with $(3-4P_N)/(12(1-P_N)))$ (see section 3.1)

$$P_{NO,SO} = 1 - \frac{(\sigma(\gamma') - (1-1/4))(1-(3-4P_N)/(12(1-P_N)))}{1-1/4} - \frac{3-4P_N}{12(1-P_N)} = \frac{(9-8P_N)P_N}{9(1-P_N)}$$

4.23 a is natural and odd, b is none squarefree and even

Experimental Data:

Table: a is natural and odd, b is none squarefree and even

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.810556	0.000013	0.003549	+/- 0.000591	+/- 0.001774
40000	2e+6	0.810548	0.000001	0.000825	+/- 0.000137	+/- 0.000412
10000	1.6e+7	0.811172	0.000014	0.003676	+/- 0.000613	+/- 0.001838
90000	1.6e+7	0.810546	0.000000	0.000583	+/- 0.000097	+/- 0.000291

Analytical Calculation: We have

$$f(a) = \begin{cases} \{0,1,0,1\}, & p=2 \\ \{1,1,\dots,1\}, & \text{otherwise} \end{cases}$$

and

$$f(b) = \begin{cases} \{g,0,1,0\}, & p=2 \\ \{d,1,\dots,1\}, & \text{otherwise} \end{cases}$$

where $g = \frac{3}{3-4P_N}$ and $d = \frac{p^2-1}{(3-2P_N)p^2/3-1}$ (see section 4.43).

Case $p=2$: We have $2(g+1)$ possible pairs and 0 sufficient pair and therefore 0.
Case $p>2$: We have $dp^2+p^2(p^2-1)$ possible pairs and $d+p^2-1$ sufficient pairs.

We get $\gamma = \{0, 1/p_2^2, 1/p_3^3, \dots\}$ and (see section 3.3)

$$P_{NO,FE} = \frac{4P_N}{3}$$

4.24 a is natural and odd, b is none squarefree and odd

Experimental Data:

Table: a is natural and odd, b is none squarefree and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.405619	0.000011	0.003377	+/- 0.000563	+/- 0.001688
40000	2e+6	0.404687	0.000001	0.000936	+/- 0.000156	+/- 0.000468
10000	1.6e+7	0.404736	0.000010	0.003111	+/- 0.000519	+/- 0.001556
90000	1.6e+7	0.405482	0.000000	0.000557	+/- 0.000093	+/- 0.000278

Analytical Calculation: We have

$$f(a) = \begin{cases} \{0,1,0,1\}, & p=2 \\ \{1,1,\dots,1\}, & \text{otherwise} \end{cases}$$

and

$$f(b) = \begin{cases} \{0,1,0,1\}, & p=2 \\ \{g,1,\dots,1\}, & \text{otherwise} \end{cases}$$

where $g = \frac{p^2-1}{(\frac{3-4P_N}{3})p^2-1}$ (see section 4.45).

Case $p=2$: We have 4 possible pairs and 2 sufficient pairs and therefore 1/2.
Case $p>2$: We have $p^2(p^2-1)+gp^2$ possible pairs and p^2-1+g sufficient pairs.

We get $\gamma = \{1/2, 1/p_2^2, 1/p_3^3, \dots\}$ and (see section 3.2)

$$P_{NO,FO} = \frac{2P_N}{3}$$

4.25 a is squarefree, b is squarefree

Experimental Data:

Table: a is squarefree, b is squarefree

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.531186	0.000042	0.006474	+/- 0.001079	+/- 0.003237
40000	2e+6	0.530610	0.000012	0.003490	+/- 0.000582	+/- 0.001745
10000	1.6e+7	0.531978	0.000039	0.006277	+/- 0.001046	+/- 0.003138
90000	1.6e+7	0.530763	0.000004	0.002033	+/- 0.000339	+/- 0.001017

Numerical Calculation: Since a is squarefree, $a \equiv 0 \mod p^2$ does not exist and we have $f(a) = \{0, 1, \dots, 1\}$ (see [PRE02]). We get

$$P_{S,S} = 1 - \sigma\left(\frac{1}{p^2 - 1}\right)$$

Calculation of the Error-Term R_n: Let $\gamma = (1/(p^2 - 1))$

$$\sigma^n(\gamma) \leq \sigma(\gamma) \leq \sigma^n(\gamma) + \sum_{i=\frac{p_n+1}{2}}^{\infty} \frac{1}{(2i+1)^2 - 1} = \sigma^n(\gamma) + \frac{1}{2} \sum_{i=\frac{p_n+1}{2}}^{\infty} \left(\frac{1}{2i} - \frac{1}{2i+2}\right)$$

and we get

$$R_n = \frac{1}{2} \sum_{i=\frac{p_n+1}{2}}^{\infty} \left(\frac{1}{2i} - \frac{1}{2i+2}\right) = \frac{1}{2} \cdot \frac{1}{p_n + 1}$$

Summation over the first 360 primes gives

$$0.5303553651 \leq P_{S,S} = 1 - \sigma(\gamma) \leq 0.5308540687$$

4.26 a is squarefree, b is squarefree and even

Experimental Data:

Table: a is squarefree, b is squarefree and even

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.527811	0.000014	0.003702	+/- 0.000617	+/- 0.001851
40000	2e+6	0.530033	0.000033	0.005726	+/- 0.000954	+/- 0.002863
10000	1.6e+7	0.525722	0.000093	0.009650	+/- 0.001608	+/- 0.004825
90000	1.6e+7	0.530206	0.000008	0.002878	+/- 0.000480	+/- 0.001439

Analytical Calculation: We have $f(a) = \{0, 1, \dots, 1\}$ and

$$f(b) = \begin{cases} \{0, 0, 1, 0\}, & p = 2 \\ \{0, 1, \dots, 1\}, & \text{otherwise} \end{cases}$$

Case $p = 2$: We have 3 possible pairs and 1 sufficient pair therefore $1/3 = 1/(p^2 - 1)$.
Case $p > 2$: We have $(p^2 - 1)^2$ possible pairs and $p^2 - 1$ sufficient pairs.

We get $\gamma = \{1/(p_1^2 - 1), 1/(p_2^2 - 1), \dots\}$ and (see section 4.25)

$$P_{S,SE} = P_{S,S}$$

4.27 a is squarefree, b is squarefree and odd

Experimental Data:

Table: a is squarefree, b is squarefree and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.530883	0.000017	0.004155	+/- 0.000692	+/- 0.002077
40000	2e+6	0.530867	0.000005	0.002332	+/- 0.000389	+/- 0.001166
10000	1.6e+7	0.530831	0.000021	0.004556	+/- 0.000759	+/- 0.002278
90000	1.6e+7	0.530759	0.000005	0.002332	+/- 0.000389	+/- 0.001166

Analytical Calculation: We have $f(a) = \{0, 1, \ldots, 1\}$ and

$$f(b) = \begin{cases} \{0,1,0,1\}, & p = 2 \\ \{0,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

Case $p = 2$: We have 6 possible pairs and 2 sufficient pair therefore $1/3 = 1/(p^2 - 1)$.
Case $p > 2$: We have $(p^2 - 1)^2$ possible pairs and $p^2 - 1$ sufficient pairs.

We get $\gamma = \{1/(p_1^2 - 1), 1/(p_2^2 - 1), \ldots\}$ and (see section 4.25)

$$P_{S,SO} = P_{S,S}$$

4.28 a is squarefree, b is none squarefree

Experimental Data:

Table: a is squarefree, b is none squarefree

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.727867	0.000004	0.002028	+/- 0.000338	+/- 0.001014
40000	2e+6	0.727876	0.000001	0.000852	+/- 0.000142	+/- 0.000426
10000	1.6e+7	0.727628	0.000005	0.002206	+/- 0.000368	+/- 0.001103
90000	1.6e+7	0.727629	0.000000	0.000669	+/- 0.000111	+/- 0.000334

Numerical Calculation: We have $f(a) = \{0, 1, 1, \ldots\}$ and $f(b) = \{g, 1, 1, \ldots\}$ where $g = \frac{p^2-1}{(1-P_N)p^2-1}$ (see section 4.40).
We get $(p^2 - 1)^2 + g(p^2 - 1)$ possible pairs, $p^2 - 1$ sufficient pairs and $\gamma = \{((1 - P_N)p_i^2 -$

$1)/((1-P_N)p_i^2(p_i^2-1),\dots\}$
Let $a=(1-P_N)$ and $b=-1$ then

$$R_n=\left(1+\frac{b}{a}\right)\frac{1}{2(p_n+1)}-\frac{b}{a}\left(\frac{\pi^2}{8}-\sum_{i=1}^{\frac{p_n-1}{2}}\frac{1}{(2i+1)^2}\right)$$

Summation over the first 260 primes give

$$0.7322374064\le 1-\sigma\left(\frac{(1-P_N)p^2-1}{(1-P_N)p^2(p^2-1)}\right)\le 0.7326794903$$

Analytical Calculation: We have

$$P_{S,N}=P_N=P_N P_{S,S}+(1-P_N)P_{S,F}$$

and therefore

$$P_{S,F}=\frac{P_N(1-P_{S,S})}{1-P_N}$$

4.29 a is squarefree, b is none squarefree and even

Experimental Data:

Table: a is squarefree, b is none squarefree and even

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.774317	0.000005	0.002256	+/- 0.000376	+/- 0.001128
40000	2e+6	0.774031	0.000002	0.001303	+/- 0.000217	+/- 0.000651
10000	1.6e+7	0.774928	0.000004	0.001972	+/- 0.000329	+/- 0.000986
90000	1.6e+7	0.774126	0.000000	0.000682	+/- 0.000114	+/- 0.000341

Numerical Calculation: We have $f(a)=\{0,1,1,\dots\}$ and

$$f(b)=\begin{cases}\{g,0,1,0\}, & p=2\\ \{d,1,\dots,1\}, & \text{otherwise}\end{cases}$$

where $g=\frac{3}{3-4P_N}$ and $d=\frac{p^2-1}{(3-2P_N)p^2/3-1}$ (see section 4.43).

Case $p=2$: We have $3(g+1)$ possible pairs and 1 sufficient pair and therefore $(3-4P_N)/(3(6-4P_N))$
Case $p>2$: We have $d(p^2-1)+(p^2-1)^2$ possible pairs and p^2-1 sufficient pairs.

We get $\gamma = \{(3-4P_N)/(3(6-4P_N)), ((3-2P_N)p_i^2-3)/((3-2P_N)p_i^2(p_i^2-1)), \ldots\}$ ($R_n(a,b)$ where $a = 3-2P_N$ and $b = -3$ see section 4.28).
Summation over the first 260 primes give

$$0.7766532048 \leq P_{N,FE} \leq 0.7770586546$$

Analytical Calculation: We have (note: $P_{S,SE} = P_{S,S}$)

$$P_{NE,S} = \frac{10P_N}{9} = \frac{2P_N}{3}P_{S,S} + (1 - \frac{2P_N}{3}P_{S,FE})$$

and therefore

$$P_{S;FE} = \frac{2P_N(5-3P_{S,S})}{3(1-2P_N)}$$

4.30 a is squarefree, b is none squarefree and odd

Experimental Data:

Table: a is squarefree, b is none squarefree and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.582350	0.000019	0.004391	+/- 0.000732	+/- 0.002195
40000	2e+6	0.580461	0.000011	0.003261	+/- 0.000543	+/- 0.001630
10000	1.6e+7	0.578378	0.000013	0.003667	+/- 0.000611	+/- 0.001834
90000	1.6e+7	0.582240	0.000001	0.000886	+/- 0.000148	+/- 0.000443

Numerical Calculation: We have $f(a) = \{0, 1, 1, \ldots\}$ and

$$f(b) = \begin{cases} \{0,1,0,1\}, & p = 2 \\ \{g, 1, \ldots, 1\}, & \text{otherwise} \end{cases}$$

where $g = \frac{p^2-1}{(\frac{3-4P_N}{3})p^2-1}$ (see section 4.45).

Case $p = 2$: We have 6 possible pairs and 2 sufficient pair and therefore 1/3.
Case $p > 2$: We have $(p_i^2-1)^2 + g(p^2-1)$ possible pairs and $p_i^2 - 1$ sufficient pairs and $1/(g + p^2 - 1)$.

We get $\gamma = \{1/3, ((3-4P_N)p_i^2-3)/((3-4P_N)p_i^2(p_i^2-1)), \ldots\}$ and (start with $\gamma' = \{((3-4P_N)p_i^2-3)/((3-4P_N)p_i^2(p_i^2-1))\}$) and replace $((3-4P_N)4-3)/((3-4P_N)12)$ with 1/3), ($R_n(a,b)$ where $a = 3-4P_N$, $b = -3$ see section 4.28)
Summation over the first 260 primes give

$$0.5818668833 \leq P_{S,FO} \leq 0.5823840943$$

Analytical Calculation: We have (note: $P_{S,SO} = P_{S,S}$)

$$P_{NO,S} = \frac{8P_N}{9} = \frac{4P_N}{3} P_{S,S} + (1 - \frac{4P_N}{3}) P_{S,FO}$$

and therefore

$$P_{S_FO} = \frac{4P_N(2 - 3P_{S,S})}{3(3 - 4P_N)}$$

4.31 a is squarefree and even, b is squarefree and even

Experimental Data:

Table: a is squarefree and even, b is squarefree and even

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.000000	0.000000	0.000000	+/- 0.000000	+/- 0.000000
40000	2e+6	0.000000	0.000000	0.000000	+/- 0.000000	+/- 0.000000
10000	1.6e+7	0.000000	0.000000	0.000000	+/- 0.000000	+/- 0.000000
90000	1.6e+7	0.000000	0.000000	0.000000	+/- 0.000000	+/- 0.000000

Numerical Calculation: We have

$$f(a) = \begin{cases} \{0,0,1,0\}, & p = 2 \\ \{0,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

and $\gamma = \{1, 1/(p_2^2 - 1), 1/(p_3^2 - 1), \ldots\}$ (R_n see section 4.25).
Summation over the first 360 primes gives

$$-0.0008902158 \leq P_{SE,SE} = 1 - \sigma(\gamma) \leq 0.0005031794$$

Analytical Calculation: We have $\gamma = \{1, 1/(p_2^2 - 1), 1/(p_3^2 - 1), \ldots\}$ and with $\gamma' = \{1/(p_1^2 - 1), 1/(p_2^2 - 1), \ldots\}$
$\gamma_i = 1/(p_i^2 - 1)$, $B_- = 1/3$ and $C_- = 1$ (i.e. replace 1/3 with 1)

$$P_{SE,SE} = 1 - \frac{((1 - P_{S,S}) - 1/3)(1 - 1)}{1 - 1/3} - 1 = 0$$

A second approach:
Since all natural even squarefree numbers a are $a \equiv 2 \mod 4$, all sums c of two even squarefree numbers are $c \equiv 0 \mod 4$. Therefore we get $P_{SE,SE} = 0$.

4.32 a is squarefree and even, b is squarefree and odd

Experimental Data:

Table: a is squarefree and even, b is squarefree and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.796167	0.000003	0.001719	+/- 0.000286	+/- 0.000859
40000	2e+6	0.796140	0.000000	0.000592	+/- 0.000099	+/- 0.000296
10000	1.6e+7	0.795775	0.000003	0.001584	+/- 0.000264	+/- 0.000792
90000	1.6e+7	0.796129	0.000000	0.000323	+/- 0.000054	+/- 0.000161

Numerical Calculation: Since

$$f(a) = \begin{cases} \{0,0,1,0\}, & p = 2 \\ \{0,1,\dots,1\}, & \text{otherwise} \end{cases}$$

and Since

$$f(b) = \begin{cases} \{0,1,0,1\}, & p = 2 \\ \{0,1,\dots,1\}, & \text{otherwise} \end{cases}$$

we have $\gamma = \{0, 1/(p_2^2 - 1), 1/(p_3^2 - 1), \dots\}$ (R_n see section 4.25).
Summation over the first 360 primes gives

$$0.7958892988 \le P_{SE,SO} = P_{SO,SE} \le 0.7961498412$$

Analytical Calculation: We have $\gamma = \{0, 1/(p_2^2 - 1), 1/(p_3^2 - 1), \dots\}$.
$\gamma'(1/(p^2 - 1))$, $B_- = 1/3$ (i.e. delete 1/3).

$$P_{SE,SO} = P_{SO,SE} = (1 - \sigma(\gamma)) = 1 - \frac{((1 - P_{S,S}) - 1/3)}{1 - 1/3} = \frac{3P_{S,S}}{2}$$

Remark 14. .
$P_{S,S} = (1/9)P_{SE,SE} + (4/9)P_{SO,SO} + (4/9)P_{SE,SO}$

4.33 a is squarefree and even, b is none squarefree

Experimental Data:

Table: a is squarefree and even, b is none squarefree

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.727558	0.000002	0.001348	+/- 0.000225	+/- 0.000674
40000	2e+6	0.727696	0.000001	0.000813	+/- 0.000136	+/- 0.000407
10000	1.6e+7	0.727847	0.000002	0.001545	+/- 0.000258	+/- 0.000773
90000	1.6e+7	0.727594	0.000000	0.000327	+/- 0.000055	+/- 0.000164

Numerical Calculation: We have Since

$$f(a) = \begin{cases} \{0,0,1,0\}, & p = 2 \\ \{0,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

and $f(b) = \{g, 1, 1, \ldots\}$ where $g = \frac{p^2-1}{(1-P_N)p^2-1}$ (see section 4.40).
Case $p = 2$: We have $g + 3$ possible pairs and 1 sufficient pair therefore $1/(g + p^2 - 1)$.
Case $p > 2$: We have $(p^2 - 1)^2 + g(p^2 - 1)$ possible pairs and $p^2 - 1$ sufficient pairs.

We get $\gamma = \{((1 - P_N)p_i^2 - 1)/((1 - P_N)p_i^2(p_i^2 - 1), \ldots\}$ (see section 4.28)

Analytical Calculation: We have (note: $P_{N,SE} = P_N$ and $P_{S,SE} = P_{S,S}$)

$$P_{N,SE} = P_N P_{S,SE} + (1 - P_N)P_{F,SE}$$

and

$$P_{F,SE} = \frac{P_N(1 - P_{S,S})}{1 - P_N}$$

4.34 a is squarefree and even, b none squarefree and even

Experimental Data:

Table: a is squarefree and even, b is none squarefree and even

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.681475	0.000000	0.000703	+/- 0.000117	+/- 0.000352
40000	2e+6	0.681488	0.000000	0.000254	+/- 0.000042	+/- 0.000127
10000	1.6e+7	0.681592	0.000000	0.000634	+/- 0.000106	+/- 0.000317
90000	1.6e+7	0.681453	0.000000	0.000146	+/- 0.000024	+/- 0.000073

Numerical Calculation: We have

$$f(a) = \begin{cases} \{0,0,1,0\}, & p = 2 \\ \{0,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

and

$$f(b) = \begin{cases} \{g,0,1,0\}, & p = 2 \\ \{d,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

where $g = \frac{3}{3-4P_N}$ and $d = \frac{p^2-1}{(3-2P_N)p^2/3-1}$ (see section 4.43).

Case $P = 2$: We have $g + 1$ possible pairs and 1 sufficient pair and therefore $1/(g + 1)$.
Case $p > 2$: We have $d(p^2 - 1) + (p^2 - 1)^2$ possible pairs and $p^2 - 1$ sufficient pairs and

therefore $1/(d+p^2-1)$.

We get $\gamma = \{(9-8P_N)/((3-2_PN)12), ((3-2P_N)p_i^2-3)/((3-2P_N)p_i^2(p_i^2-1)), \dots\}$ ($R_n(a,b)$ where $a = 3-2P_N$, $b = -3$ see section 4.28)
Summation over the first 260 primes give

$$0.6894892439 \leq P_{SE,FE} \leq 0.689970712$$

Analytical Calculation: We have (note: $P_{NE,SE} = 2P_N/3$ and $P_{SE,SE} = 0$)

$$P_{NE,SE} = \frac{2P_N}{3} P_{SE,SE} + (1 - \frac{2P_N}{3}) P_{SE,FE}$$

and

$$P_{SE,FE} = \frac{2P_N}{3-2P_N}$$

4.35 a is squarefree and even, b is none squarefree and odd

Experimental Data:

Table: a is squarefree and even, b is none squarefree and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.872742	0.000005	0.002243	+/- 0.000374	+/- 0.001121
40000	2e+6	0.872748	0.000001	0.001029	+/- 0.000172	+/- 0.000515
10000	1.6e+7	0.872592	0.000005	0.002340	+/- 0.000390	+/- 0.001170
90000	1.6e+7	0.872565	0.000000	0.000619	+/- 0.000103	+/- 0.000309

Numerical Calculation: We have

$$f(a) = \begin{cases} \{0,0,1,0\}, & p=2 \\ \{0,1,\dots,1\}, & \text{otherwise} \end{cases}$$

and

$$f(b) = \begin{cases} \{0,1,0,1\}, & p=2 \\ \{g,1,\dots,1\}, & \text{otherwise} \end{cases}$$

where $g = \frac{p^2-1}{(\frac{3-4P_N}{3})p^2-1}$ (see section 4.45).

Case $P = 2$: We have 2 possible pairs and 0 sufficient pair and therefore 0.
Case $p > 2$: We have $g(p^2-1) + (p^2-1)^2$ possible pairs and p^2-1 sufficient pairs and therefore $1/(g+p^2-1)$.

We get $\gamma' = \{((3-4P_N)p_i^2-1)/((3-4P_N)p_i^2(p_i^2)^2), \dots\}$ and $B_- = ((3-4P_N)-3)/((3-4P_N)12)$ ($R_n(a,b)$ where $a = 3-4P_N$, $b = -1$ see section 4.28)
Summation over the first 360 primes five

$$0.8731328127 \leq P_{SE,FO} \leq 0.8733692493$$

Analytical Calculation: We have (note: $P_{SE,F} = P_N(1-P_{S,S})/(1-P_N)$ and $P_{SE,FE} = 2P_N/(3-2P_N)$)

$$P_{SE,F} = \frac{3-2P_N}{6(1-P_N)}P_{SE,FE} + \frac{3-4P_N}{6(1-PN)}P_{SE,FO}$$

and

$$P_{SE,FO} = \frac{2P_N(2-3P_{S,S})}{3-4P_N}$$

4.36 a is squarefree and odd, b is squarefree and odd

Experimental Data:

Table: a is squarefree and odd, b is squarefree and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.398914	0.000005	0.002163	+/- 0.000360	+/- 0.001081
40000	2e+6	0.398166	0.000000	0.000590	+/- 0.000098	+/- 0.000295
10000	1.6e+7	0.398308	0.000005	0.002273	+/- 0.000379	+/- 0.001136
90000	1.6e+7	0.398081	0.000000	0.000367	+/- 0.000061	+/- 0.000183

Numerical Calculation: Since

$$f(a) = f(b) = \begin{cases} \{0,1,0,1\}, & p=2 \\ \{0,1,\dots,1\}, & \text{otherwise} \end{cases}$$

, we have $\gamma = \{1/2, 1/(p_2^2-1), 1/(p_3^2-1), \dots\}$ (R_n see section 4.25).
Summation over the first 360 primes gives

$$0.3975903312 \leq P_{SO,SO} \leq 0.3982051918$$

Analytical Calculation: We have $\gamma = \{1/2, 1/(p_2^2-1), 1/(p_3^2-1), \dots\}$.
$\gamma'(1/(p^2-1))$, $B_- = 1/3$ and $C_- = 1/2$ (i.e. replace 1/3 with 1/2).

$$P_{SO,SO} = 1-\sigma(\gamma') = 1 - \frac{((1-P_{S,S})-1/3)(1-1/2)}{1-1/3} - 1/2 = \frac{3P_{S,S}}{4}$$

4.37 a is squarefree and odd, b is none squarefree

Experimental Data:

Table: a is squarefree and odd, b is none squarefree

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.727450	0.000004	0.001920	+/- 0.000320	+/- 0.000960
40000	2e+6	0.727774	0.000001	0.000801	+/- 0.000134	+/- 0.000401
10000	1.6e+7	0.727456	0.000006	0.002498	+/- 0.000416	+/- 0.001249
90000	1.6e+7	0.727673	0.000000	0.000477	+/- 0.000079	+/- 0.000238

Numerical Calculation: We have

$$f(a) = f(b) = \begin{cases} \{0,1,0,1\}, & p = 2 \\ \{0,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

and $f(b) = \{g, 1, 1, \ldots\}$ where $g = \frac{p^2-1}{(1-P_N)p^2-1}$ (see section 4.40).
Case $p = 2$: We have $g + 3$ possible pairs and 1 sufficient pair therefore $1/(g + p^2 - 1)$.
Case $p > 2$: We have $(p^2 - 1)^2 + g(p^2 - 1)$ possible pairs and $p^2 - 1$ sufficient pairs.

We get $\gamma = \{((1 - P_N)p_i^2 - 1)/((1 - P_N)p_i^2(p_i^2 - 1), \ldots\}$ (see section 4.28)

Analytical Calculation: We have (note: $P_{S,F} = P_N(1 - P_{S,S})/(1 - P_N)$ and $P_{SE,F} = 2P_N P_{S,F}/3$)

$$P_{S,F} = \frac{P_N}{3} P_{SE,F} + \frac{2P_N}{3} P_{SO,F}$$

and

$$P_{SO,F} = \frac{P_N(1 - P_{S,S})}{1 - P_N}$$

4.38 a is squarefree and odd, b is none squarefree and even

Experimental Data:

Table: a is squarefree and odd, b is none squarefree and even

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.820447	0.000002	0.001497	+/- 0.000250	+/- 0.000749
40000	2e+6	0.820608	0.000000	0.000701	+/- 0.000117	+/- 0.000350
10000	1.6e+7	0.820581	0.000002	0.001576	+/- 0.000263	+/- 0.000788
90000	1.6e+7	0.820294	0.000000	0.000462	+/- 0.000077	+/- 0.000231

Numerical Calculation: We have

$$f(a) = f(b) = \begin{cases} \{0,1,0,1\}, & p=2 \\ \{0,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

and

$$f(b) = \begin{cases} \{g,0,1,0\}, & p=2 \\ \{d,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

where $g = \frac{3}{3-4P_N}$ and $d = \frac{p^2-1}{(3-2P_N)p^2/3-1}$ (see section 4.43).

Case $p = 2$: We have $2g + 2$ possible pairs and 0 sufficient pairs and therefore 0.
Case $p > 2$: We have $d(p^2 - 1) + (p^2 - 1)^2$ possible pairs and $p^2 - 1$ sufficient pairs.

We get $\gamma = \{0, ((3-2P_N)p_i^2 - 3)/((3-2PN)p_i^2(p_i^2-1)), \ldots\}$ ($R_n(a,b)$ where $a = 3-2P_N$, $b = -3$ see section 4.28)
Summation over the first 260 primes give

$$0.8202351854 \le P_{SO,FE} \le 0.8206026258$$

Analytical Calculation: We have (note: $P_{NE,SO} = 4P_N/3$ and $P_{SE,SO} = 3P_{S,S}/2$)

$$P_{NE,SO} = \frac{2P_N}{3} P_{SE,SO} + (1 - \frac{2P_N}{3}) P_{SO,FE}$$

and

$$P_{SO,FE} = \frac{P_N(4-3P_{S,S})}{3-2P_N}$$

4.39 a is squarefree and odd, b is none squarefree and odd

Experimental Data:

Table: a is squarefree and odd, b is none squarefree and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.436369	0.000004	0.002024	+/- 0.000337	+/- 0.001012
40000	2e+6	0.436217	0.000001	0.001087	+/- 0.000181	+/- 0.000544
10000	1.6e+7	0.436206	0.000003	0.001663	+/- 0.000277	+/- 0.000831
90000	1.6e+7	0.436393	0.000000	0.000543	+/- 0.000091	+/- 0.000272

Numerical Calculation: We have

$$f(a) = f(b) = \begin{cases} \{0,1,0,1\}, & p = 2 \\ \{0,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

and

$$f(b) = \begin{cases} \{0,1,0,1\}, & p = 2 \\ \{g,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

where $g = \frac{p^2-1}{(\frac{3-4P_N}{3})p^2-1}$ (see section 4.45).

Case $P = 2$: We have 4 possible pairs and 2 sufficient pair and therefore 1/2.
Case $p > 2$: We have $g(p^2-1)+(p^2-1)^2$ possible pairs and p^2-1 sufficient pairs and therefore $1/(g+p^2-1)$.

We get $\gamma' = \{1/2, ((3-4P_N)p_i^2-3)/((3-4PN)p_i^2(p_i^2-1)), \ldots\}$ ($R_n(a,b)$ where $a = 3-4P_N$, $b = -3$ see section 4.28)
Summation over the first 260 primes give

$$0.4362711899 \le P_{SO,FO} \le 0.4368743826$$

Analytical Calculation: We have (note: $P_{NO,SO} = 2P_N/3$ and $P_{SO,SO} = 3P_{S,S}/4$))

$$P_{NO,SO} = \frac{4P_N}{3}P_{SO,SO} + \frac{3-4P_N}{3}P_{SO,FO}$$

and

$$P_{SO,FO} = \frac{P_N(2-3P_{S,S})}{3-4P_N}$$

4.40 a is none squarefree, b is none squarefree

Experimental Data:

Table: a is none squarefree, b is none squarefree

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.423864	0.000037	0.006100	+/- 0.001017	+/- 0.003050
40000	2e+6	0.423737	0.000010	0.003151	+/- 0.000525	+/- 0.001576
10000	1.6e+7	0.422072	0.000012	0.003405	+/- 0.000568	+/- 0.001703
90000	1.6e+7	0.422254	0.000005	0.002146	+/- 0.000358	+/- 0.001073

Numerical Calculation: We have $f(a) = \{g, 1, 1, \dots\}$ where $g = \frac{p^2-1}{(1-P_N)p^2-1}$.
Since for natural numbers $f(a) = \{1, \dots, 1\}$ and for squarefree numbers $f(a) = \{0, 1, \dots, 1\}$ we can calculate the remainder distribution of the none squarefree numbers. Let $f(0) := g'$, $f(m) := d$, $0 < m < p$ and we have:

$$(1 - P_N)g' = 1/p^2$$

and

$$1/p^2 = P_N/(p^2 - 1) + (1 - P_N)d$$

Calculate the ratio g'/d and set $d = 1$:

$$d = \frac{(1 - P_N)p^2 - 1}{(1 - P_N)p^2(p^2 - 1)}$$

$$g := \frac{g'}{d} = \frac{p^2 - 1}{(1 - P_N)p^2 - 1}$$

. Finally we get

$$f(a) = \{g, 1, \dots, 1\}$$

The number off all sufficient pairs is: $g^2 + (p^2 - 1)$ and the number of all possible pairs is $(g + (p^2 - 1))^2$ and therefore

$$\gamma_i = \frac{(1 - P_N)^2 p_i^2 + 2P_N - 1}{(1 - P_N)^2 p_i^2 (p_i^2 - 1)}.$$

and

$$P_{F,F} = 1 - \sigma\left(\frac{(1 - P_N)^2 p^2 + 2P_N - 1}{(1 - P_N)^2 p^2 (p^2 - 1)}\right)$$

Calculation of R_n:
Let $a = (1 - P_N)^2$ and $b = 2P_N - 1$:

$$\sum_{i=1}^{\infty} \frac{ap_i^2 + b}{ap_i^2(p_i^2 - 1)} = \sum_{i=1}^{\infty} \frac{1}{p_i^2 - 1} + \sum_{i=1}^{\infty}\left(\frac{b}{a(p_i^2 - 1)} - \frac{b}{ap_i^2}\right) = \left(1 + \frac{b}{a}\right)\sum_{i=1}^{\infty} \frac{1}{p_i^2 - 1} - \frac{b}{a}\sum_{i=1}^{\infty} \frac{1}{p_i^2}$$

and therefore

$$R_n = \left(1 + \frac{b}{a}\right)\frac{1}{2(p_n + 1)} - \left(\frac{\pi^2}{8} - \sum_{i=0}^{\frac{p_n-1}{2}} \frac{1}{(2i + 1)^2}\right)$$

Summation over the first 160 primes gives

$$0.4257533714 \le P_{F,F} \le 0.4267550866$$

Analytical Calculation: We have (note: $P_{S,F} = \frac{P_N(1-P_{S,S})}{1-P_N}$)

$$P_{N,N} = P_N^2 P_{S,S} + 2P_N(1-P_N)P_{S,F} + (1-P_N)^2 P_{F,F}$$

and

$$P_{F,F} = \frac{P_N(1-2P_N+P_N P_{S,S})}{(1-P_N)^2}$$

4.41 a is none squarefree, b is none squarefree and even

Experimental Data:

Table: a is none squarefree, b is none squarefree and even

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.347094	0.000030	0.005467	+/- 0.000911	+/- 0.002733
40000	2e+6	0.351240	0.000003	0.001749	+/- 0.000291	+/- 0.000874
10000	1.6e+7	0.352394	0.000090	0.009494	+/- 0.001582	+/- 0.004747
90000	1.6e+7	0.352690	0.000006	0.002535	+/- 0.000423	+/- 0.001268

Analytical Calculation: We have (note: $P_{NE,F} = \frac{(9-10P_N)P_N}{9(1-P_N)}$ and $P_{SE,F} = \frac{P_N(1-P_{S,S})}{1-P_N}$)

$$P_{NE,F} = \frac{2P_N}{3} P_{SE,F} + \frac{3-2P_N}{3} P_{F,FE}$$

and

$$P_{F,FE} = \frac{P_N(9-16P_N+6P_N P_{S,S})}{3(1-P_N)(3-2P_N)}$$

4.42 a is none squarefree, b is none squarefree and odd

Experimental Data:

Table: a is none squarefree, b is none squarefree and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.650142	0.000076	0.008715	+/- 0.001452	+/- 0.004357
40000	2e+6	0.646808	0.000007	0.002702	+/- 0.000450	+/- 0.001351
10000	1.6e+7	0.648811	0.000027	0.005180	+/- 0.000863	+/- 0.002590
90000	1.6e+7	0.648515	0.000004	0.002012	+/- 0.000335	+/- 0.001006

Analytical Calculation: We have (note: $P_{NO,F} = \frac{(9-8P_N)P_N}{9(1-P_N}$ and $P_{SO,F} = \frac{P_N(1-P_{S,S})}{1-P_N}$)

$$P_{NO,F} = \frac{4P_N}{3}P_{SO,F} + \frac{3-4P_N}{3}P_{F,FO}$$

and

$$P_{F,FO} = \frac{P_N(9-20P_N+12P_NP_{S,S})}{3(3-4P_N)(1-P_N)}$$

4.43 a is none squarefree and even, b is none squarefree and even

Experimental Data:

Table: a is none squarefree and even, b is none squarefree and even

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.218044	0.000089	0.009416	+/- 0.001569	+/- 0.004708
40000	2e+6	0.216811	0.000007	0.002647	+/- 0.000441	+/- 0.001323
10000	1.6e+7	0.210961	0.000020	0.004489	+/- 0.000748	+/- 0.002244
90000	1.6e+7	0.215647	0.000001	0.001203	+/- 0.000200	+/- 0.000601

Numerical Calculation: We have

$$f(a) = \begin{cases} \{g,0,1,0\}, & p=2 \\ \{d,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

where $g = \frac{3}{3-4P_N}$ and $d = \frac{p^2-1}{(3-2P_N)p^2/3-1}$.

Since, for natural even number is

$$f(a) = \begin{cases} \{1,0,1,0\}, & p=2 \\ \{1,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

, for squarefree even numbers is

$$f(a) = \begin{cases} \{0,0,1,0\}, & p=2 \\ \{0,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

and for even none squarefree numbers is

$$f(a) = \begin{cases} \{g'',0,g',0\}, & p=2 \\ \{d,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

First we calculate d. We have $\kappa = (2/3)P_N$, $\kappa' = 1-\kappa$,

$$\frac{1}{p^2} = \frac{\kappa}{p^2-1} + \kappa' d$$

and therefore $d = (p^2 - 1)/(\kappa' p^2 - 1)$ and

$$d = \frac{p^2 - 1}{(1 - \frac{2P_N}{3})p^2 - 1}$$

Next we calculate the ratio $g'' : g'$. Observe that all numbers $a \equiv 0 \mod 4$ are none squarefree, but not all numbers $a \equiv 2 \mod 4$ are squarefree. Therefore we have

$$\frac{1}{2} = \kappa + g' \; ; \; \frac{1}{2} = g''$$

and $g'' : g' = \frac{1}{1-2\kappa} : 1$,

$$\frac{g''}{g'} = \frac{3}{3 - 4P_N}$$

and we get (Note, (sufficient pairs) : (possible pairs) is $(g''^2 + g'^2) : (g'' + g')^2$)

$$\gamma' = \left\{ \frac{1 + (1 - (4/3)P_N)^2}{4(1 - (2/3)P_N)^2}, \frac{(1 - \frac{2}{3}P_N)^2 p_i^2 + \frac{4}{3}P_N - 1}{(1 - \frac{2}{3}P_N)^2 p_i^2 (p_i^2 - 1)}, \dots \right\}, i = 2, 3, \dots$$

With $\gamma = \left\{ \frac{(1-\frac{2}{3}P_N)^2 p_i^2 + \frac{4}{3}P_N - 1}{(1-\frac{2}{3}P_N)^2 p_i^2 (p_i^2 - 1)}, \dots \right\}$, $i = 1, 2, \dots$, $B_- = \frac{4(1-\frac{2}{3}P_N)^2 + \frac{4}{3}P_N - 1}{12(1-\frac{2}{3}P_N)^2} =: \beta_1$ and $C_- = \frac{1+(1-(4/3)P_N)^2}{4(1-(2/3)P_N)^2} =: \beta_2$ we get

$$P_{FE,FE} = 1 - \frac{(\sigma(\gamma) - \beta_1)(1 - \beta_2)}{1 - \beta_1} - \beta_2$$

Summation over the first 260 primes gives:

$$0.2146142543 \leq P_{FE,FE} \leq 0.2155672963$$

Analytical Calculation: We have (note: $P_{SE,FE} = \frac{2P_N}{3-2P_N}$)

$$P_{NE,FE} = \frac{2P_N}{3} P_{SE,FE} + \frac{3 - 2P_N}{3} P_{FE,FE}$$

and

$$P_{FE,FE} = \frac{2P_N(3 - 4P_N)}{(3 - 2P_N)^2}$$

4.44 a is none squarefree and even, b is none squarefree and odd

Experimental Data:

Table: a is none squarefree and even, b is none squarefree and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.769814	0.000009	0.003021	+/- 0.000504	+/- 0.001511
40000	2e+6	0.768051	0.000006	0.002381	+/- 0.000397	+/- 0.001190
10000	1.6e+7	0.767217	0.000009	0.003041	+/- 0.000507	+/- 0.001521
90000	1.6e+7	0.768070	0.000002	0.001384	+/- 0.000231	+/- 0.000692

Analytical Calculation: We have (note: $P_{F,FE} = \frac{P_N(9-16P_N+6P_N P_{S,S})}{3(1-P_N)(3-2P_N)}$ and $P_{FE,FE} = \frac{2P_N(3-4P_N)}{(3-2P_N)^2}$)

$$P_{F,FE} = \frac{3-2P_N}{6(1-P_N)} P_{FE,FE} + \frac{3-4P_N}{6(1-P_N)} P_{FE,FO}$$

and

$$P_{FE,FO} = \frac{12P_N(1-2P_N+P_N P_{S,S})}{(3-2P_N)(3-4P_N)}$$

4.45 a is none squarefree and odd, b is none squarefree and odd

Experimental Data:

Table: a is none squarefree and odd, b is none squarefree and odd

w	N	mean	Var	Std.Var	68.3 %	99.7 %
10000	2e+6	0.270936	0.000023	0.004808	+/- 0.000801	+/- 0.002404
40000	2e+6	0.271267	0.000001	0.001081	+/- 0.000180	+/- 0.000540
10000	1.6e+7	0.275794	0.000008	0.002863	+/- 0.000477	+/- 0.001432
90000	1.6e+7	0.272264	0.000001	0.001023	+/- 0.000171	+/- 0.000512

Numerical Calculation: We have

$$f(a) = f(b) = \begin{cases} \{0,1,0,1\}, & p=2 \\ \{g,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

where $g = \frac{p^2-1}{(\frac{3-4P_N}{3})p^2-1}$.

Since for natural odd number is

$$f(a) = \begin{cases} \{0,1,0,1\}, & p=2 \\ \{1,1,\ldots,1\}, & \text{otherwise} \end{cases}$$

, for squarefree odd numbers is

$$f(a) = \begin{cases} \{0,1,0,1\}, & p=2 \\ \{0,1,\dots,1\}, & \text{otherwise} \end{cases}$$

and for even none squarefree numbers is

$$f(a) = \begin{cases} \{0,1,0,1\}, & p=2 \\ \{g,1,\dots,1\}, & \text{otherwise} \end{cases}$$

Set $\kappa = (4/3)P_N$ (see P_{NO}) and $\kappa' = 1-\kappa$ we get (same calculation as for $P_{F,F}$)

$$\gamma' = \left\{1/2, \frac{(1-\frac{4}{3}P_N)^2 p_i^2 + \frac{8}{3}P_N - 1}{(1-\frac{4}{3}P_N)^2 p_i^2 (p_i^2-1)}, \dots\right\}, i = 2,3,\dots$$

With $\gamma = \left\{\frac{(1-\frac{4}{3}P_N)^2 p_i^2 + \frac{8}{3}P_N - 1}{(1-\frac{4}{3}P_N)^2 p_i^2 (p_i^2-1)}, \dots\right\}$, $i = 1,2,\dots$, $B_- = \frac{4(1-\frac{4}{3}P_N)^2 + \frac{8}{3}P_N - 1}{12(1-\frac{4}{3}P_N)^2} =: \beta_1$ and $C_- = 1/2 =: \beta_2$ we get

$$P_{FO,FO} = 1 - \frac{(\sigma(\gamma)-\beta_1)(1-\beta_2)}{1-\beta_1} - \beta_2$$

Summation over the first 260 primes gives:

$$0.2757437889 \le P_{FO,FO} \le 0.2775894378$$

Analytical Calculation: We have (note: $P_{SO,FO} = \frac{P_N(2-3P_N)}{3-4P_N}$)

$$P_{NO,FO} = \frac{4P_N}{3} P_{SO,FO} + \frac{3-4P_N}{3} P_{FO,FO}$$

and

$$P_{FO,FO} = \frac{2P_N(3-8P_N+6P_N P_{S,S})}{(3-4P_N)^2}$$

References

[PRE02] H. Preininger, *Distribution of the Residues and Cycle Counting*, 2017, http://vixra.org/pdf/1705.0289v1.pdf